On the Equality of Relative Velocities Between Two Objects for Observers in Different Rest Frames

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Applications of the theory of special relativity (SRT) are discussed in which the relative velocity of two objects needs to be determined from the vantage point of two different rest frames. It has generally been assumed without proof that such relative velocities are the same for all observers. In the present work, a general proof based on Einstein's light-speed postulate is given. It is shown to be consistent with applications of the relativistic velocity transformation (RVT) introduced by Einstein in his 1905 paper. One consequence is that the average range of decay of meta-stable particles is directly proportional to the measured lifetime in each rest frame. The fact that an observer traveling with the particles will measure a shorter range of decay than his counterpart on the earth's surface is traced to the increase in length of measuring devices in the former rest frame that results from the acceleration. It is pointed out that this conclusion runs counter to the SRT prediction of FitzGerald-Lorentz length contraction accompanying time dilation.

Keywords: light-speed postulate, relativistic velocity transformation (RVT), meta-stable particle lifetimes, average range before decay, FitzGerald-Lorentz length contraction

I. Introduction

In the original studies of meson decays in the atmosphere [1-3], the quantity which was determined experimentally is the average range before decay L. It was assumed that the value of L is proportional to both the speed u of the particles and their lifetime T measured in a given rest frame. Consequently, L is shorter from the vantage point of the particles' rest frame than for an observer on the earth's surface since the latter's measured value for T is longer. Implicit in this conclusion is the assumption that the speed u of the particles relative to the earth's surface is the same for observers in all rest frames, and thus that the ratio L/T has a constant value for each of them. The results of the latter experiments have subsequently been used in textbooks [4,5] to illustrate the relationship between time dilation and length contraction in the special theory of relativity (SRT [6]).

The assumption that the relative velocity $\mathbf{u}_2 - \mathbf{u}_1$ of two rest frames is the same for all observers is easily justified using the Galilean transformation. However, since relativistic velocities are involved in the above experiments, this means of verifying the above result theoretically is not sufficient. A satisfactory justification of the equality of relative velocities from the vantage point of different rest frames must involve Einstein's relativistic velocity transformation (RVT [6]), as will be discussed below.

II. Implications of the Light-speed Postulate

According to the modern-day definition of the meter [7], the length L of an object is to be determined by measuring the elapsed time T for a light pulse to pass between its two end-points and multiplying this result with c, which is defined to have a value of 2.99792458×10^8 ms^{-1}. When observers in two different rest frames S_1 and S_2 apply this definition, the following two general equations result for the length of the object: $L_1 = c_1 T_{1c}$ and $L_2 = c_2 T_{2c}$, where T_{1c} and T_{2c} are the respective elapsed times. For the time being, it is assumed that the corresponding two values of the light speed, c_1 and c_2, are not equal. An alternative means of measuring the object's length is to measure the elapsed times T_{1v} and T_{2v} for another object to move between the same two end-points. If one assumes that the corresponding measured speeds are v_1 and v_2, respectively, the following two equations result: $L_1 = v_1 T_{1v}$ and $L_2 = v_2 T_{2v}$. The key point is that the length of the object does not depend on whether a light pulse is used to measure it or some other quantity such as a car or a train. The ratio of the two measured lengths is then given by the following set of equations:

$$\frac{L_1}{L_2} = \frac{c_1 T_{1c}}{c_2 T_{2c}} = \frac{v_1 T_{1v}}{v_2 T_{2v}}. \tag{1}$$

On the other hand, the ratio of elapsed times measured by the two observers must be the same no matter what common event is considered, i.e. $\frac{T_{1c}}{T_{2c}} = \frac{T_{1v}}{T_{2v}}$. Therefore, if the light-speed postulate is now assumed, i.e. $c_1 = c_2$, it follows that

$$\frac{L_1}{L_2} = \frac{T_{1c}}{T_{2c}} = \frac{T_{1v}}{T_{2v}} = \frac{v_1 T_{1v}}{v_2 T_{2v}}. \tag{2}$$

The conclusion is therefore that $v_1 = v_2$, as was to be proved. In short, the light-speed postulate implies that two observers must also agree on the speed of any object moving between the two end-points.

In the case of the meta-stable particles, the two end-points are the location in which they are first accelerated by radiation and some other later position such as on the earth's surface. The equality of the measured relative velocities obtained by two observers leads to the following proportionality between their respective measured values of time and distance on a general basis:

$$\frac{L_1}{L_2} = \frac{T_1}{T_2}. \tag{3}$$

In short, the faster an observer's clock runs, the longer the distance L he will measure for the distance an object travels.

III. Using the RVT to Compare Relative Velocities

The RVT relates the velocity components u_i and u'_i of an object from the standpoint of observers in two different rest frames (S and S') that are moving with relative speed v along the x axis of the coordinate system. The three equations for the respective x, y, z components are given below:

$$u'_x = \left(u_x - v\right)\left(1 - v u_x c^{-2}\right)^{-1} = \eta\left(u_x - v\right). \tag{4a}$$

$$u'_y = \left(1 - v^2 c^{-2}\right)^{0.5}\left(1 - v u_x c^{-2}\right)^{-1} u_y = \eta\left[\gamma\left(v\right)\right]^{-1} u_y \tag{4b}$$

$$u'_z = \left(1 - v^2c^{-2}\right)^{0.5}\left(1 - vu_xc^{-2}\right)^{-1}u_z = \eta\left[\gamma(v)\right]^{-1}u_z \qquad (4c)$$

where $\gamma(v) = \left(1 - v^2c^{-2}\right)^{-0.5}$ and c is the speed of light in free space. Note the occurrence of the quantity $\eta = \left(1 - vu_xc^{-2}\right)^{-1}$ in each case. It is a function of both the relative speed v of S and S' as well as the parallel component u_x of the object's velocity relative to S.

A well-known special case will first be considered in which a light pulse moves on a train travelling with speed v along the x axis relative to the station platform [8]. To give the example more general significance, assume that the object is other than a light pulse and moves parallel to the train with speed $u'_{21} = u'_x = w$ for a stationary observer riding on the train. An observer on the platform finds according to the RVT of eqs. (4a-c) that the object on the train moves with speed $u_x = (v+w)\left(1 + vwc^{-2}\right)^{-1}$. The relative speed of the particle and the mid-point of the train from the vantage point of S is therefore:

$$\begin{aligned} u_{21} &= \left[(v+w)\left(1+vwc^{-2}\right)^{-1} - v\right]\left[1 - (v+w)\left(1+vwc^{-2}\right)^{-1}vc^{-2}\right]^{-1} \\ &= c^2\left[v + w - v\left(1+vwc^{-2}\right)\right]\left[c^2\left(1+vwc^{-2}\right) - vw - v^2\right]^{-1} \qquad (5) \\ &= \left(c^2w - v^2w\right)\left(c^2 - v^2\right)^{-1} = w\,. \end{aligned}$$

If the object moves in a perpendicular direction to the train's velocity, the corresponding calculation using the RVT proceeds as follows. From the vantage point of the stationary observer on the train, the object travels with relative speed $u'_{21} = u' = u'_y = w$, with

$u'_x = 0$. The observer on the platform therefore finds on the basis of the RVT that $u_x = v$ and $u_y = \left(1 - v^2 c^{-2}\right)^{0.5} u'_y = \gamma^{-1} w \; (\eta' = 1)$. The velocity of the particle relative to the train u_{21} from the vantage point of the platform observer is therefore obtained from the RVT as:

$$u_{21} = \left(1 - v u_x c^{-2}\right)^{-1} \gamma^{-1} u_y = \gamma^2 \gamma^{-1} \left(\gamma^{-1} w\right) = w, \tag{6}$$

the same value as for the observer on the train. This result is therefore also in agreement with the general conclusion reached in Sect. II on the basis of the SRT light-speed postulate, i.e. that observers in different rest frames always agree on the relative velocity of two objects.

The aberration of starlight at the zenith provides a concrete example for the latter result. Assume that the light moves along the y axis in S′ (a rest frame in the neighboring sky). Thus, $u'_x = 0$, $u'_y = u'_{21y} = c$ and $\eta' = 1$ in the RVT. The observer at rest on the earth (S) finds that S′ moves with speed v along the x axis, i.e. $u_x = v$, and $u_y = c\gamma^{-1}$. Therefore, $\eta = \left(1 - v u_x c^{-2}\right)^{-1} = \gamma^2$. The light appears to be moving at an angle for the observer at rest in S, namely $\tan^{-1}\left(u_x u_y^{-1}\right) = \tan^{-1}\left(\gamma v c^{-1}\right)$, consistent with the aberration phenomenon. It is important to see that this velocity connects the initial location of the light pulse with its final location *at some later time*, and is therefore not equal to the *relative* velocity $\mathbf{u}_{21}$ observed in S. To obtain the latter, it is necessary to use the RVT with the following input: $u_{1x} = u_{2x} = v$ and $u_{2y} = c\gamma^{-1}$. The result is $u_{21x} = 0$ and $u_{21y} = \eta\gamma^{-1} u_{2y} = \eta\gamma^{-2} c = c$

since $\eta = \gamma^2$. Thus, $\mathbf{u}_{21} = \mathbf{u}'_{21}$, in agreement with the result in eqs. (5,6).

The above equality for relative velocities also holds for any other direction of an object's motion [9]. A general case is discussed below in which an object is observed to travel at an angle θ relative to **v**, the velocity of separation of S and S'. The stationary observer in S' finds that the object moves with speed w, with components u_x'=wcosθ in the parallel (x) direction and u_y' = wsinθ in the perpendicular direction (y). The RVT of eqs. (4a-c) can be used to obtain the corresponding velocity **u** for the object's motion *relative to the S' observer* from the vantage point of the stationary observer in S. There are two steps to be followed for each component of **u**: first, the velocity $\mathbf{w}^{\mathbf{S}}$ of the object relative to S is computed; secondly, the desired relative velocity **u** of the object to S' is obtained based on the value of $\mathbf{w}^{\mathbf{S}}$**.** The RVT eq. (4a) gives the following value for the x-component of $\mathbf{w}^{\mathbf{S}}$, namely $w_x^S = (w\cos\theta + v)(1 + wvc^{-2}\cos\theta)^{-1}$. Next, the difference u_x between w_x^S and v needs to be computed, again using eq. (4a), and not simply subtracting v from w_x^S:

$$\begin{aligned}
u_x &= (w_x^S - v)(1 - vw_x^S c^{-2})^{-1} \\
&= [(w\cos\theta + v)(1 + wvc^{-2}\cos\theta)^{-1} - v][1 - vc^{-2}(v + w\cos\theta)(1 + wvc^{-2}\cos\theta)^{-1}]^{-1} \qquad (7) \\
&= (w\cos\theta + v - v - wv^2c^{-2}\cos\theta)(1 + wvc^{-2}\cos\theta - v^2c^{-2} - vwc^{-2}\cos\theta)^{-1} \\
&= w\cos\theta(1 - v^2c^{-2})(1 - v^2c^{-2})^{-1} = w\cos\theta \\
&= u_x'.
\end{aligned}$$

The corresponding calculation of u_y using the RVT eq. 4b in two steps is given next, whereby

$$w_y^S = w\sin\theta(1 - v^2c^{-2})^{0.5}(1 + wvc^{-2}\cos\theta)^{-1}:$$

$$u_y = w_y^S(1 - v^2c^{-2})^{0.5}[1 - vc^{-2}(v + w\cos\theta)(1 + wvc^{-2}\cos\theta)^{-1}]^{-1}$$

$$= w\sin\theta(1-v^2c^{-2})(1 + wvc^{-2}\cos\theta - v^2c^{-2} - vwc^{-2}\cos\theta)^{-1} \quad (8)$$
$$= w\sin\theta(1-v^2c^{-2})(1- v^2c^{-2})^{-1} = w\sin\theta$$
$$= u_y'.$$

It is therefore seen that both components of **u** and **u'** are equal, as was to be proved.

In other words, the RVT leads to the conclusion that two observers always agree on the *relative speed of an object to its starting point or to another object.* This finding is thus consistent with the above conclusion based on the light-speed postulate that the unit of velocity is the same for the respective stationary observers in S and S', even though their units of time generally differ.

IV. Deductions for Time and Length Variations

The realization that observers in different rest frames always agree on the relative velocity of two objects has important consequences. One of these is the proportionality relation of eq. (3). In the context of the muon experiments [1-3], it simply verifies the conclusion that the range of the meta-stable particles is proportional to their lifetime as measured in a given rest frame. A simple way of understanding this relationship is by introducing the concept of units into the discussion. The equality of relative velocities is a direct indication that the unit of speed is the same for all observers (assuming they are at the same gravitational potential [10]). The unit of time increases when clocks slow down systematically. The only way this can be consistent with a constant unit of velocity is if the unit of distance increases by exactly the same fraction. The ratios in eq. (3) are conversion factors between the two sets of units in rest frames S_1 and S_2.

As simple as this idea is, it has nonetheless been misconstrued in textbooks when it comes to the interpretation of the muon-decay experiments. There is general agreement that the range of the particles increases with their lifetime, in agreement with eq. (3). The problem comes when the discussion turns to the fact that the observer moving with the accelerated muons measures all distances to be smaller than does his counterpart at rest on the earth's surface. The explanation is easily found to be the increase in the unit of distance in the rest frame of the accelerated muons (S_1), which in concrete terms means that the *meter stick* employed there is larger than on the earth's surface (in rest frame S_2). Yet textbooks [4,5] typically claim instead that the shorter distances observed in S_1 are due to the length-contraction phenomenon of SRT. However, the terms "time dilation" and "length contraction" refer to changes in objects that are stationary in a moving rest frame (S_1), and not the distances and elapsed times measured with these instruments. Measured values are inversely proportional to the units in which they are expressed. Consequently, the fact that smaller distance values are measured in S_1 actually is an indication that the lengths of the meter stick and all other stationary objects there have increased at the same time that clocks have slowed down relative to those in S_2. Eq. (3) shows that the effect on lengths is the same in all directions, which also runs contrary to the length-contraction prediction of SRT. In summary, the muon-decay experiments indicate that time dilation is accompanied by *isotropic length expansion* in the rest frame of the accelerated particles. This state of affairs is the inevitable consequence of Einstein's light-speed postulate [6] and the equality of measured relative velocities that follows from it.

V. Conclusion

The velocity of a single object is generally different for two observers in relative motion, and the goal of velocity transformations is to specify the relationship between their respective measured values. However, the same two observers agree on the relative velocity of two different rest frames, or alternatively, on the relative velocity of any two objects moving with respect to one another. The latter relationship is obvious when the Galilean transformation is used because the relative speed v of the observers simply cancels in making this determination. Although Einstein's relativistic velocity transformation (RVT) is more complicated, it has been shown in the present work to lead to the same equality for relative velocities.

The above result has an important application for the decay of meta-stable particles. It leads directly to the relationship given in eq. (3). Accordingly, the average range of decay for a given observer is proportional to both his measured values for the lifetime of the particles and their velocity relative to the point of their initial acceleration in the atmosphere. Since the latter value is the same for all observers, it follows that the average range of decay is directly proportional to the measured lifetime in each case. Moreover, the same value for the range is obtained independent of the direction in which the particles move. Therefore, an observer traveling with the particles will measure a shorter range of decay than his counterpart in the laboratory on the earth's surface, and by the same factor (γ^{-1}), whether the particles are moving toward the earth or in some other direction. The reason the former's value for the range of the particles is shorter is because the measuring device he employs has increased in dimension as a result of its acceleration. It is thus an example of isotropic length expansion accompanying time dilation in a given rest frame, the opposite of what is expected on the basis of the length-contraction prediction derived from the Lorentz transformation of SRT.

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