A "Super Atom" Dedicated to Prof. Walter Greiner who is a rare combination of a great teacher and a brilliant researcher

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We consider an atom with a very large number of nucleons. It displays a number of novel features and could be an approximate model for heavy nuclei being studied by Greiner and others.

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"Super Atom"

In the following we would like to observe that in a hypothetical atom with Z the atomic number being very very large, due to a very large number of protons, the outer electrons would behave like an almost free bosonic assembly. Naively, we would expect that the energy levels are given by,

$$|E_n| \propto \frac{(Z\alpha)^2}{2n^2}, n = 1, 2, \cdots (Z >> 1)$$

But let us examine this situation more closely.
We consider the interesting case of a "Super Atom", one in which there are a very large number of protons in the nucleus. We use the fact that when a particle is far from a cloud of charged particles, and moves slowly, the potential is approximately spherically symmetric [1]. So if there is an atom

with atomic number $Z >> 1$, so that the outer most electrons are far away, such an atom is like a Hydrogen atom [2]. In this case, if ρ is the (reduced) distance from the nucleus then we have (Cf.ref.[2])

$$\langle \rho \rangle = \frac{3N^2 - l(l+1)}{2Z} \leq N^2 \tag{1}$$

where N represents the number of states and remembering that Z itself is large. We also have that the energy of the Nth level is given instead by

$$\epsilon_N = \frac{-Z^2}{2N^2} \leq -\frac{1}{N^2} \tag{2}$$

From (1) it follows that the reduced volume V is given by

$$V \leq N^6 \tag{3}$$

It then follows using (3) that the Fermi energy is given by [3]

$$\epsilon_F \propto \left(\frac{N}{V}\right)^{2/3} \leq \frac{1}{N^{3.3}} \tag{4}$$

We can see from (2) and (4) that the outer electron cloud for large N behaves as if it is a degenerate electron gas below the Fermi energy (or temperature). The electrons in such a "Super Atom" are like a Fermi gas except for their negative energy which is approximately zero in any case.

Novel Dimensionality

The situation mimics a White Dwarf star [4]. In particular we argue that such an electron assembly would show bosonic characteristics in that each energy level is filled and in momentum space the particles fill a sphere of radius p_F, the Fermi momentum, just as bosonic phonons fill up a sphere of a similar radius [5].

Let us consider a nearly mono energetic collection of Fermions. We next use the well known formula of the occupation number of a Fermi gas [3]

$$\bar{n}_p = \frac{1}{z^{-1} e^{bE_p} + 1} \tag{5}$$

where, $z' \equiv \frac{\lambda^3}{v} \equiv \mu z \approx z$ because, here, as can be easily shown $\mu \approx 1$,

$$v = \frac{V}{N}, \lambda = \sqrt{\frac{2\pi\hbar^2}{m/b}}$$

$$b \equiv \left(\frac{1}{KT}\right), \quad \text{and} \quad \sum \bar{n}_p = N \tag{6}$$

Let us consider in particular a collection of Fermions which is somehow made nearly mono-energetic, that is, given by the distribution,

$$n'_p = \delta(p - p_0)\bar{n}_p \tag{7}$$

where $\bar{n}_p$ is given by (5).
This is not possible in general - here we consider a special situation of a collection of mono-energetic particles in equilibrium which is the idealization of a contrived experimental set up.
By the usual formulation we have,

$$N = \frac{V}{\hbar^3}\int d\vec{p}n'_p = \frac{V}{\hbar^3}\int \delta(p - p_0)4\pi p^2 \bar{n}_p dp = \frac{4\pi V}{\hbar^3}p_0^2 \frac{1}{z^{-1}e^\theta + 1} \tag{8}$$

where $\theta \equiv bE_{p_0}$.
It must be noted that in (8) there is a loss of dimension in momentum space, due to the δ function in (7) - in fact such a fractal two dimensional situation would in the relativistic case lead us back to the anomalous behaviour (Cf.ref.[6] for details). In the non relativistic case two dimensions would imply that the coordinate ψ of the spherical polar coordinates (r, ψ, ϕ) would become constant, $\pi/2$ in fact. In this case the usual Quantum numbers l and m of the spherical harmonics [7] no longer play a role in the usual radial wave equation

$$\frac{d^2u}{dr^2} + \left\{\frac{2m}{\hbar^2}[E - V(r)] - \frac{l(l+1)}{r^2}\right\}u = 0, \tag{9}$$

The coefficient of the centrifugal term $l(l+1)$ in (9) is replaced by m^2 as in Classical Theory [8].
To proceed, in this case, $KT =< E_p >\approx E_p$ so that, $\theta \approx 1$. But we can continue without giving θ any specific value.
Using the expressions for v and z given in (6) in (7), we get

$$(z^{-1}e^\theta + 1) = (4\pi)^{5/2}\frac{z^{'-1}}{p_0}; \text{whence}$$

$$z^{'-1}A \equiv z^{'-1}\left(\frac{(4\pi)^{5/2}}{p_0} - e^\theta\right) = 1, \tag{10}$$

where we use the fact that in (6), $\mu \approx 1$ as can be easily deduced.
A number of conclusions can be drawn from (10). For example, if,

$$A \approx 1, i.e.,$$

$$p_0 \approx \frac{(4\pi)^{5/2}}{1+e} \tag{11}$$

where A is given in (10), then $z' \approx 1$. Remembering that in (6), λ is of the order of the de Broglie wave length and v is the average volume occupied per particle, this means that the gas gets very densely packed for momenta given by (11). Infact for a Bose gas, as is well known, this is the condition for Bose-Einstein condensation at the level $p = 0$ (cf.ref.[3]).
On the other hand, if,

$$A \approx 0(\text{that} \quad \text{is} \quad \frac{(4\pi)^{5/2}}{e} \approx p_0)$$

then $z' \approx 0$. That is, the gas becomes dilute, or V increases.
More generally, equation (10) also puts a restriction on the energy (or momentum), because $z' > 0$, viz.,

$$A > 0(i.e. p_0 < \frac{(4\pi)^{5/2}}{e})$$

$$\text{But} \quad \text{if} A < 0, (i.e. p_0 > \frac{(4\pi)^{5/2}}{e})$$

then there is an apparent contradiction.
The contradiction disappears if we realize that $A \approx 0$, or

$$p_0 = \frac{(4\pi)^{5/2}}{e} \tag{12}$$

(corresponding to a temperature given by $KT = \frac{p_0^2}{2m}$) is a threshold momentum (phase transition). For momenta greater than the threshold given by (12), the collection of Fermions behaves like Bosons. In this case, the occupation number is given by

$$\bar{n}_p = \frac{1}{z^{-1} e^{bE_p} - 1},$$

instead of (5), and the right side equation of (10) would be given by $' - 1'$ instead of $+1$, so that there would be no contradiction. Thus in this case there is an anomalous behaviour of the Fermions.
We could consider a similar situation for Bosons also where an equation like (7) holds. In this case we have equations like (11) and (12):

$$p_0 \approx \frac{(4\pi)^{5/2}}{1.4e - 1} \tag{13}$$

$$p_0 \approx \frac{(4\pi)^{5/2}}{e} \tag{14}$$

(14) is the same as (12), quite expectedly. It gives the divide between the Fermionic and Bosonic behaviour in the spirit of the earlier remarks. At the momentum given by (13) we have a densely packed Boson gas rather as in the case of Bose Einstein condensation. On the other hand at the momentum given by (14) we have infinite dilution, while at lower momenta than in (14) there is an anomalous Fermionisation.
So there is a fractal dimensionality 1.5 or 2 [9], so that we can expect $|\epsilon_N| \sim |\epsilon_F|$. In any case, due to this dense packing, the electron cloud would experience the pressure due to the exclusion principle as in the case of a degenerate Fermi gas. Also we have taken the spin to be zero in equation (4), but this merely contributes a numerical factor.

Remark:

This is a case where super critical potentials, studied by Greiner and coworkers [10] appears.

Appendix:

Super "Nuclei" and White Dwarf Stars [11]:

It is known that Neutron stars or Pulsars have strong magnetic fields of $\sim 10^8$ Tesla in their vicinity, while certain White Dwarfs have magnetic fields $\sim 10^2$ Tesla. If we were to use conventional arguments that when a sun type star with a magnetic field $\sim 10^{-4}$ Tesla contracts, there is conservation of magnetic flux, then we are lead to magnetic fields for Pulsars and White Dwarfs which are a few orders of magnitude less than the required values

[12].
We will now argue, that in the light of the above results that below the Fermi temperature, the degenerate electron gas obeys a semionic statistics, that is a statistics in between the Fermi-Dirac and Bose-Einstein, it is possible to deduce the correct magnetic fields for Neutron stars and White Dwarfs. Moreover this will also enable us to deduce the correct magnetic field of a planet like the earth.
We have for the energy density e, in case of sub Fermi temperatures,

$$e \propto \int_o^{p_F} \frac{p^2}{2m} d^3p \propto T_F^{2.5} \tag{15}$$

where p_F is the Fermi momentum and T_F is the Fermi temperature. On the other hand, it is known that [13,14] in n dimensions we have,

$$e \propto T_F^{n+1} \tag{16}$$

(For the case $n = 3$, (16) is identical to the Stefan-Boltzmann law). Comparison of (16) and (15) shows that the assembly behaves with the fractal dimensionality 1.5.
Let us now consider an assembly of N electrons. As is known, if N_+ is the average number of particles with spin up, the magnetisation per unit volume is given by

$$M = \frac{\mu(2N_+ - N)}{V} \tag{17}$$

where μ is the electron magnetic moment. At low temperatures, in the usual theory, $N_+ \approx \frac{N}{2}$, so that the magnetisation given in (17) is very small. On the other hand, for Bose-Einstein statistics we would have, $N_+ \approx N$. With the above semionic statistics we have,

$$N_+ = \beta N, \frac{1}{2} < \beta < 1, \tag{18}$$

If N is very large, this makes an enormous difference in (17).
Let us first use (17) and (18) for the case of Neutron stars. In this case, as is well known, we have an assembly of degenerate electrons at temperatures $\sim 10^7 K$, (cf.for example [3]). So our earlier considerations apply. In the case of a Neutron star we know that the number density of the degenerate electrons, $n \sim 10^{31}$ per c.c.[15,16]. So using (17) and (18) and remembering that $\mu \approx 10^{-20} G$, the magnetic field near the Pulsar is $\sim 10^{11} G \lesssim 10^8$ Tesla,

as required.
Some White Dwarfs also have magnetic fields. If the White Dwarf has an interior of the dimensions of a Neutron star, with a similar magnetic field, then remembering that the radius of a White Dwarf is about 10^3 times that of a Neutron star, its magnetic field would be 10^{-6} times that of the neutron star, which is known to be the case.
Such considerations have been used by the author for the earth's magnetic field too [17].

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