On the relativity of shapes

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We show that the extrinsic curvature behaves as an independent spin-2 field, solution of the Einstein-Gupta equations, whose source is the vacuum energy density of quantum gauge fields.

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Introduction

As has been widely discussed [1], the presence of extra dimensions may provide a possible explanation not only for the hierarchy problem of the fundamental interactions, but also for the late time cosmic acceleration, with no need to invoke either a cosmological constant Λ or a quintessence field Φ. In this paper we use a model independent formulation of the so-called brane-world program [2, 3]. Although the extrinsic curvature must be taken into account are well known and have been studied at length in the literature, nonetheless it is still ignored by using junctions conditions and the embedding process is commonly neglected. Rather than just reflecting the discontinuity of the bulk geometry across the brane-world, the extrinsic curvature assumes the important role of driving the propagation of gravitation along the extra dimensions of the bulk space. In the following, we present a mathematically correct structure of space-time embedding based on Nash's theorem. Secondly, we derive the dynamical equation required to determine the extrinsic curvature and analyzing the cosmological constant problem which does not hold in our model. The objective of our paper is to present a more general theory of embedded space-times, which is more general than the usual brane-world setting present in most current models.

Embedded spaces

To start with, instead of assuming the string inspired embedding of a 3-dimensional hypersurface generating a four-dimensional embedded volume, we must first look at the conditions for the existence of the embedding of the space-time itself. The embedding of a manifold into another is a non-trivial problem and the resulting embedded geometry resulting from of an evolving 3-surface must comply with these conditions. For instance, the astronomical observations of our Universe are made as if the objects, stars galaxies, clusters are located in a 3-space with Euclidean geometry. However, when it comes to theorise these observations, say to describe the shapes of the observed objects, another concept of curvature is used namely that of Riemannian geometry as applied to General relativity. These are not equivalent geometrical representations of the observations, because Riemannian geometry lacks some detail in the description of the local shape of objects. Indeed, the Riemannian curvature can describe shapes that are different from those we observe on astronomy. Reviewing the concept, given a basis $\{e_\mu\}$ the the Riemann tensor describes the curvature of a manifold by displacing a vector field e_ρ along a closed parallelogram defined by e_μ and e_ν and comparing the result with the original vector obtaining: $R(e_\mu, e_\nu)e_\rho = R_{\mu\nu\rho\sigma}e^\sigma = [\nabla_\mu, \nabla_\rho]e^\sigma$. When the difference is zero, the manifold is said to be flat. Such Riemannian flat space is not necessarily equal to a flat space in Euclidean geometry. It could likewise be a cylinder or a helicoid. After Riemann conceptualized a manifold intrinsically, the question if the geometry of a Riemannian manifold has the same geometry of a manifold embedded in an Euclidean space soon arose. Today we know that every Riemannian manifold defined intrinsically can be embedded isometrically, locally or globally, in a Euclidean space with *appropriate* dimensions. Schlaefli in 1851 [4] was the first to conjecture that a flat Riemannian manifold with analytic and positive defined metric can be locally and isometrically embedded in an Euclidean E^m of dimensions $m = n(n+1)/2$. This means that a 4-dimensional Riemannian manifold is embedded in $m = 4(4+1)/2 = 10$-dimensional Euclidean space [5].

In the particular case of General relativity the arbitrariness of the tangent space was resolved by use of an additional assumption outside Riemannian geometry, the Poincaré symmetry of Maxwell's equations, applied to the tangent space. This is a well know problem and we do not intend to indulge an such philosophical discussion on this theme, so that we jump directly to our point: Riemannian geometry has proven quite satisfactory to compare with the observations at the level of the classical tests. However, the new physical problems today, like supernovae Ia observations indicate for the dark energy problem [6], seem to be telling us that something is missing in the geometrical side of Einstein's equations.

The analytic methods and Nash's theorem

Since Schlaefli's conjecture, the embedding problem was regarded as an open question. Only in the first half of the 1900's some efforts were proposed by using the analytic methods. In 1926, a perturbative embedding mechanism was made by J.E. Campbell based on the analytic methods[7]. In the subsequent years, another studies using analytical methods were made by M. Janet[8], Cartan [9] and Burstin [10].

The general solution for this problem was given by J. Nash [11] in 1956. Nash showed how any Riemannian geometry can be generated by metric perturbations against a bulk space (which he assumed to be Euclidean, but it was soon extended to a pseudo Riemannian bulk by R. Greene [12]). As it happens, any embedded metric geometry can be generated by a continuous sequence of small metric perturbations of a given geometry with a metric of the embedded manifold defined by the extrinsic curvature as

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta y \, \bar{k}_{\mu\nu} + (\delta y)^2 \, \bar{g}^{\rho\sigma} \bar{k}_{\mu\rho} \bar{k}_{\nu\sigma} \cdots \quad . \tag{1}$$

The embedding apparently introduces fixed background geometry as opposed to a completely intrinsic and self contained geometry in General relativity. One way out of the necessity of having a fixed background is to transfer the dynamical structure to the embedding space. As it happens, Nash's theorem is based on a smooth (differentiable) manifold structure. For that purpose Nash introduced a theory of smoothing operators to guarantee that differentiable embeddings are sufficient. Here we suggest that the geometry of the bulk is given by the Einstein-Hilbert action principle. This has the meaning that the embedding space has the smoothest possible curvature. Together with the Gauss-Codazzi and Ricci equations this implies that the embedding is necessarily differentiable. With this definition the embedding space becomes the main dynamical structure.

Nash's perturbation method innovates in two basic aspects: first, there is no need to apply the restrictive convergent series power of analytical function hypothesis to make an embedding between manifolds. Second, in a physical sense, the perturbational nature of the process can be obtained in the same fashion as Cauchy's problem in Mechanics: by dynamical equations; besides, it also gives a prescription on how to construct geometrical structures by deforming simpler ones. It seems that this geometric perturbation process may have to do with the formation of structures in the early universe.

When Nash's theorem is applied to physics, it provides a general mathematical tool appropriated to the brane-world program. The covariant formulation of the brane-world uses basically three essential postulates: (a) The D-dimensional bulk is a solution of Einstein's equations; (b) The brane-world is a differentiable submanifold embedded in that bulk; (c) Gauge fields and ordinary matter are confined to the brane-world, whereas gravity propagates along the extra-dimensions. The most general covariant equations of motion are derived from the former conditions and can be reviewed in [2, 13, 14] and can be applied to specific models, as long as the particularities of the model are set after the formal development of the theory. On the other hand, on a model independent of a covariant formulation, the extrinsic curvature appears as an independent symmetric tensor field which evolves together with the brane-world dynamics. This is an important result due to Nash's theorem because the extrinsic curvature becomes independent of the matter content on the locally embedded brane-world. Thus, any junction condition imposed on the brane-world can be dispensed. Interestingly, the presence of the independent symmetric rank-two tensor field has been considered long before the observation of the accelerated expansion of the universe under different motivations and circumstances as a possible repulsive gravitational field [15]. As we shall see, brane-world gravity presents one such field in the form of the extrinsic curvature.

Einstein's equations for the bulk geometry can be written as

$$\mathcal{R}_{AB} - \frac{1}{2}\mathcal{R}\mathcal{G}_{AB} + \Lambda_* \mathcal{G}_{AB} = \alpha_* T^*_{AB} \ \ , \tag{2}$$

where α^* denotes the fundamental energy scale (Tev) and T^*_{AB} is the stress energy-momentum tensor of all possible sources, which we assume here to be made of ordinary matter and gauge fields, which means that it is essentially a confined source. In order to introduce the 4-dimensional cosmological constant to those equations we have to consider a constant D-dimensional, generic curvature to the bulk, where the Riemann curvature is given by

$$\mathcal{R}_{ABCD} = K_*(\mathcal{G}_{AC}\mathcal{G}_{BD} - \mathcal{G}_{AD}\mathcal{G}_{BC}), \quad A..D = 4 + N \ \ ,$$

where the constant curvature K_* can be positive, in a deSitter bulk, or negative, in a anti-deSitter bulk, and is related to the cosmological constant Λ_* of the bulk by $K_* = \frac{2}{(2+N)(3+N)}\Lambda_*$, where N is the number of extra dimensions. The local embedding is defined by an embedding map $\mathcal{Z} : \bar{V}_n \to V_D$ $(n < D)$ admitting that $\mathcal{Z}^\mu$ is a regular and differentiable map. The components $\mathcal{Z}^A = f^A(x^1, ..., x^n)$ associate to each point of V_n a point in V_D with coordinates $\mathcal{Z}^A$. As it happens, the isometric condition given by the direct relation between the manifolds, that is, the bulk and the embedded manifold can be written as

$$g_{\alpha\beta} = \mathcal{G}_{AB}\mathcal{Z}^A_{,\alpha}\mathcal{Z}^B_{,\beta}, \tag{3}$$

where $\mathcal{Z}^A_{,\alpha}$ are the components of the tangent vectors of V_n, $g_{\alpha\beta}$ is the metric of the embedded manifold and $\mathcal{G}_{AB}$ is the metric of the bulk. It is important to notice that we must have $D - n$ vectors normal to V_n. If η^A are the components of these vectors, then they satisfy the conditions of orthogonality

$$\mathcal{G}_{AB}\mathcal{Z}^A_{,\mu}\eta^B_b = 0 \ . \tag{4}$$

Finally choosing the vectors η^A_a to be mutually orthogonal and of norm ± 1, we can also write the following condition

$$\mathcal{G}_{AB}\eta^A_a\eta^B_b = g_{ab} = \epsilon_a\delta_{ab}, \tag{5}$$

where $\epsilon_a = \pm 1$ are the signs associated to the two possible signatures of the extra dimensions. So under the supposition that $\mathcal{Z}^A$ defines a new Riemannian geometry V_n embedded in V_D. Concerning notation $\mu, \nu = 1...4$; $a, b = 5...D$.

Taking the tangent, vector and scalar components of eq.(2), the components of the Riemann tensor of the *bulk* $\mathcal{R}_{ABCD}$ defined in the Gaussian frame embedding veilbein $\{\mathcal{Z}^A_{,\mu}, \eta^A\}$, we can find the integrability equations of the embedding given by the Gauss, Codazzi and Ricci equations:

$$R_{\alpha\beta\gamma\sigma} = g^{ab}(k_{a\alpha\gamma}k_{b\sigma\beta} - k_{a\alpha\sigma}k_{b\beta\gamma}) + \mathcal{R}_{ABCD}\mathcal{Z}^A_{,\alpha}\mathcal{Z}^B_{,\beta}\mathcal{Z}^C_{,\gamma}\mathcal{Z}^D_{,\sigma} \ , \quad (6)$$

$$k_{a\alpha\delta;\gamma} - k_{a\alpha\gamma;\delta} = g^{cd}(A_{cd\gamma}k_{c\alpha\delta} - A_{cd\delta}k_{c\alpha\gamma}) + \mathcal{R}_{ABCD}\mathcal{Z}^A_{,\alpha}\eta^B_b\mathcal{Z}^C_{,\gamma}\mathcal{Z}^D_{,\delta} \ , \quad (7)$$

$$A_{ba\gamma;\delta} - A_{ba\delta;\gamma} = g^{cd}(A_{cb\delta}A_{da\gamma} - A_{da\gamma}A_{cb\delta}) + g^{cd}(k_{c\gamma\delta}k_{d\delta\gamma} - k_{c\delta\gamma}k_{d\gamma\delta}) + \mathcal{R}_{ABCD}\eta^A_a\eta^B_b\mathcal{Z}^C_{,\gamma}\mathcal{Z}^D_{,\delta} \ , \quad (8)$$

where $A_{\mu ba}$ is the "torsion" vector and is given by $A_{\mu ba} = \eta^A_{a,\mu}\eta^B_b\mathcal{G}_{AB}$.

In contrast with the extra dimensional perturbative behaviour of the gravitational field, all gauge fields of the standard model remain confined to the four-dimensional space-time. This is a direct consequence of the gauge field structure. Just as a reminder, the Yang- Mills equations can be written as $D \wedge F = 0$, $D \wedge F^* = 4\pi J^*$, where $F = F_{\mu\nu}dx^\mu \wedge dx^\nu$, $F_{\rho\sigma} = [D_\rho, D_\sigma]$, $D_\mu = I\partial_\mu + A_\mu$, $F^* = F^*_{\mu\nu}dx^\mu \wedge dx^\nu$ and $F^*_{\mu\nu} = \epsilon_{\mu\nu\rho\sigma}F^{*\ \rho\sigma}$. The duality operation $F \to F^*$ requires the existence of an isomorphism between 3-forms and 1-forms, which can only be realized only in a four dimensional space-time manifold. Therefore, the confinement of gauge fields, matter and vacuum states is a property that is independent of the perturbation of the brane-world geometry. There are two relevant consequences of the confinement. In the first place, it implies that all ordinary matter which interacts with the gauge fields, and also the vacuum states and its energy-momentum tensor associated with the confined fields also remain confined to the four-dimensional brane-world. Secondly, the diffeomorphism invariance of General relativity cannot apply to the bulk manifold V_D, for it would imply in breaking the confinement. Of course, such limitation could be fixed by applying a coordinate gauge, but then we will be imposing a modification to Nash's theorem.

Nash's theorem demands the embedding to be differentiable and regular, so that there is a 4×4 non-singular sub-matrix of the Jacobian determinant of the embedding map, thus guaranteeing the diffeomorphism invariance in the four-dimensional embedded submanifold only. Admitting that the original (on-embedded) space-time is a solution of Einstein's equations, the gauge fields, matter and its vacuum states keep a 1:1 correspondence with the source fields in the embedded space-time structure. Consequently, the confinement can be generally set as a condition on the embedding map such that

$$\alpha_* T^*_{\mu\nu} = 8\pi G T_{\mu\nu}; \quad T^*_{\mu 5} = T^*_{55} = 0; \; \mu, \nu = 1...4$$

The gravitational field equations for the brane-world are necessarily more complicated than Einstein's equations in General relativity, because they involve the extrinsic geometry.

Added to the confinement condition and the Einstein-Hilbert action for the bulk geometry, we can obtain the equations of motion for a generic brane-world written as

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} - Q_{\mu\nu} - g^{ab} \mathcal{R}_{AB} \eta^A_a \eta^B_b g_{\mu\nu} = -8\pi G T_{\mu\nu} \quad , \quad (9)$$

$$k_{\mu}{}^{\rho}{}_{a;\rho} - h_{a;\mu} + A_{\rho c a} k^{\rho}{}_{\mu}{}^{c} - A_{\mu c a} h^c - \frac{1}{2}\left[R - (K^2 - h^2)\right] g_{\mu a} - g^{ab} \mathcal{R}_{AB} \eta^A_a \eta^B_b g_{\mu a} = 0 \quad , \quad (10)$$

$$R - (K^2 - h^2) = \mathcal{R} - 2 g^{ab} \mathcal{R}_{AB} \eta^A_a \eta^B_b \quad , \quad (11)$$

which are referred respectively as gravitational tensor, vector and scalar equations of the brane-world, where we have denoted

$$Q_{\mu\nu} = \; g^{cd}\left(g^{\rho\sigma}k_{\mu\rho c}k_{\nu\sigma d} - k_{\mu\nu d}h_c\right) - \frac{1}{2}\left(K^2 - h^2\right)g_{\mu\nu} \;\; . \quad (12)$$

The effective 4-dimensional cosmological constant Λ is a property of the bulk and is related to Λ_* by $\Lambda = \frac{2+3N}{2+N}\Lambda_*$. Here $k_{\mu\nu}$ denotes the extrinsic curvature and $h = g^{\mu\nu}k_{\mu\nu}$, $K^2 = k^{\mu\nu}k_{\mu\nu}$ and $h^2 = g^{ab}h_a h_b$. The term $Q_{\mu\nu}$ is a conserved quantity (which does not exist in a pure Riemannian geometry) in the sense that $Q^{\mu\nu}{}_{;\nu} = 0$ and $Q = g^{\mu\nu}Q_{\mu\nu}$. In the next section, we derive the Gupta equations in order to solve the arbitrary of the extrinsic curvature due to the homogeneity of Codazzi's equations.

Dynamical equation for the extrinsic curvature

The study of a linear massless spin-2 fields in Minkowski space-time originated from Fierz and Pauli in 1939 [16]. In 1954 S. Gupta noted that the Fierz-Pauli equation has a remarkable resemblance with the linear approximation of Einstein's equations for the gravitational field, suggesting that such equation could be just the linear approximation of a more general, non-linear equation for a massless spin-two fields. Gupta found that indeed any spin-2 field in Minkowski space-time must satisfy an equation that has the same formal structure as Einstein's equations: In short, in the same way as Einstein's equations, can be obtained by an infinite sequence of infinitesimal perturbations of the linear gravitational equation, it possible to obtain the full non-linear equation for any spin-2 field by an infinite sequence of infinitesimal perturbations of the Fierz-Pauli equations. The result is an Einstein-like system of equations called the Gupta equations [17].

As it happens the extrinsic curvature which appear in any embedded geometry is a symmetric tensor of order two and by the spin-statistics theorem qualifies as a spin-two field, which promotes the propagation of the gravitational field along the extra dimension, according to Nash's theorem. This spin-2 interpretation of the extrinsic curvature together with Nash's theorem represents a profound innovation in the theory of gravitation, based on a historical development on the physics of gravitation. Of course, the Israel condition does away with the extrinsic curvature when the mirror symmetry is applied.

Therefore, the extrinsic curvature can be seen as an independent rank-2 field in V_n, acting as the generator of the gravitational perturbations along the extra dimensions of the bulk. Consequently, $k_{\mu\nu}$ must satisfy Gupta's equation defined on the embedded brane-world V_n (instead of Minkowski's space-time as in the original theorem of Gupta).

To obtain Gupta's equation in a brane-world, we can make use of an analogy with Riemannian's geometry, defining a "connection" associated with $k_{\mu\nu}$ and consequently the corresponding Riemann tensor, keeping in mind that the geometry of the embedded space-time has been already defined by the metric $g_{\mu\nu}$.

Since $k_{\mu\nu}k^{\mu\nu} = K^2 \neq 4$, $k^{\mu\nu}$ is not the inverse of $k_{\mu\nu}$. However, we may re-scale $k_{\mu\nu}$, by defining

$$f_{\mu\nu} = \frac{2}{K}k_{\mu\nu}, \text{ and } f^{\mu\nu} = \frac{2}{K}k^{\mu\nu}, \quad \text{so that} \quad f^{\mu\rho}f_{\rho\nu} = \delta^{\mu}_{\nu} \quad (13)$$

Next we construct the "Levi-civita connection" associated with $f_{\mu\nu}$, based on a similarity with the "metricity condition". Let us denote by $||$ the covariant derivative with respect to $f_{\mu\nu}$ (while keeping the usual $(;)$ notation for the covariant derivative with respect to $g_{\mu\nu}$) so that $f_{\mu\nu||\rho} = 0$. Therefore, the f-connection is

$$\Upsilon_{\mu\nu\sigma} = \frac{1}{2}(\partial_\mu f_{\sigma\nu} + \partial_\nu f_{\sigma\mu} - \partial_\sigma f_{\mu\nu}) \text{ and } \text{ and} \Upsilon_{\mu\nu}{}^{\lambda} = f^{\lambda\sigma}\Upsilon_{\mu\nu\sigma}$$

The Riemann-like curvature tensor for $f_{\mu\nu}$ is

$$\mathcal{F}_{\nu\alpha\lambda\mu} = \partial_\alpha \Upsilon_{\mu\lambda\nu} - \partial_\lambda \Upsilon_{\mu\alpha\gamma} + \Upsilon_{\alpha\sigma\mu}\Upsilon^{\sigma}_{\lambda\nu} - \Upsilon_{\lambda\sigma\mu}\Upsilon^{\sigma}_{\alpha\nu}$$

Likewise the the Ricci-like tensor and the Ricci-like scalar curvature, respectively are consistently defined as

$$\mathcal{F}_{\mu\nu} = f^{\alpha\lambda}\mathcal{F}_{\nu\alpha\lambda\mu} \text{ and } \mathcal{F} = f^{\mu\nu}\mathcal{F}_{\mu\nu}$$

Finally, the contracted Bianchi identities for $f_{\mu\nu}$ gives Gupta's equations

$$\mathcal{F}_{\mu\nu} - \frac{1}{2}\mathcal{F} f_{\mu\nu} + \Lambda_f \, f_{\mu\nu} = \alpha_f \tau_{\mu\nu} \tag{14}$$

where $\tau_{\mu\nu}$ stands for the "source" of the $f_{\mu\nu}$-field, with coupling constant α_f. For generality we have included Λ_f as the equivalent to the cosmological constant (remembering that the contracted Bianchi identity allows for such constant).

Since $k_{\mu\nu}$ are the coefficients of Nash's perturbations of the gravitational field according to (1), the source of Gupta's equations must be the quantum fluctuations of the vacuum represented by a plasma fluid with constant density with energy-momentum tensor [19, 20] $\tau_{\mu\nu} = 8\pi G < \rho_v > f_{\mu\nu}$. Replacing this in (1) and comparing the constant terms we find that

$$\Lambda_f \stackrel{def}{=} 8\pi G < \rho_v >$$

Where we notice that Λ_f is not the same observed cosmological constant associated with Einstein's equations. In fact we may take the above expression as the definition of Λ_f. With this choice Gupta's equations for $f_{\mu\nu}$ becomes simply

$$\mathcal{F}_{\mu\nu} = 0 \tag{15}$$

With this definition the cosmological constant problem does appear and the vacuum energy density is indeed a source of gravitational perturbations, but only through $k_{\mu\nu}$. Thus, the cosmological constant problem does not propagate in the bulk due to the lack of either *extrinsic* Casimir effect or *extrinsic* Cosmological Constant observational data. The underlying physical idea is that the gravity can be produced by self-interaction field with non-massive gravitons.

After analyzing the structure of the mixing term $Q_{\mu\nu}$, it can be separated into two parts in the sense that

$$Q_{\mu\nu} = q_{\mu\nu}^{pure} + q_{\mu\nu}^{mix} \ ,$$

where $q_{\mu\nu}^{pure} = k_\mu^\rho k_{\rho\nu}$ is the pure extrinsic term and fully propagates in the bulk and $q_{\mu\nu}^{mix} = -k_{\mu\nu}H - \frac{1}{2}\left(K^2 - H^2\right) g_{\mu\nu}$ is the truly mixing term which oscillates on the brane-world and also propagates in the bulk. In fact, we have two manifolds which are communicated by the mixing term $Q_{\mu\nu}$. As we have shown, $Q_{\mu\nu}$ is a conserved quantity with respect to the covariant derivative (;) but it does not hold true when using (13) and expressing $Q_{\mu\nu}$ in terms of $f_{\mu\nu}$. As it happens, we obtain

$$Q_{\mu\nu}^f = f_\mu^\rho f_{\rho\nu} - f_{\mu\nu} h_f - \frac{1}{2}\left(4 - H_f^2\right) g_{\mu\nu} \ ,$$

and

$$Q_{\mu\nu}^f = \frac{4}{K^2} Q_{\mu\nu} \ ,$$

where $h_f = g^{\mu\nu} f_{\mu\nu}$ and $H_f^2 = h_f.h_f$. It can be directly verified that $Q^f_{\mu\nu||\nu} \neq 0$.

In addition to the Einstein-Gupta's equation, the conservation of $Q_{\mu\nu}$ suggests that it works as an effective gravitational energy flux in the sense that propagates from the brane-world into the bulk. Thus, it appears that there is some exchange of energy-momentum between the bulk and the brane-world. This energy flux is generated by the intrinsic perturbations on the brane-world geometry. It follows that the cosmological constant problem does not hold true in the brane-world theory because in a four-dimensional observer confined to the brane-world interprets the difference $\Lambda g_{\mu\nu} - Q_{\mu\nu}$ in (9) as being the vacuum energy of the confined fields

$$< \rho_v >_{\text{confined}} = \Lambda - Q/4 \ ,$$

showing that the extrinsic curvature composing Q compensates the difference between the observed cosmological constant and the general relativistic (intrinsic) vacuum.

Final Remarks

The existence of a background geometry is necessary to fix the ambiguity of the Riemann curvature of a given manifold, without a reference structure. General relativity solves this ambiguity problem by specifying that the tangent Minkowski space is a flat plane, as decided by the Poincaré symmetry, and not by the Riemann geometry itself. Such difficulty was known by Riemann himself, when he acknowledged that his curvature tensor defines a class of objects and not just one [18]. Unlike the case of string theory the bulk geometry is a solution of Einstein's equations, acting as a dynamic reference of shape for all embedded Riemann geometries. This generality follows from the remarkable accomplishment of Nash's theorem on embedded geometries. Nash showed that any Riemannian geometry can be generated by continuous sequence of infinitesimal perturbations defined by the extrinsic curvature. It seems natural that this result provides the required geometrical structure to describe a dynamically changing universe. The four-dimensionality of the embedded space-times is determined by the dualities of the gauge fields, which corresponds to the equivalent concept of confinement gauge fields and ordinary matter in the brane-world program. However, this confinement implies that the extrinsic curvature cannot be completely determined, simply because Codazzi's equations becomes homogeneous. Since the extrinsic curvature assumes a fundamental role in Nash's theorem, an additional equation is required. We have noted that the extrinsic curvature is an independent rank-2 symmetric tensor, which corresponds to a spin-2 field defined on the embedded space-time. However, as it was demonstrated by Gupta, any spin-2 field satisfy an Einstein-like equation.

Such comfortable situation between particle physics and Einstein's gravity was shaken by the observations of a small cosmological constant Λ, and the emergence of the cosmological constant problem. Our paper calls to the attention to the fact that unless this problem is solved, the definition of the gravitational ground state becomes again dubious: Either we have Minkowski or else we have deSitter. This leads us back to the necessity of the embedding of space-times as the only known solution of the Riemann curvature ambiguity. The general solution of the embedding problem is shown in the paper to be given by Nash's theorem on the differentiable embedding. The applications of Nash's theorem to gravitation is a landmark of our paper, providing an exciting new tool to the gravitational perturbation issue. We have explained the necessity of these embedding conditions, and explained the meaning of the Gauss-Codazzi-Ricci equations as well as the necessity of a dynamical equation for extrinsic curvature in order to obtain a correct mathematical tool which can help us in the search for a new quantum gravity theory.

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