Quantum Entanglement Through Quaternions

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Using quaternions, we study the geometry of the single and two qubit states of quantum entanglement. Through the Hopf fibrations, we identify geometric manifestations of the separability and entanglement of two qubit quantum systems.

Keywords: Quantum Entanglement, Quaternions, Hopf fibrations

Introduction

Ever since the invention of "quaternions [1-7]" in 1843 by Sir William Hamilton to model the three dimensional motion of rigid bodies, these magic numbers have fascinated mathematicians and physicists worldwide with application growing by the day. Quaternions have provided a successful and elegant means for the representation of three dimensional rotations, Lorentz transformations of special relativity, robotics, computer vision, problems of electrical engineering and so on. Quaternionic Quantum Mechanics has aso shown potential of possible unification with General Relativity. In fact, there is belief in some schools of thought that the conventional quantum mechanics in complex spacetime is an asymptotic version of the Quaternionic Quantum Mechanics.

In this paper, an attempt is made to apply these "quaternions" in quantum entanglement theory.

The Geometry of Two Qubit States & Quantum Entanglement

The state space of a two level quantum system is conventionally taken as the tensor product Hilbert space $\mathrm{H} \equiv \mathbb{C} \otimes \mathbb{C}$ [8-11] which in the projective ray representation is isomorphic to the complex projective space $\mathbb{C}P^3$. The conventional Hopf map $\pi : S^3 \to S^2$ [12-15] can easily be generalized to $\pi : S^7 \to S^4$. This motivates us to examine the geometry of a two qubit quantum state using the formalism of the Hopf map. However, when addressing multiple qubit states, one needs to carefully consider the issue of quantum entanglement. The "quaternions" again come in handy in studying the two qubit state.

We can write a generic pure state of a two qubit system in the computational basis as

$$\left|\Upsilon\right\rangle = \alpha\left|00\right\rangle + \beta\left|01\right\rangle + \chi\left|10\right\rangle + \delta\left|11\right\rangle \tag{1}$$

where

$$\left|ij\right\rangle \equiv \left|i\right\rangle \otimes \left|j\right\rangle, \left|i\right\rangle \in \mathrm{H}_A, \left|j\right\rangle \in \mathrm{H}_B, \alpha, \beta, \chi, \delta \in \mathbb{C},$$

$$|\alpha|^2 + |\beta|^2 + |\chi|^2 + |\delta|^2 = 1, \; \alpha = \alpha_{\mathrm{Re}} + i\alpha_{\mathrm{Im}},$$

$$\beta = \beta_{\mathrm{Re}} + i\beta_{\mathrm{Im}}, \; \chi = \chi_{\mathrm{Re}} + i\chi_{\mathrm{Im}}, \; \delta = \delta_{\mathrm{Re}} + i\delta_{\mathrm{Im}}.$$

This normalization condition translates to a sphere S^7 embedded in R^8. Now, if the two qubit state is a composition of two separable one qubit states, then it should be possible to write the composite state as the tensor product of the two single qubit states. Writing

$$\left|\phi\right\rangle_A = a_1\left|0\right\rangle_A + a_2\left|1\right\rangle_A \tag{2a}$$

$$\left|\phi\right\rangle_B = b_1\left|0\right\rangle_B + b_2\left|1\right\rangle_B \tag{2b}$$

we have, for separable states

$$\left|\Upsilon\right\rangle = \left|\phi\right\rangle_A \otimes \left|\phi\right\rangle_B = a_1 b_1\left|00\right\rangle + a_1 b_2\left|01\right\rangle + a_2 b_1\left|10\right\rangle + a_2 b_2\left|11\right\rangle \tag{3}$$

whence, from eqs. (1) & (3), the separability condition can be inferred as

$$\alpha\delta - \beta\chi = 0 \tag{4}$$

To introduce the Hopf fibration $\pi : S^7 \to S^4$ through the quaternions, we write the probability amplitudes $\alpha, \beta, \chi, \delta \in \mathbb{C}$ in the form of two quaternions using the symplectic decomposition as $q_1 = \alpha_{\mathrm{Re}} + \alpha_{\mathrm{Im}}\mathbf{i} + \beta_{\mathrm{Re}}\mathbf{j} + \beta_{\mathrm{Im}}\mathbf{k}$ and $q_2 = \chi_{\mathrm{Re}} + \chi_{\mathrm{Im}}\mathbf{i} + \delta_{\mathrm{Re}}\mathbf{j} + \delta_{\mathrm{Im}}\mathbf{k}$. Obviously, the normalization condition implies that $\left|q_1\right|^2 + \left|q_2\right|^2 = 1$. Parametrizing the sphere S^4 as $\sum_{l=1}^{5} \xi_l^2 = 1$, we obtain the Hopf map $\pi : S^7 \to S^4$ by the mapping

$$\xi_1 = Q_0, \xi_2 = Q_1, \xi_3 = Q_2, \xi_4 = Q_3 \ \& \ \xi_5 = \sqrt{\left(1 - \left|Q\right|^2\right)} \tag{5}$$

where

$$\pi\left(q_1, q_2\right) = Q = Q_0 + Q_1\mathbf{i} + Q_2\mathbf{j} + Q_3\mathbf{k} = 2\overline{\left(q_1\overline{q}_2\right)} \tag{6}$$

Explicit computation using the values of the quaternions q_1 and q_2 yield

$$\xi_1 = 2\left(\alpha_{\rm Re}\chi_{\rm Re} + \beta_{\rm Re}\delta_{\rm Re} + \alpha_{\rm Im}\chi_{\rm Im} + \beta_{\rm Im}\chi_{\rm Im}\right) = 2\,{\rm Re}\left(\bar{\alpha}\chi + \bar{\beta}\delta\right) \quad (7)$$

$$\xi_2 = 2\left(\alpha_{\rm Re}\chi_{\rm Im} - \alpha_{\rm Im}\chi_{\rm Re} + \beta_{\rm Re}\delta_{\rm Im} - \beta_{\rm Im}\delta_{\rm Re}\right) = 2\,{\rm Im}\left(\bar{\alpha}\chi + \bar{\beta}\delta\right) \quad (8)$$

$$\xi_3 = 2\left(\alpha_{\rm Re}\delta_{\rm Re} - \alpha_{\rm Im}\delta_{\rm Im} - \beta_{\rm Re}\chi_{\rm Re} + \beta_{\rm Im}\chi_{\rm Im}\right) = 2\,{\rm Re}\left(\alpha\delta - \beta\chi\right) \quad (9)$$

$$\xi_4 = 2\left(\alpha_{\rm Re}\delta_{\rm Im} + \alpha_{\rm Im}\delta_{\rm Re} - \beta_{\rm Re}\chi_{\rm Im} - \beta_{\rm Im}\chi_{\rm Re}\right) = 2\,{\rm Im}\left(\alpha\delta - \beta\chi\right) \quad (10)$$

$$\xi_5 = 1 - 2\left|q_1 q_2\right| = \left|q_1\right|^2 - \left|q_2\right|^2 \quad (11)$$

The Hopf map $\pi : S^7 \to S^4$ is equivalent to the mapping of S^7 onto a fibre bundle with the base space being the unit sphere S^4 and the fibres being spheres S^3 (this is evidenced by the invariance of this map under the transformation $\left(q_1, q_2\right) \mapsto \left(q_1\lambda, q_2\lambda\right)$, where λ is a unit quaternion i.e. $\left|\lambda\right| = 1$ for $\pi\left(q_1\lambda, q_2\lambda\right) = 2\left(\overline{q_1\lambda\,\overline{q_2\lambda}}\right)$ $= 2\left(\overline{q_1\lambda\,\bar{\lambda}\,\bar{q}_2}\right) = 2\left(\overline{q_1\,\bar{q}_2}\right) = \pi\left(q_1, q_2\right)$).

A perusal of the above expressions reveals an intriguing feature of the Hopf map. If the two qubit states are separable i.e. $\alpha\delta - \beta\chi = 0$, then

$$\xi_3 = \xi_4 = 0 \quad (12)$$

and the base space reduces to S^2 which is the Bloch sphere. This Bloch sphere (the base space) constitutes the state space of one of the qubits of the two qubit separable system. The obvious question to be posed, then is – What about the state space of the other qubit of this separable system? To investigate this issue further, we invert the Hopf map to obtain the set of points in S^7 that project to the quaternion Q in S^4 by the Hopf map $\pi : S^7 \to S^4$. The inverse mapping gives

$$\pi^{-1}(Q) = \left\{ \left(w, 2^{-1} Q w \right) : w \in \mathbb{Q}, |w| = 1 \right\} \tag{13}$$

where w is a unit quaternion that spans the S^3 fibre. Now, in the unentangled joint state characterized by $\alpha\delta - \beta\chi = 0$, we find, as mentioned above, that $\xi_3 = \xi_4 = 0$ whence $Q_2 = Q_3 = 0$ so that such states are mapped to the set of pure complex numbers by $\pi : S^7 \to S^4$. Further, writing $w = u + \mathbf{j}\bar{v}$ with $u, v \in \mathbb{C}$ and $|u|^2 + |v|^2 = 1$, we obtain $\pi^{-1}(Q)$ in four complex component form as

$$\pi^{-1}(Q) = \left\{ \left(u, \bar{v}, \frac{1}{2}(\xi_0 + i\xi_1) u, \frac{1}{2}(\xi_0 + i\xi_1) \bar{v} \right) : u, v, \in \mathbb{C}, |u|^2 + |v|^2 = 1 \right\}$$
(14)

Writing $\xi_0 + i\xi_1 \equiv 2e^{i\theta}$ and making use of the freedom of ray representation, we can write the unentangled state corresponding to this $\pi^{-1}(Q)$ as the tensor product

$$\left| \Psi \right\rangle_{\text{unentangled}} = \left(e^{-i\theta/2} \left| 0 \right\rangle_1 + e^{i\theta/2} \left| 1 \right\rangle_1 \right) \otimes \left(u \left| 0 \right\rangle_2 + \bar{v} \left| 1 \right\rangle_2 \right) \tag{15}$$

thereby recovering the standard definition in this framework. It is pertinent to mention here that the coefficients of the second qubit state u, v do not depend on the coordinates of the base space (that corresponds to the first qubit) and so if we introduce a second Hopf map that fibres out the fibrings of the first Hopf map i.e. if by means of another Hopf map $\pi' : S^3 \to S^2$ we further, fibrate the fibres of the first map into a base space (the two sphere S^2) and fibres (being the one dimensional sphere), then the corresponding coordinates of the second map would not depend on the coordinates of the base space of the map $\pi : S^7 \to S^4$ in the case of an unentangled system.

To obtain explicit expressions for the second Hopf map, we make use of the canonical representation of the quaternion units by the well known Pauli matrices $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ as $i \equiv -i\sigma_1$, $j \equiv -i\sigma_2, i \equiv -i\sigma_3$. In terms of these matrices, acting as the basis, the second Hopf mapping is defined by

$$\mathbf{x} = \pi'(w) = \begin{pmatrix} \bar{u} & \bar{v} \end{pmatrix} \boldsymbol{\sigma} \begin{pmatrix} u & v \end{pmatrix}^T \tag{16}$$

yielding

$$\mathbf{x} = \left(\bar{v}u + \bar{u}v, i\left(\bar{v}u - \bar{u}v\right), |u|^2 - |v|^2 \right) \tag{17}$$

Let us take an element of the unitary group $U(1)$, say, $\phi = \begin{pmatrix} \eta & 0 \\ 0 & \bar{\eta} \end{pmatrix} = \tau \mathbf{I} + \vartheta \sigma_3$. We, then, have $\pi'(w\phi) = (w\phi)^\dagger \boldsymbol{\sigma} w\phi$ $= \phi^\dagger \mathbf{x} \phi = \mathbf{x}$ confirming, thereby that $\pi'(w) = \pi'(w\phi)$ for $\phi \in U(1)$ and hence, establishing the projective nature of this Hopf map taking all elements of S^3 connected through a unitary transformation to a single image. The image set is confirmed to be S^2 since $\mathbf{x}^2 = 1$ as can be easily verified. Thus, this Hopf map creates a principal bundle structure for S^3 with the base manifold being S^2 and the fibres being circles S^1 (members of the unitary group $U(1)$).

The second fibration creates another Bloch sphere that can be considered as the state space of the second qubit in the two qubit separable composite system. It needs be emphasized here that such a construction is not permissible in an entangled system because of the non vanishing of the coordinates ξ_3, ξ_4.

To obtain further insights into the geometry of the two state separable system, let us introduce $\alpha = z_0$, $\beta = z_1 e^{i\varphi_1}$, $\chi = z_2 e^{i\varphi_2}$ & $\delta = z_3 e^{i\varphi_3}$ with $z_i, \varphi_i \in \mathbb{R}$, $z_i \geq 0$ whence the normalization condition translates to $z_0^2 + z_1^2 + z_2^2 + z_3^2 = 1$. The separability condition $\alpha\delta - \beta\chi = 0$ in the new coordinates becomes equivalent to the pair of real equations $z_0 z_3 - z_1 z_2 = 0$ & $\varphi_1 + \varphi_2 - \varphi_3 = 0$. The first of these equations implies that $\frac{z_0}{z_1} = \frac{z_2}{z_3} = k\left(\text{say}\right)$. It follows from this that corresponding to a fixed state of one of the qubits, the states of the other qubit trace out a straight line in the space spanned by the basis vectors corresponding to the new set of coordinates. In such a space, therefore, the separable two qubit states would manifest themselves as two families of straight lines. Interestingly, each of these families of straight lines corresponds to a one parameter family of Hopf circles in the above Hopf fibration framework. The above is an illustration of an embedding of the space of separable states $\mathbb{C}P^1 \times \mathbb{C}P^1$ in the complex projective space of the composite two qubit system $\mathbb{C}P^3$ (Segre embedding).

Having established the compatibility of the Hopf fibration representation with the conventional theory for unentangled states, let us, now, address the issue of measurability of entanglement in this formalism. In the context, "Wootters' Concurrence" and the related "Entanglement of Formation" constitute well accepted measures of entanglement, particularly so, for pure states.

To introduce briefly, the concept of "Wootters Concurrence" [16-18] insofar as it relates to pure states, we consider a normalized state vector of a pure state of two qubits represented as

$$|\psi\rangle = \sum_{i,j} C_{ij} |i\rangle |j\rangle \tag{18}$$

with $\sum_{i,j} |C_{ij}|^2 = 1$. We can write its Schmidt decomposition in the form

$$|\psi\rangle = \sum_i \sqrt{\mu_i} |e_i\rangle \otimes |f_i\rangle \tag{19}$$

where the basis sets $\{|e_i\rangle\}$, $\{|f_i\rangle\}$ are obtained from the bases $\{|i\rangle\}$, $\{|j\rangle\}$ by unitary transformations and hence, retain their orthonormality. It can be easily shown, using the singular value decomposition of the coefficient matrix (C_{ij}) that the μ_i's are eigenvalues of $CC^\dagger$ and they satisfy $\mu_1 + \mu_2 = 1$ as can be seen from the characteristic equation for $CC^\dagger$. The von Neumann entropy is, then, $S(\mu_i) = -\sum_i \mu_i \ln \mu_i$. Further, $(\det C)^2 = \mu_1\mu_2 = \mu_1(1-\mu_1)$ whence $\mu_1 = \frac{1}{2}\left(1-\sqrt{1-4(\det C)^2}\right)$.

Let us, now, represent the state $|\psi\rangle$ in the so called "magic basis" [16-18] $\{|\Phi^+\rangle, i|\Phi^-\rangle, i|\Psi^+\rangle, |\Psi^-\rangle\}$ where $|\Phi^+\rangle, |\Phi^-\rangle$, $|\Psi^+\rangle$ & $|\Psi^-\rangle$ constitute the "Bell basis" for the space $\mathbb{C}^2 \otimes \mathbb{C}^2$ of a pair of qubits. We write

$$|\psi\rangle = a_1|\Phi^+\rangle + a_2 i|\Phi^-\rangle + a_3 i|\Psi^+\rangle + a_4|\Psi^-\rangle$$

$$= \frac{1}{\sqrt{2}} \left[\left(a_1 + i a_2 \right) \left| 00 \right\rangle + \left(i a_3 + a_4 \right) \left| 01 \right\rangle + \left(i a_3 - a_4 \right) \left| 10 \right\rangle + \left(a_1 - i a_2 \right) \left| 11 \right\rangle \right] \tag{20}$$

so that

$$C = \frac{1}{\sqrt{2}} \begin{pmatrix} a_1 + i a_2 & i a_3 + a_4 \\ i a_3 - a_4 & a_1 - i a_2 \end{pmatrix} \text{ and } \det C = \frac{1}{2} \sum_k a_k^2 \tag{21}$$

The quantity

$$\kappa = \left| \sqrt{2 \left(1 - Tr \left(\rho^{(1)} \right)^2 \right)} \right| = \left| \sqrt{2 \left(1 - \sum_j \mu_j^2 \right)} \right|$$

$$= 2 \left| \sqrt{\mu_1 \mu_2} \right| = 2 \left| \det C \right| = \left| \sum_k a_k^2 \right| = \left| \left\langle \psi \middle| \tilde{\psi} \right\rangle \right| \tag{22}$$

is called "Wootter's concurrence" in the literature [16-18], where, $\left| \tilde{\psi} \right\rangle = \sigma_y \otimes \sigma_y \left| \psi^* \right\rangle$ is the spin flipped state. κ ranges from 0 to 1 and is monotonically related to the entanglement. It can, therefore, be considered a measure of entanglement. It follows from eq. (22) that any real linear combination of the "magic basis" would result in a fully entangled state with unit concurrence. Conversely, any completely entangled state can be written as a linear combination in the "magic basis" with real components, upto an overall phase factor. In fact, these properties are not unique to a state description in the "magic basis" and hold in any other basis that is obtained from the "magic basis" by an orthogonal transformation since orthogonal transformations do not disturb the norm of a state i.e. $\sum_k a_k^2 = \sum_k a'^2_k$ so that concurrence is not affected by any transformation $O \in SO(4)$

[18].

For the generic two qubit state with coefficients α,β,χ,δ in the usual computational basis, the characteristic equation for $CC^{\dagger}$ takes the form

$$\mu^2 - \mu + \left(\alpha\delta - \beta\chi\right)\left(\bar{\alpha}\bar{\delta} - \bar{\beta}\bar{\chi}\right) = 0 \tag{23}$$

whence the Concurrence is

$$\kappa = 2\left|\left(\alpha\delta - \beta\chi\right)\right| = \sqrt{\xi_3^2 + \xi_4^2} \tag{24}$$

It follows that states with the same Concurrence get mapped into concentric circles of equal radius in the two dimensional projective subspace of the base space that is spanned by the quaternion units $\mathbf{j},\mathbf{k}$. The separable states get mapped to the centre of the circle whereas the states with maximal entanglement constitute the boundary of the unit circle. All states constitute the unit disc in this subspace. Similarly in the three dimensional projective subspace of the base space spanned by $\left(\xi_1,\xi_2,\xi_5\right)$, the set of states will manifest as a ball of unit radius. States of equal concurrence appear as concentric spherical shells of radius $\sqrt{1-\kappa^2}$, separable states get mapped into the boundary shell of unit radius while maximally entangled states form the centre of the ball.

The above expression for the concurrence can be vindicated in another manner. The density matrix corresponding to our two qubit system is

$$\rho_{AB} = \left|\Psi\right\rangle_{AB\,AB}\left\langle\Psi\right| = \left(\bar{\alpha},\bar{\beta},\bar{\chi},\bar{\delta}\right)\otimes\left(\bar{\alpha},\bar{\beta},\bar{\chi},\bar{\delta}\right)^T \tag{25}$$

Taking a partial trace over the variables of the second qubit, we obtain the reduced density matrix for the first qubit as

$$\rho_{AB}^{A} = \begin{pmatrix} |\alpha|^2 + |\beta|^2 & \alpha\bar{\chi} + \beta\bar{\delta} \\ \bar{\alpha}\chi + \bar{\beta}\delta & |\chi|^2 + |\delta|^2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+\xi_5 & \xi_1 - i\xi_2 \\ \xi_1 + i\xi_2 & 1-\xi_5 \end{pmatrix} \tag{26}$$

so that $\det \rho_{AB}^{A} = \frac{1}{4}\left(\xi_3^2 + \xi_4^2\right)$ and Concurrence

$$\kappa = 2\sqrt{\rho_{AB}^{A}} = \sqrt{\xi_3^2 + \xi_4^2} \tag{27}$$

It is easily seen by taking a partial trace over the Hilbert space of the first qubit that Concurrence calculated with the reduced density matrix of the second qubit is the same.

Conclusion

It is shown that the "quaternions" provide an attractive and efficient machinery to study the geometry of the two qubit systems. One is led to the conclusion, through the Hopf map $\pi : S^3 \to S^2$, that the one qubit system has a geometrical representation as the Bloch sphere S^2 which the base space of a principal bundle with fibres consisting of the one dimensional sphere S^1. In the case of the two qubit composite system, a similar over fibration $\pi : S^7 \to S^4$ implies that the system has the geometry of a fibre bundle with the base space being the four dimensional sphere S^4 fibres consisting of S^3. As a fallout of the Hopf map analysis, we also find that unentangled two qubit systems admit a geometry as a direct product of two Bloch spheres as is intuitively to be expected. However, the Bloch sphere corresponding to one of the qubits in an unentangled system must be extracted from the S^3 fibres of the $\pi : S^7 \to S^4$ by invoking a second Hopf fibration of these S^3 fibres as $\pi : S^3 \to S^2$. We also obtain a measure of Wootters' Concurrence in terms of the coordinates of the base space of the two qubit fibration.

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