An Attempt To Explain Dark Energy In Terms Of Statistical Anisotropy

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An attempt to explain dark energy is made in terms of a modelling error introduced by the simplifications used to derive the Friedman equations of cosmology. A demonstration is given that small, statistically-fluctuating anisotropic terms in the Robertson-Walker line element could effect an unexpected additional expansion. A modelling error might therefore account for the cosmological constant, and anisotropies explain the anomalous acceleration in the expansion of the Cosmos. Such an interpretation has the advantage of not requiring any new fundamental physics. By assuming a flat Friedmann cosmology and then adding some statistical anisotropies the need for a locally hyperbolic geometry rather than the original flat geometry results. Hyperbolic geometry is more expansive and open than flat geometry in some sense, and so the tendency towards openness is suggestive of dark energy. Unfortunately (and for several reasons necessarily) an actual acceleration term does not appear in this local geometry in spite of this feature.

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Introduction

An attempt to explain dark energy is made in terms of a modelling error introduced by the simplifications used to derive the Friedman equations of cosmology. A demonstration is given that small, statistically-fluctuating anisotropic terms in the Robertson-Walker line element could effect an unexpected additional expansion. A modelling error might therefore account for the cosmological constant, and anisotropies explain the anomalous acceleration in the expansion of the Cosmos. Such an interpretation has the advantage of not requiring any new fundamental physics. By assuming a flat Friedmann cosmology and then adding some statistical anisotropies the need for a locally hyperbolic geometry rather than the original flat geometry results. Hyperbolic geometry is more expansive and open than flat geometry in some sense, and so the tendency towards openness is suggestive of dark energy. Unfortunately (and for several reasons necessarily) an actual acceleration term does not appear in this local geometry in spite of this feature.

The averaging problem is the problem that the time evolution of an averaged metric need not be the average result of the time evolution of the original metric. There is a reasonably large amount of literature on the averaging problem of general relativity which includes possible explanations for dark energy in similar terms to those used here i.e. as a result of a modelling error, that is, inaccuracies in the assumption that the universe is perfectly homogeneous and isotropic. There appear to be a wide range of possible phenomena that have been invoked to achieve this, the most current and prominent being the idea that the Earth is in a special volume of space-time that is expanding differently from the mean [7]. Other phenomena have also been invoked including the idea of a 'backreaction' from inhomogeneities [8]. Here a similar argument is presented based on a statistical distribution of anisotropies.

In order to investigate whether substantially similar works have already been undertaken a literature search was undertaken. First, the review paper by Célérier [8], but as the title suggests this deals mainly with inhomogeneous models, rather than anisotropic ones. Similarly and perhaps more relevantly the review paper by Buchert [9] on dark energy from structure. This in turn references a further review paper [13]. It turns out that similar but not identical ideas are expressed by Buchert et al [9], [10] "Dark Energy emerges as unbalanced kinetic and potential energies due to structural inhomogeneities." The second of these references being in preparation. A similar line element has been used in the Lemaitre-Tolman-Bondi solution [11], but again has been exploited for purposes of explaining the cosmological constant primarily in terms of inhomogeneities [8][12]. Similarly in [14]. [14] does however quote: .".. it is suggested that the cosmic acceleration might originate from the violation of the cosmological principle, homogeneity and isotropy." There are many other attempts to use inhomogeneity with respect to the averaging problem to derive or imply a cosmic acceleration e.g. [16][17]. And anisotropies are also considered in [15],[17] and [18]. Some of the steps in the present argument are approximative and also statistical in nature. Standard texts such as Wald [4] refer to anisotropic cosmologies as well as perturbations, but without any statistical element.

It is assumed that the k term in the Robertson-Walker line element is 0 giving an approximately flat cosmology and that real space-time varies slightly from that. The mathematics is based on a standard derivation of the Friedman equations from the Robertson-Walker line element given in *A Short Course in General Relativity* [1] and some notation, definitions and conventions (e.g. sign conventions) were taken from this for ease.

Surveys of dark energy, quintessence and related matters can be found in [2], [5], [6] and [13].

Conventions

Following the notation similar to [1] we have:
Space-time with signature [+, –, –, –] during the calculation.
The speed of light c = 1
Ricci tensor defined as follow:

$$R_{mv} = \Gamma^{s}_{ms,v} - \Gamma^{s}_{mv,s} + \Gamma^{r}_{ms}\Gamma^{s}_{rv} - \Gamma^{r}_{mv}\Gamma^{s}_{rs} \tag{1}$$

Calculation

In the standard Robertson-Walker [1] cosmology we have line element:

$$d\tau^2 = dt^2 - \left(R(t)\right)^2\left(\left(1-kr^2\right)^{-1}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\varphi^2\right) \tag{2}$$

where R(t) is a dimensionless scale factor depending only on the time t. We have k = –1,0 or 1. We have timelike coordinate t and spatial coordinates r, θ and Φ. Here the assumption is made that it is possible to model anisotripic terms, to within close approximation, using an adjusted Robertson-Walker line element with small variations from the above equation. This is done using the following line element:

$$\begin{aligned} d\tau^2 = dt^2 &- \left(R_1(t,r,\theta,\varphi)\right)^2\left(1-kr^2\right)^{-1}dr^2 \\ &+ \left(R_2(t,r,\theta,\varphi)\right)^2 r^2 d\theta^2 + \left(R_3(t,r,\theta,\varphi)\right)^2 r^2\sin^2\theta d\varphi^2 \end{aligned} \tag{3}$$

where the Ri represent small variations from R(t) as a function of all coordinates. For simplicity we will generally only write Ri (–)

to remind us that these are functions of all coordinates or just Ri where this is not necessary.
Using lagrangian methods [1] [3] we can define the lagrangian:

$$L(\dot{x}^s, x^s) = \frac{1}{2} g_{ab}(x^s)\dot{x}^a \dot{x}^b \tag{4}$$

where the superscript on x from the set {0, 1, 2, 3} indicates {t, r, θ, Φ} respectively. And superscript dot is a partial derivative with respect to an affine parameter. The above lagrangian equals:

$$\begin{gathered}(\dot{t}^2 - (R_1(t,r,\theta,\phi))^2(1-kr^2)^{-1}\dot{r}^2 \\ +(R_2(t,r,\theta,\phi))^2 r^2 \dot{\theta}^2 + (R_3(t,r,\theta,\phi))^2 r^2 \sin^2\theta\,\dot{\phi}^2)/2\end{gathered} \tag{5}$$

The Euler-Lagrange equation can then be used to calculate the Christoffel symbols [1][3] since the Euler-Lagrange equations,

$$\frac{d}{du}\frac{\partial L}{\partial \dot{x}^c} - \frac{dL}{dx^c} = 0 \tag{6}$$

give a set of equations of the form:

$$\ddot{x} + \Gamma^c_{ab}\dot{x}^a x^b = 0 \tag{7}$$

From this the Christoffel symbols can simply be read off. Although this follows the standard derivation of the Friedmann equations [1] the extra terms generated by the Ri are important here and so they will also be derived in detail. First we write down the derivatives of the lagrangian:

$$\frac{\partial L}{\partial t} = -R_1(-)\frac{\partial R_1(-)}{\partial t}\left[(1-kr^2)^{-1}\dot{r}^2 + r^2\dot{\theta}^2 + r^2(\sin^2\theta)\dot{\phi}^2\right]$$

$$\frac{\partial L}{\partial \dot{t}} = \dot{t}$$

$$\frac{\partial L}{\partial \dot{r}} = -R_1(-)^2 (1-kr^2)^{-1} \dot{r}$$

$$\begin{aligned}\frac{\partial L}{\partial r} = &- R_1(-)^2 (1-kr^2)^{-2} kr\, \dot{r}^2 - R_2(-)^2 r\, \dot{\theta}^2 - R_3(-)^2 r(\sin^2\theta)\, \dot{\phi}^2 \\ &- R_1(-)\frac{\partial R_1(-)}{\partial r}(1-kr^2)^{-1}\dot{r}^2 - R_2(-)\frac{\partial R_2(-)}{\partial r} r^2 \dot{\theta}^2 \\ &- R_3(-)\frac{\partial R_3(-)}{\partial r} r^2 (\sin^2\theta)\, \dot{\phi}^2\end{aligned} \tag{10}$$

Where all terms after the first row above are new terms with respect to the original derivation and not present in the Friedmann cosmology.

$$\frac{\partial L}{\partial \dot{\theta}} = -R_2(-)^2 r^2 \dot{\theta}$$

$$\begin{aligned}\frac{\partial L}{\partial \theta} = &- R_2(-)^2 r^2 \sin\theta\cos\theta\, \dot{\phi}^2 \\ &- R_1(-)\frac{\partial R_1(-)}{\partial \theta}(1-kr^2)^{-1}\dot{r}^2 - R_2(-)\frac{\partial R_2(-)}{\partial \theta} r^2 \dot{\theta}^2 \\ &- R_3(-)\frac{\partial R_3(-)}{\partial \theta} r^2 (\sin^2\theta)\, \dot{\phi}^2\end{aligned} \tag{9}$$

Where all terms after the first row above are new with respect to the original derivation.

$$\begin{aligned}\frac{\partial L}{\partial \phi} = &- R_1(-)\frac{\partial R_1(-)}{\partial \phi}(1-kr^2)^{-1}\dot{r}^2 - R_2(-)\frac{\partial R_2(-)}{\partial \phi} r^2 \dot{\theta}^2 \\ &- R_3(-)\frac{\partial R_3(-)}{\partial \phi} r^2 (\sin^2\theta)\, \dot{\phi}^2\end{aligned} \tag{10}$$

Where all terms above are new.

$$\frac{\partial L}{\partial \dot{\phi}} = -R_3(-)^2 r^2 (\sin^2\theta)\dot{\phi} \tag{10}$$

Using these terms we can then find 4 differential equations corresponding to the Euler-Lagrange equations by algebraic manipulation and by taking partial derivatives:

$$0 = \ddot{t} + R_1(-)\frac{\partial R_1(-)}{\partial t}\left[(1-kr^2)^{-1}\dot{r}^2 + r^2\dot{\theta}^2 + r^2(\sin^2\theta)\dot{\phi}^2\right]$$

$$0 = \ddot{r} + \left[(1-kr^2)^{-1}kr + R_1(-)^{-1}\frac{\partial R_1(-)}{\partial r}\right]\dot{r}^2 + 2\mathrm{R}_1(-)^{-1}\frac{\partial R_1(-)}{\partial t}\dot{r}\dot{t}$$

$$-\left[\frac{R_2(-)^2}{R_1(-)^2}(1-kr^2)r + \frac{R_2(-)}{R_1(-)^2}\frac{\partial R_2(-)}{\partial r}(1-kr^2)r^2\right]\dot{\theta}^2$$

$$-\left[\frac{R_3(-)^2}{R_1(-)^2}\sin^2\theta(1-kr^2)r + \frac{R_3(-)}{R_1(-)^2}\frac{\partial R_3(-)}{\partial r}\sin^2\theta(1-kr^2)r^2\right]\dot{\phi}^2$$

$$+2\mathrm{R}_1(-)^{-1}\frac{\partial R_1(-)}{\partial \theta}\dot{r}\dot{\theta} \quad +2\mathrm{R}_1(-)^{-1}\frac{\partial R_1(-)}{\partial \theta}\dot{r}\dot{\theta}$$

$$0 = \ddot{\theta} - \frac{R_1(-)}{R_2(-)^2}\frac{\partial R_1(-)}{\partial \theta}(1-kr^2)^{-1}r^{-2}\dot{r}^2$$

$$-\left[\sin\theta\cos\theta + \frac{R_3(-)}{R_2(-)^2}\frac{\partial R_3(-)}{\partial \theta}\sin^2\theta\right]\dot{\phi}^2$$

$$+R_2(-)^{-1}\frac{\partial R_2(-)}{\partial\theta}\dot{\theta}^2 \qquad +2\mathrm{R}_2(-)^{-1}\frac{\partial R_2(-)}{\partial\phi}\dot{\theta}\dot{\phi}$$

$$+\left[\frac{2}{r}+2\mathrm{R}_2(-)^{-1}\frac{\partial R_2(-)}{\partial r}\right]\dot{r}\dot{\theta} \qquad +2\mathrm{R}_2(-)^{-1}\frac{\partial R_2(-)}{\partial\theta}\dot{t}\dot{\theta}$$

$$0 \;=\; \ddot{\phi} \;+\; R_3(-)^{-1}\frac{\partial R_3(-)}{\partial\phi}\dot{\phi}^2$$

$$-\;\frac{R_1(-)}{R_3(-)^2}\frac{\partial R_1(-)}{\partial\phi}(1-kr^2)^{-1}(r^2\sin^2\theta)^{-1}\dot{r}^2$$

$$-\;\frac{R_2(-)}{R_3(-)^2}\frac{\partial R_2(-)}{\partial\phi}\dot{\theta}^2\sin^{-2}\theta \;+\; 2\mathrm{R}_3(-)^{-1}\frac{\partial R_3(-)}{\partial t}\dot{t}\dot{\phi}$$

$$+\left[2\cos\theta\sin^{-1}\theta \;\; +2\mathrm{R}_3(-)^{-1}\frac{\partial R_3(-)}{\partial\theta}\right]\dot{\theta}\dot{\phi}$$

$$+\left[\frac{2}{r}+2\mathrm{R}_3(-)^{-1}\frac{\partial R_3(-)}{\partial r}\right]\dot{r}\dot{\theta} \tag{11}$$

These equations can then be used as in [1] to read off the non-zero Christoffel symbols. In the sequel those terms to the far right after a long space are the extra terms not present in the original calculation and have been separated here for ease of comparison.

$$\Gamma^0_{11} \;=\; R_1(-)\frac{\partial R_1(-)}{\partial t}(1-kr^2)^{-1}$$

$$\Gamma^0_{22} \;=\; R_1(-)\frac{\partial R_1(-)}{\partial t}r^2 \qquad \Gamma^0_{33} \;=\; R_1(-)\frac{\partial R_1(-)}{\partial t}r^2(\sin^2\theta)$$

$$\Gamma^1_{11} = (1-kr^2)^{-1}kr + R_1^{-1}(-)\frac{\partial R_1(-)}{\partial r}$$

$$\Gamma^1_{22} = -\frac{R_2(-)^2}{R_1(-)^2}(1-kr^2)r - \frac{R_2(-)}{R_1(-)^2}\frac{\partial R_2(-)}{\partial r}(1-kr^2)r^2$$

$$\Gamma^1_{33} = -\frac{R_3(-)^2}{R_1(-)^2}\sin^2\theta(1-kr^2)r - \frac{R_3(-)}{R_1(-)^2}\frac{\partial R_3(-)}{\partial r}\sin^2\theta(1-kr^2)r^2$$

$$\Gamma^1_{10} = R_1(-)^{-1}\frac{\partial R_1(-)}{\partial t}$$

$$\Gamma^1_{12} = R_1(-)^{-1}\frac{\partial R_1(-)}{\partial \theta}$$

$$\Gamma^1_{13} = R_1(-)^{-1}\frac{\partial R_1(-)}{\partial \phi}$$

$$\Gamma^2_{02} = R_2(-)^{-1}\frac{\partial R_2(-)}{\partial t}$$

$$\Gamma^2_{12} = \frac{1}{r} + R_2(-)^{-1}\frac{\partial R_2(-)}{\partial r}$$

$$\Gamma^2_{23} = R_2(-)^{-1}\frac{\partial R_2(-)}{\partial \phi}$$

$$\Gamma^2_{22} = R_1(-)^{-1}\frac{\partial R_1(-)}{\partial \theta}$$

$$\Gamma^2_{11} = -\frac{R_1(-)}{R_2(-)^2}\frac{\partial R_1(-)}{\partial\theta}(1-kr^2)^{-1}r^{-2}$$

$$\Gamma^2_{33} = -\sin\theta\cos\theta - \frac{R_3(-)}{R_2(-)^2}\frac{\partial R_3(-)}{\partial\theta}\sin^2\theta$$

$$\Gamma^3_{23} = \sin^{-1}\theta\cos\theta + R_3(-)^{-1}\frac{\partial R_3(-)}{\partial\theta}$$

$$\Gamma^3_{33} = R_3(-)^{-1}\frac{\partial R_3(-)}{\partial\phi}$$

$$\Gamma^3_{11} = -\frac{R_1(-)}{R_3(-)^2}\frac{\partial R_1(-)}{\partial\phi}(1-kr^2)^{-1}(r^2\sin^2\theta)^{-1}$$

$$\Gamma^3_{22} = -\frac{R_2(-)}{R_3(-)^2}\frac{\partial R_2(-)}{\partial\phi}\sin^{-2}\theta$$

$$\Gamma^3_{13} = 1/r + R_3(-)^{-1}\frac{\partial R_3(-)}{\partial r}$$

$$\Gamma^3_{03} = R_3(-)^{-1}\frac{\partial R_3(-)}{\partial t} \qquad (12)$$

Now collecting the terms and using the definition of the Ricci tensor and whilst making certain simplifying (physically reasonable) assumptions we can obtain the terms of the Ricci tensor.

The simplifying assumptions are as follows:

(i) k = 0

(ii) That the Ri follow the same distribution (normal or normal-squared, or something similar) with mean R(t) and that the mean of the square of Ri/Rj (where j is not equal to i) is 1+μ where μ is a positive constant.

(iii) That the anisotropy is sufficiently local that we can put in these mean and mean squared terms where they arise in the Ricci tensor.

(iv) That space-time is sufficiently isotropic and homogeneous that we can ignore spatial derivatives at a large (ie cosmological) scale.

The 1+μ term in Assumption (II) is justified as for any two non-zero, non-equal reals *a* and *b*, the following equation holds:

$$\frac{a^2}{b^2}+\frac{b^2}{a^2} > 2 \tag{13}$$

which follows from $a^2 + b^2 > 2ab$ for a and b non-equal by inspecting the quadratic equation $x^2 – 2x+1=0$. So the mean of the square of a/b where a and b follow the same distribution independently is more than 1. This is also true for the mean of a/b where a and b are positive, but it just turns out that this is not the term needed, so we use squared terms throughout.

Assumption (IV) happens to be strong enough to get rid of all the 'new' terms in the Christoffel symbols leaving the old terms but adjusted by the varying R_i. The mean squared terms are present only in 2 of the Christoffel symbol components:

$$\Gamma^1_{22} \quad and \quad \Gamma^1_{33} \tag{14}$$

This only changes two terms in the final Ricci tensor. Both are diagonal components.

The working proceeds as follows: consider single components of the new Ricci tensor R_{mv}

We can look to see which terms might be different from that calculated for a Robertson-Walker cosmology without statistical fluctuations. We can look at the definition of the Ricci tensor: $R_{mv}=\Gamma^{s}_{ms,v}-\Gamma^{s}_{mv,s}+\Gamma^{r}_{ms}\Gamma^{s}_{rv}-\Gamma^{r}_{mv}\Gamma^{s}_{rs}$, for varying values of m and n to see if we can find terms that contain

$$\Gamma^{1}_{22} \quad or \quad \Gamma^{1}_{33} \tag{15}$$

Call these 'differing' terms.

For example when m is 0 or 1 there are no differences, because in the Ricci tensor definition the bottom row of the indices of the first two 'single' Christoffel symbols can't be 33,? or 22,? because m can neither be 2 nor 3. The last two 'doubles' can't contain a differing term unless s or r=1. In the case r = 1 we must have m = 2 or 3 (a contradiction again). In the case s=1 we must have v= 2 or 3 = r in which case any differing doubles contain one of the following terms:

$$\Gamma^{2}_{m1} \quad or \quad \Gamma^{3}_{m1} \tag{16}$$

which are 0 for m = 0 or 1 as can be read off from the previous working above, and therefore the differring term gets multiplied by 0 as so contributes no difference.

By symmetry this argument also discounts v = 0 or 1. This leaves only the terms R22, R32 and R33.

Take the case m=3

$$R_{3\mathrm{v}}=\Gamma^{s}_{3\mathrm{s},v}-\Gamma^{s}_{3\mathrm{v},s}+\Gamma^{r}_{3\mathrm{s}}\Gamma^{s}_{rv}-\Gamma^{r}_{3v}\Gamma^{s}_{rs} \tag{17}$$

The first term vanishes for all s, giving

$$R_{3\mathrm{v}}=-\Gamma^{s}_{3\mathrm{v},s}+\Gamma^{r}_{3\mathrm{s}}\Gamma^{s}_{rv}-\Gamma^{r}_{3v}\Gamma^{s}_{rs} \tag{18}$$

In order for there to be a differing term we must have either s=1 or r=1. They give the following possible differing terms (being careful not to count the case s=r=1 twice). The second and third terms below are for s=1, the rest are for r=1, but excluding a repeat of s=1.

$$-\Gamma^{1}_{3v,1}+\Gamma^{r}_{31}\Gamma^{1}_{rv}-\Gamma^{r}_{3v}\Gamma^{1}_{r1}$$
$$+\Gamma^{1}_{30}\Gamma^{0}_{1v}-\Gamma^{1}_{3v}\Gamma^{0}_{10}+\Gamma^{1}_{32}\Gamma^{2}_{1v}-\Gamma^{1}_{3v}\Gamma^{2}_{12}+\Gamma^{1}_{33}\Gamma^{3}_{1v}-\Gamma^{1}_{3v}\Gamma^{3}_{13} \tag{19}$$

Expanding for index r and cancelling 0 terms,

$$-\Gamma^{1}_{3v,1}+\Gamma^{3}_{31}\Gamma^{1}_{3v}-\Gamma^{0}_{3v}\Gamma^{1}_{01}-\Gamma^{1}_{3v}\Gamma^{2}_{12}+\Gamma^{1}_{33}\Gamma^{3}_{1v}-\Gamma^{1}_{3v}\Gamma^{3}_{13} \tag{20}$$

Cancelling terms,

$$-\Gamma^{1}_{3v,1}-\Gamma^{0}_{3v}\Gamma^{1}_{01}-\Gamma^{1}_{3v}\Gamma^{2}_{12}+\Gamma^{1}_{33}\Gamma^{3}_{1v} \tag{21}$$

The second term can not be differing, so:

$$-\Gamma^{1}_{3v,1}-\Gamma^{1}_{3v}\Gamma^{2}_{12}+\Gamma^{1}_{33}\Gamma^{3}_{1v} \tag{22}$$

For v= 2 these are all 0, so look at case v=3 for differing terms.

$$-\Gamma^{1}_{33,1}-\Gamma^{1}_{33}\Gamma^{2}_{12}+\Gamma^{1}_{33}\Gamma^{3}_{13} \tag{23}$$

This gives a differing term of:

$$+(1+\mu)\sin^{2}\theta \tag{24}$$

So the diagonal term can be written as:

$$R_{33}= \quad (-(R\ddot{R}+2\dot{R}^{2})r^{2}+\mu)\sin^{2}\theta \tag{25}$$

The only other term that turns out to be different is the remaining diagonal component:

$$R_{22}= \quad -(R\ddot{R}+2\dot{R}^{2})r^{2}+\mu \tag{26}$$

Where the unindexed Rs in the above equation are not scalar curvatures but the R(t) parameter in the Robertson-Walker line element. And the difference to the usual curvature is simply the addition of the μ term.

The term arises as follows:

$$R_{22}=\Gamma^{s}_{2s,2}-\Gamma^{s}_{22,s}+\Gamma^{r}_{2s}\Gamma^{s}_{r2}-\Gamma^{r}_{22}\Gamma^{s}_{rs} \tag{27}$$

The differing terms being contained in the second single and the two doubles as follows:

$$-\Gamma^{1}_{22,1}+\Gamma^{r}_{2s}\Gamma^{s}_{r2}-\Gamma^{r}_{22}\Gamma^{s}_{rs} \tag{28}$$

Where for the first double we have either r=2, s=1 or r=1, s=2...

$$-\Gamma^{1}_{22,1}+\Gamma^{1}_{22}\Gamma^{2}_{12}+\Gamma^{2}_{21}\Gamma^{1}_{22}-\Gamma^{r}_{22}\Gamma^{s}_{rs}$$

$$-\Gamma^{1}_{22,1}+2\Gamma^{1}_{22}\Gamma^{2}_{12}-\Gamma^{r}_{22}\Gamma^{s}_{rs} \tag{29}$$

By looking at the components of the first two terms above this simplifies to:

$$-(1+\mu)-\Gamma^{r}_{22}\Gamma^{s}_{rs} \tag{30}$$

The last term differs only when index r = 1 giving

$$-(1+\mu)+(1+\mu)r(\Gamma^{0}_{10}+\Gamma^{1}_{11}+\Gamma^{2}_{12}+\Gamma^{3}_{13}) \tag{31}$$

Of the gamma terms the first 2 are easily verified to be 0 and the second 2 are 1/r. The terms cancel except for one, giving: +(1+ μ)

The differing term is +(1+ μ) instead of simply a 1 (when μ=0) . And so we can simply add the μ-term to the components of the Ricci tensor of the Robertson-Walker cosmology as follows:

$$R_{22}= \ -(R\ddot{R}+2\dot{R}^{2})r^{2}+\mu$$

and

$$R_{33}= \ (-(R\ddot{R}+2\dot{R}^{2})r^{2}+\mu)\sin^{2}\theta \tag{32}$$

The Cosmological Constant

The argument that leads to the open more expansive geometry can now be given. The idea is that the Cosmos is approximately Robertson-Walker (k=0, flat) but with small local anisotropies following a statistical distribution with constant standard deviation across the entire manifold. So a line element with slightly more irregularity than the Robertson-Walker line element, a statistically fluctuating Roberston-Walker line element, was used. The result was then smoothed or averaged by certain simplifying assumptions to give an averaged Ricci tensor which is different from the original Robertson-Walker line element only in two terms. This was done in the calculation. This argument relied on a statistical effect. The argument now takes a different course.

Additional assumptions:

(v) There is no cosmological constant in the underlying field equations.

(vi) The Universe is approximately a perfect fluid with negligible pressure.

Once again these are physically reasonable.

The usual Ricci tensor for a Robertson-Walker line element is as follows [1]:

$$R_{00} = 3\ddot{R}/R \qquad R_{11} = -(R\ddot{R}+2\dot{R}^2+2\mathrm{k})/(1-kr^2)$$

$$R_{22} = -(R\ddot{R}+2\dot{R}^2+2\mathrm{k})r^2$$

$$R_{33} = -(R\ddot{R}+2\dot{R}^2+2\mathrm{k})r^2\sin^2\theta \tag{33}$$

In a Friedmann cosmology k is constrained to be –1,0 or 1. However if we set k=0 (flat space) in the anisotropic case and then compare its Ricci tensor with the Robertson-Walker Ricci tensor we can note that the last two terms coincide when we make the substitution $\mathrm{k} = -\mu/2r^2$:

$$R_{22} = -(R\ddot{R}+2\dot{R}^2)r^2+\mu \quad == \quad R_{22} = -(R\ddot{R}+2\dot{R}^2+2\mathrm{k})r^2$$

$$R_{33} = \left(-(R\ddot{R}+2\dot{R}^2)r^2+\mu\right)\sin^2\theta$$

$$==$$

$$R_{33} = -(R\ddot{R}+2\dot{R}^2+2\mathrm{k})r^2\sin^2\theta \tag{34}$$

The second diagonal term R_{11} is however quite different, but nevertheless the standard mathematics can be borrowed to some extent and this partial correspondence used as a guide. To do this we follow the derivation of the two differential equations of the Robertson-Walker space-time that can be derived from the Ricci tensor, the metric and the stress-energy tensor of a perfect fluid following the assumptions above. We have as in [1]:

$$T_{mv}-(1/2)T\,g_{mv} = (\rho+p)\delta^0_m\delta^0_v-(1/2)(\rho-p)g_{mv} \tag{35}$$

Extracting the metric from the line element as in [1], we see that:

$$T_{00}-(1/2)T\,g_{00} = (1/2)(\rho+3p)$$

$$T_{11}-(1/2)T\,g_{11} = (1/2)(\rho-p)R_1^2$$

$$T_{22}-(1/2)T\,g_{22} = (1/2)(\rho-p)R_2^2r^2$$

$$T_{33}-(1/2)T\,g_{33} = (1/2)(\rho-p)R_3^2r^2\sin^2\theta$$

$$T_{mv}-(1/2)T\,g_{mv} = 0 \quad \text{for other m and v} \tag{36}$$

The only difference between this and the standard equations are therefore the anisotropic R_i. The field equations (without cosmological constant) can be written as:

$$R_{mv} = \kappa(T_{mv}-(1/2)T\,g_{mv}) \tag{37}$$

where κ is –8πG and not to be confused with the Robertson-Walker k parameter which is 0. And so the following 4 differential equations can be read off for the averaged R(t) by equating the Ricci tensor in terms of R(t) with the Ricci tensor in terms of the stress-energy expression in terms of a perfect fluid:

$$\begin{aligned} 3\ddot{R}/R &= \kappa(1/2)(\rho+3p) \\ (R\ddot{R}+2\dot{R}^2) &= -\kappa(1/2)(\rho-p)R_1^2 \\ (R\ddot{R}+2\dot{R}^2)r^2-\mu &= -\kappa(1/2)(\rho-p)R_2^2 r^2 \\ ((R\ddot{R}+2\dot{R}^2)r^2-\mu)\sin^2\theta &= -\kappa(1/2)(\rho-p)R_3^2 r^2\sin^2\theta \end{aligned} \quad (38)$$

The last 3 of these can be added together in the following way:

$$\begin{aligned} 2(R\ddot{R}+2\dot{R}^2+(-\mu/r^2))+(R\ddot{R}+2\dot{R}^2) &= \\ -\kappa(1/2)(\rho-p)R_3^2-\kappa(1/2)(\rho-p)R_2^2-\kappa(1/2)(\rho-p)R_1^2 \end{aligned}$$

$$\begin{aligned} 2(-\mu/r^2)+3(R\ddot{R}+2\dot{R}^2) &= \\ -\kappa(1/2)(\rho-p)3\mathrm{R}^2[(R_3^2+R_2^2+R_1^2)/3\mathrm{R}^2] \end{aligned} \quad (39)$$

Motivated by,

$$(R_3^2+R_2^2+R_1^2)/3\mathrm{R}^2 \cong 1 \quad (40)$$

We can write this as:

$$\begin{aligned} 2(-\mu/r^2)+3(R\ddot{R}+2\dot{R}^2) &= \\ -\kappa(1/2)(\rho-p)3\mathrm{R}^2[1-(1-(R_3^2+R_2^2+R_1^2)/3\mathrm{R})^2] \end{aligned}$$

$$\begin{aligned} 2(-\mu/r^2)+3(R\ddot{R}+2\dot{R}^2) &= -\kappa(1/2)(\rho-p)3\mathrm{R}^2 \\ &+\kappa(1/2)(\rho-p)3\mathrm{R}^2[1-(R_3^2+R_2^2+R_1^2)/3\mathrm{R}^2] \end{aligned}$$

$$(R\ddot{R}+2\dot{R}^2)+(2/3)(-\mu/r^2)$$
$$-\kappa(1/2)(\rho-p)R^2[1-(R_3^2+R_2^2+R_1^2)/3\mathrm{R}^2]=-\kappa(1/2)(\rho-p)R^2 \tag{41}$$

Together with,

$$3\ddot{R}/R \;=\; \kappa(1/2)(\rho+3p) \tag{42}$$

This last equation when written as:

$$(R\ddot{R}+2\dot{R}^2)+$$
$$2\{(1/3)(-\mu/r^2)-\kappa(1/4)(\rho-p)R^2[1-(R_3^2+R_2^2+R_1^2)/3\mathrm{R}^2]\}$$
$$=\;-\kappa(1/2)(\rho-p)R^2 \tag{43}$$

...is none other than the Friedmann equation with the term in braces being the parameter k. The parameter k, here however, is not a constant, changing for example with r, but also from point to point via the anisotropies. And so it certainly is not constrained to be 0, 1 or –1 as occurs in the Friedmann model. Nevertheless much of the mathematics can be copied verbatim (with the minor proviso that the pressure term is assumed to be negligible with respect to the density term). The resulting Friedmann-like equation can be written:

$$\dot{R}^2+\{(1/3)(-\mu/r^2)-\kappa(1/4)(\rho-p)R^2[1-(R_3^2+R_2^2+R_1^2)/3\mathrm{R}^2]\}$$
$$=\;(8\pi G/3)\rho R^2 \tag{44}$$

Where we have some horrible terms depending on the coordinates. By taking the mean of this equation, the term in braces becomes always negative.

If further we make the extra assumption (which is valid locally to arbitrary accuracy) that the term in braces is constant then this leads, via identical mathematics to the Friedman equations, to an open solution, essentially the same as the hyperbolic Friedmann equation.

Attempts to solve the above equations for a varying term in braces were not fruitful, and it is clear from the equation:

$$3\ddot{R}/R = \kappa(1/2)(\rho+3p) \tag{45}$$

...that in any case the model started implicitly with the assumption of a deceleration (rather than an acceleration) term, and so the discovery of a cosmological constant would not be expected. Nevertheless a quality suggestive of acceleration, that is hyperbolicity, and the requirement (due to anisotropies) for a locally open cosmology rather than a flat cosmology *was* found.

Conclusion

The process laid out in the abstract was completed and a possible explanation for dark energy explored. This was done by finding an unexpected source of hyperbolicity, identified in terms of errors in the isotropic assumptions of the Friedmann Equations. This is similar to the idea of backreaction from inhomogeneities. The methods used were approximate and statistical in nature using a Robertson-Walker line element with local anisotropic terms added afterwards, along with a statistical distribution for those anisotropies. The derivation of hyperbolicity is suggestive of cosmological acceleration without the need to change or adjust Einstein's equations by adding a cosmological constant, dark energy or quintessence. However the presence of an explicit constant, actual acceleration, was not discovered.

The model used here started implicitly with the assumption of a deceleration (rather than an acceleration) term, and so the discovery of a cosmological constant would not be expected. Nevertheless a quality suggestive of acceleration, that is hyperbolicity, and the requirement (due to anisotropies) for a locally open cosmology rather than a flat cosmology *was* found.

Referring to the cosmological constant, its introduction and subsequent removal, Wald says … " Λ has been reintroduced in numerous occasions when discrepancies have arisen between theory and observations, only to be abandoned again when these discrepancies have been resolved" [4] A possible mathematical explanation for this reappearance and disappearance of the cosmological constant lies in the fact that the Einstein tensor with cosmological constant is the most general second order tensor that is source free, a concomitant of the metric tensor (to 2^{nd} order) and which satisfies other nice energy-like properties, so it is difficult to escape during adjustments to Einstein's field equations.

Whilst this study has not solved the problem of the cosmological constant, it does show that the averaging problem has more to say about the exact nature of cosmological expansion, and in this particular case that local anisotropy may be experimentally significant.

Dedication

This paper is dedicated to Ilaria for her patience during the Summer vacation I used to write this paper.

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