# Time-like Involutes of a space-like helix in Minkowski space-time

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In this work, we deal with a classical differential geometry topic in Minkowski space-time. First, we prove that there are no time-like involutes of a time-like evolute. In the light of this result, we observed that involute curve transforms to a time-like curve when evolute is a space-like helix with a time-like principal normal. Then, we investigated relationships among Frenet-Serret apparatus of involute and evolute curves by the method expressed as in [11]. Moreover, we also proved that the time-like involute cannot be a helix, a general helix or a type-3 slant helix, respectively.

*Keywords*: Minkowski space-time; Classical differential geometry; Frenet-Serret equations; Involute-evolute curve couple.

## Introduction

By the 20th century, researchers discovered the bridge between theory of relativity and Lorentzian manifolds in the sense of differential geometry. Since, they adapted the geometrical models to relativistic motion of charged particles. Consequently, the theory of the curves has been one of the most fascinating topic for such modeling process. As it stands, the Frenet-Serret (FS) formalism of a relativistic motion describes the dynamics of the charged particles [5]. In this process, the theory of degenerate submanifolds has been treated by the researchers and some classical differential geometry topics have been extended to Lorentzian manifolds. For instance, in classical manner, one can see [1, 3, 7–9]. And recently, geometers investigated FS apparatus of the space-like and time-like curves according to signature $-,+,+,+$ and $+,+,+,-$ in space-time. Moreover, there exists a vast literature on this subject, see [10, 11].

The idea of a string involute is due to C. Huygens (1658), who is also known for his work in optics. He discovered involutes while trying to build a more accurate clock *for details, see* [2]. The involute of a given curve is a well-known concept in Euclidean-3 space. In [7], authors extended this topic to Minkowski space-time and studied relations between FS apparatus of space-like involute-evolute curve couples. In a further work [8], same authors defined and presented some characterizations of type-3 slant helices in terms of FS equations. In this work, we also use their definition.

This paper deals with time-like involutes. We observed that involute of a time-like evolute cannot be a time-like curve. To motivate this problem, we deal with a space-like helix (with constant FS curvatures) as evolute curve. Thereafter, we obtained a time-like involute and investigated relations among Frene-Serret apparatus of involute-evolute curve couple. We also proved that the time-like involute cannot be a helix, a general helix or a type-3 slant helix, respectively.

## Preliminaries

To meet the requirements in the next sections, here the basic elements of the theory of curves in the space $\mathrm{E}_1^4$ are briefly presented (A more complete elementary treatment can be found in [6, 9].)

Minkowski space-time $\mathrm{E}_1^4$ is a pseudo-Euclidean space $\mathrm{E}^4$ provided with the standard flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2, \tag{1}$$

where $(x_1, x_2, x_3, x_4)$ is a rectangular coordinate system in $\mathrm{E}_1^4$. Since $g$ is an indefinite metric, recall that a vector $v \in \mathrm{E}_1^4$ can have one of the three causal characters; it can be space-like if $g(v, v) > 0$ or $v = 0$, time-like if $g(v, v) < 0$ and null (light-like) if $g(v, v)$=0 and $v \neq 0$. Similarly, an arbitrary curve $\alpha = \alpha(s)$ in $\mathrm{E}_1^4$ can be locally space-like, time-like or null (light-like), if all of its velocity vectors $\alpha'(s)$ are respectively space-like, time-like or null. Also, recall the norm of a vector $v$ is given by $\|v\| = \sqrt{|g(v, v)|}$. Therefore, $v$ is a unit vector if $g(v, v) = \pm 1$. Next, vectors $v, w$ in $E_1^4$ are said to be orthogonal if $g(v, w) = 0$. The velocity of the curve $\alpha(s)$ is given by $\|\alpha'(s)\|$ . Let $\vartheta = \vartheta(s)$

be a curve in $E_1^4$. If tangent vector field of this curve forms a constant angle with a constant vector field $U$, then this curve is called a *general helix*. Let $\varphi$ and $\delta$ be space-like curves in $E_1^4$. $\varphi$ is an involute of $\delta$ if $\varphi$ lies on the tangent line to $\delta$ at $\delta(s_0)$ and the tangents to $\delta$ and $\varphi$ at $\delta(s_0)$ and $\varphi$ are perpendicular for each $s_0$. $\varphi$ is an evolute of $\delta$ if $\delta$ is an involute of $\varphi$. And this curve couple defined by $\varphi = \delta + \lambda T$.

Denote by $\{T(s), N(s), B_1(s), B_2(s)\}$ the moving FS frame along the curve $\alpha(s)$ in the space $E_1^4$. Then $T, N, B_1, B_2$ are, respectively, the tangent, the principal normal, the binormal and the trinormal vector fields. Space-like or time-like curve $\alpha(s)$ is said to be parametrized by arc length function $s$, if $g(\alpha'(s), \alpha'(s)) = \pm 1$.

Let $\alpha(s)$ be a curve in the space-time $E_1^4$, parametrized by arc length function $s$. Then for the curve $\alpha$ the following FS equations are given in [3, 9] as follows:

**Case 1.** $\alpha$ is a space-like curve with a time-like principal normal. In this case the FS equations can be written

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ \kappa & 0 & \tau & 0 \\ 0 & \tau & 0 & \sigma \\ 0 & 0 & -\sigma & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix} \tag{2}$$

where $T, N, B_1$ and $B_2$ are mutually orthogonal vectors satisfying equations

$$g(T,T) = g(B_1,B_1) = g(B_2,B_2) = 1,\ g(N,N) = -1.$$

**Case 2.** $\alpha$ is a time-like curve. Then, the FS formulae have the form

$$\begin{bmatrix} T' \\ N' \\ B_1' \\ B_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 & 0 \\ \kappa & 0 & \tau & 0 \\ 0 & -\tau & 0 & \sigma \\ 0 & 0 & -\sigma & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix} \tag{3}$$

where $T, N, B_1$ and $B_2$ are mutually orthogonal vectors satisfying equations

$$g(N, N) = g(B_1, B_1) = g(B_2, B_2) = 1,\ g(T, T) = -1.$$

Here $\kappa, \tau$ and $\sigma$ are, respectively, first, second and third curvature of curve $\alpha$. In the same space, in [11], authors used a vector product (cf. [10]) and gave a method to calculate the FS frame for an arbitrary time-like curve by the following definition and theorem:

*Definition 1.* Let $a = (a_1, a_2, a_3, a_4)$, $b = (b_1, b_2, b_3, b_4)$ and $c = (c_1, c_2, c_3, c_4)$ be vectors in $\mathrm{E}_1^4$. The vector product in Minkowski space-time $E_1^4$ is defined by the determinant

$$a \wedge b \wedge c = - \begin{vmatrix} -e_1 & e_2 & e_3 & e_4 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix}, \tag{4}$$

where $e_1, e_2, e_3$ and $e_4$ are mutually orthogonal vectors (coordinate direction vectors) satisfying equations

$$e_1 \wedge e_2 \wedge e_3 = e_4\ ,\ e_2 \wedge e_3 \wedge e_4 = e_1\ ,\ e_3 \wedge e_4 \wedge e_1 = e_2\ ,$$
$$e_4 \wedge e_1 \wedge e_2 = -e_3.$$

*Theorem 2.* Let $\alpha = \alpha(t)$ be an arbitrary time-like curve in Minkowski space-time $\mathrm{E}_1^4$. The FS apparatus of $\alpha$ can be calculated by the following equations;

$$T = \frac{\alpha'}{\|\alpha'\|}, \tag{5}$$

$$N = \frac{\|\alpha'\|^2 \alpha'' + g(\alpha', \alpha'')\alpha'}{\left\| \|\alpha'\|^2 \alpha'' + g(\alpha', \alpha'')\alpha' \right\|}, \tag{6}$$

$$B_1 = \mu N \wedge T \wedge B_2, \tag{7}$$

$$B_2 = \mu \frac{T \wedge N \wedge \alpha'''}{\|T \wedge N \wedge \alpha'''\|}, \tag{8}$$

$$\kappa = \frac{\left\| \|\alpha'\|^2 \alpha'' + g(\alpha', \alpha'')\alpha' \right\|}{\|\alpha'\|^4} \tag{9}$$

$$\tau = \frac{\|T \wedge N \wedge \alpha'''\| \, \|\alpha'\|}{\left\| \|\alpha'\|^2 \alpha'' + g(\alpha', \alpha'')\alpha' \right\|} \tag{10}$$

and

$$\sigma = \frac{g(\alpha^{(IV)}, B_2)}{\|T \wedge N \wedge \alpha'''\| \, \|\alpha'\|}. \tag{11}$$

Here $\mu$ is taken $\pm 1$ to make $+1$ of determinant of $[T, N, B_1, B_2]$ matrix. By this way, FS frame becomes positively oriented. If the vector product is used in terms of FS frame, then, It is adapted according to character of FS frame vectors (space-like or time-like, i.e.). Recall that an arbitrary curve is called a $W-$curve or only a *helix* if it has constant FS curvatures [4]. And, from the view of Differential Geometry, a helix is a geometric curve with non-vanishing constant curvature $\kappa$ and non-vanishing constant torsion $\tau$ [4]. In the same space, [8], authors defined a new kind slant helix by:

*Definition 3.* A curve $\psi = \psi(s)$ is called a "type-3 slant helix" if the second binormal (trinormal) lines of $\psi$ make a constant angle with a fixed direction in $\mathrm{E}_1^4$.

## Time-like Involutes of Minkowski space-time

*Theorem 4.* There are no time-like involutes of a time-like evolute in Minkowski space-time.

*Proof.* Let us suppose $\phi$ be a time-like involute of a time-like evolute. Then, we may write

$$\phi = \theta + \lambda T, \tag{12}$$

where $\phi = \phi(s_\phi)$ is involute and $\theta = \theta(s_\theta)$ is evolute curve. By the differentiation, we may write that $\lambda = (c - s_\theta)$ (see [7]). Then, we express

$$\frac{d\phi}{ds_\phi}\frac{ds_\phi}{ds_\theta} = (c - s)\kappa N. \tag{13}$$

Taking the norm of both sides, we easily have $T_\phi = N$. This means that $\phi$ is a space-like curve. Thus, involute $\phi$ cannot be a time-like curve.

*Theorem 5.* Let $\gamma$ be involute of $\xi$. Moreover, let $\xi$ be a space-like helix (has constant curvatures, i.e.) with above FS equations. Then;

i) $\gamma$ is a time-like curve.

ii) FS apparatus of involute $\gamma$ $\{T_\gamma, N_\gamma, B_{1\gamma}, B_{2\gamma}, \kappa_\gamma, \tau_\gamma, \sigma_\gamma\}$ can be formed by the apparatus of evolute $\xi$ $\{T, N, B_1, B_2, \kappa, \tau, \sigma\}$.

iii) $\gamma(s_\gamma)$ cannot be a helix.

*Proof.* By the definition, we can write

$$\gamma = \xi + \lambda T. \tag{14}$$

Differentiation of (14) and the definition of involute-evolute curve couple, we may write

$$\lambda = (c - s). \tag{15}$$

Since, we write

$$T_\gamma \frac{ds_\gamma}{ds} = (c - s)\kappa N. \tag{16}$$

Considering above equation, we may write $T_\gamma = N$ and it follows that $g(T_\gamma, T_\gamma) = -1$, which implies that $\gamma(s_\gamma)$ is a time-like curve. So, we proved statement i. Thereafter, to determine FS apparatus of $\gamma(s_\gamma)$, we shall use presented method of [11]. First, we form the following differentiations with respect to $s$:

$$\begin{aligned} \gamma' &= (c-s)\kappa N \\ \gamma'' &= (c-s)\kappa^2 T - \kappa N + (c-s)\kappa\tau B_1 \\ \gamma''' &= \left\{ \begin{array}{c} -2\kappa^2 T + (c-s)\kappa\left[\kappa^2+\tau^2\right] N - 2\kappa\tau B_1 \\ +(c-s)\kappa\tau\sigma B_2 \end{array} \right\} \\ \gamma^{(IV)} &= \left\{ \begin{array}{c} (c-s)\kappa^2\left[\kappa^2+\tau^2\right] T - 3\kappa\left[\kappa^2+\tau^2\right] N \\ +(c-s)\kappa\tau\left[\kappa^2+\tau^2-\sigma^2\right] B_1 - 3\kappa\tau\sigma B_2 \end{array} \right\} \end{aligned} . \tag{17}$$

By the first equation, we know $\|\gamma'\| = (c-s)\kappa$. In order to calculate principal normal, we form

$$\|\gamma'\|^2 \gamma'' - g(\gamma',\gamma'')\gamma' = (c-s)^3\kappa^3(\kappa T + \tau B_1). \tag{18}$$

Thus, we obtain principal normal and the first curvature, respectively

$$N_\gamma = \frac{\kappa T + \tau B_1}{\sqrt{\kappa^2+\tau^2}} \tag{19}$$

where $\left\| \|\gamma'\|^2 \gamma'' - g(\gamma',\gamma'')\gamma' \right\| = (c-s)^3\kappa^3\sqrt{\kappa^2+\tau^2}$ and

$$\kappa_\gamma = \frac{\sqrt{\kappa^2+\tau^2}}{(c-s)\kappa}. \tag{20}$$

We express the following vector product to determine the second binormal vector field of $\gamma(s_\gamma)$

$$T_\gamma \wedge N_\gamma \wedge \gamma''' = -\frac{1}{\sqrt{\kappa^2+\tau^2}} \begin{vmatrix} T & -N & B_1 & B_2 \\ 0 & 1 & 0 & 0 \\ \kappa & 0 & \tau & 0 \\ l_1 & l_2 & l_3 & l_4 \end{vmatrix}, \tag{21}$$

where $l_i$ are the components of the differentiable vector function $\gamma'''$. We obtain

$$T_\gamma \wedge N_\gamma \wedge \gamma''' = \frac{(c-s)\kappa\tau\sigma}{\sqrt{\kappa^2+\tau^2}}(\tau T - \kappa B_1). \tag{22}$$

By means of (22), we write

$$B_{2\gamma} = \mu \frac{\tau T - \kappa B_1}{\sqrt{\kappa^2+\tau^2}} \tag{23}$$

where $\|T_\gamma \wedge N_\gamma \wedge \gamma'''\| = (c-s)\kappa\tau\sigma$. Considering (10), we have the second curvature of $\gamma$ as follows:

$$\tau_\gamma = \frac{\tau\sigma}{(c-s)\kappa\sqrt{\kappa^2+\tau^2}}. \tag{24}$$

We know that the first binormal vector may be formed by the vector product $B_{1\gamma} = \mu N_\gamma \wedge T_\gamma \wedge B_{2\gamma}$. First, we express

$$N_\gamma \wedge T_\gamma \wedge B_{2\gamma} = -\frac{1}{\kappa^2+\tau^2} \begin{vmatrix} T & -N & B_1 & B_2 \\ \kappa & 0 & \tau & 0 \\ 0 & 1 & 0 & 0 \\ \tau & 0 & -\kappa & 0 \end{vmatrix}. \tag{25}$$

We get (25) as $N_\gamma \wedge T_\gamma \wedge B_{2\gamma} = -B_2$. So, we immediately arrive at

$$B_{1\gamma} = \mu B_2. \tag{26}$$

Finally, in terms of (20), (21) and (24), we have the third curvature of the involute

$$\sigma_\gamma = \frac{\sigma}{(c-s)\sqrt{\kappa^2+\tau^2}} \tag{27}$$

Consequently, we determined FS apparatus of involute. Moreover, one can easily see that the curvature functions of the involute cannot be constant. Since, $\gamma$ cannot be a helix.

*Corollary 6.* The frame $\{T_\gamma, N_\gamma, B_{1\gamma}, B_{2\gamma}\}$ is another orthonormal frame of $E_1^4$.

*Theorem 7.* Let $\gamma(s_\gamma)$ be a time-like involute of space-like evolute helix $\xi(s)$. $\gamma(s_\gamma)$ cannot be a time-like general helix.

*Proof.* Let $\gamma(s_\gamma)$ be a time-like general helix in $E_1^4$. Then, there exist a non-zero constant time-like vector field $U$ makes a constant angle between the tangent $T_\gamma$. So that:

$$g(T_\gamma, U) = \cosh\alpha, \tag{28}$$

where $\alpha$ is a constant hyperbolic angle. By the differentiation of (28), we have

$$\kappa_\gamma g(N_\gamma, U) = 0. \tag{29}$$

Since, we express the vector $U$ as

$$U = \delta_1 T_\gamma + \delta_2 B_{1\gamma} + \delta_3 B_{2\gamma}. \tag{30}$$

We can write congruent FS frame vectors as

$$U = \frac{\delta_3\,\tau}{\sqrt{\kappa^2+\tau^2}}T + \delta_1\,N - \frac{\delta_3\,\kappa}{\sqrt{\kappa^2+\tau^2}}B_1 - \delta_2\,B_2, \tag{31}$$

where $\delta_i$ are functions of $s$. Differentiating both sides of (31) with respect to $s$, we have a system of ordinary differential equations as follows:

$$\begin{cases} \delta_1' = 0, \\ \delta_2' = \dfrac{\kappa\,\sigma}{\sqrt{\kappa^2+\tau^2}}\,\delta_3, \\ \delta_3' = \dfrac{\sqrt{\kappa^2+\tau^2}}{\kappa}(\tau\,a_1 - \sigma\,a_2), \\ \delta_3' = -\dfrac{\kappa\sqrt{\kappa^2+\tau^2}}{\tau}\delta_1. \end{cases} \tag{32}$$

The first equation of the system (32) leads to

$$\delta_1 = c_1. \tag{33}$$

The third and the fourth equations give

$$\delta_3 = \frac{\kappa^2+\tau^2}{\tau\sigma}\,c_1. \tag{34}$$

From the second equation we obtain

$$\delta_3 = 0. \tag{35}$$

The last equation, again, leads to $a_1 = 0$ and then $a_2 = 0$. The vector $U$ must equal zero, which is a contradiction.

*Theorem 8.* Let $\gamma(s_\gamma)$ be a time-like involute of space-like evolute helix $\xi(s)$. $\gamma(s_\gamma)$ cannot be a time-like type-3 slant helix.

*Proof.* Let $\gamma(s_\gamma)$ be a time-like type-3 slant helix in $E_1^4$. Then, there exist a non-zero constant space-like vector field $W$ makes a constant angle with the vector $B_{2\gamma}$, i.e.,

$$g(B_{2\gamma}, W) = \cosh\beta, \tag{36}$$

where $\beta$ is a constant hyperbolic angle. By the differentiation of (36), we have

$$\sigma_\gamma g(B_{1\gamma}, W) = 0. \tag{37}$$

Since, we know that $W$ is spanned by $\{T, N, B_1\}$ of $\xi$. Similar to above proof, substituting congruent vectors to $W$, we can obtain the following system of differential equations:

$$\begin{cases} \tau\,\varepsilon_2 - \kappa\,\varepsilon_3 = 0, \\ \varepsilon_1' + \sqrt{\kappa^2+\tau^2}\varepsilon_2 = 0, \\ \kappa\sqrt{\kappa^2+\tau^2}\varepsilon_1 + \kappa\varepsilon_2' - \tau\varepsilon_3' = 0, \\ \tau\sqrt{\kappa^2+\tau^2}\varepsilon_1 + \tau\varepsilon_2' - \kappa\varepsilon_3' = 0. \end{cases} \tag{38}$$

where $\varepsilon_i$ are the components according to $\{T, N, B_1\}$ subspace, respectively. Solving the system above, we have

$$\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = 0, \tag{39}$$

which is a contradiction.

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