# General Spin Dirac Equation

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In its bare and natural form, the Dirac Equation describes *only* spin-1/2 particles. The main purpose of this reading is to make a valid and justified mathematical modification to the Dirac Equation so that it describes any spin particle. We show that this mathematical modification is consistent with the Special Theory *of* Relativity (STR). From the vantage point of unity, simplicity and beauty, it is natural to wonder why should there exist different equations to describe particles of different spins? For example, the Klein-Gordon equation describes spin-0 particles, while the Dirac Equation describes spin-1/2, and the Rarita-Schwinger Equation describes spin-3/2. Does it mean we have to look for another equation to describe spin-2 particles, and then spin-5/2 particles *etc*? This does not look beautiful, simple, or at the very least suggest a Unification *of* the Natural Laws.

*Keywords*: boson, fermion, spin

# Introduction

This reading, is part of a series of readings on a program to modify the Dirac Equation. This on-going program – to add on to the Dirac Equation, began with the reading Nyambuya (2008) and this reading (Nyambuya 2008) is an off-spring of the reading Nyambuya (2007) where an attempt has been made on an all encampusing *Unified Field Theory* (UFT). In this reading we generalize the Dirac Equation so that it can describe a general spin-$s/2$ particle where $s = \pm 1, \pm 2, \pm 3, \ldots$ *etc.* The Dirac Equation describes only spin-1/2 particles, we ask here; why only spin-1/2 and not spin-3/2, spin-5/2 particles *etc*? For the sake of simplicity and unity, should not there be one equation to describe all fermions and peharps bosons as-well?

For the sake of completeness, we shall begin by giving a historic derivation of the Dirac Equation and its first major achievements which is its being able to describe the gyromagnetic ratio of the Electron. The Dirac Equation is a relativistic quantum mechanical wave equation invented by Paul Dirac in 1928 (Dirac 1928$a, b$) originally designed to overcome the criticism leveled against the Klein-Gordon Equation. The Klein-Gordon equation gave negative probabilities and this is considered to be physically meaningless. Despite this fact, this equation accounts well for Bosons – that is, spin zero particles. This criticism leveled against the Klein-Gordon equation, is what motivated Dirac to successfully seek an equation devoid of negative probabilities.

The Dirac Equation is consistent with Quantum Mechanics (QM) and fully consistent with the Special Theory *of* Relativity (STR). This equation accounts in a natural way for the nature of particle spin as a relativistic phenomenon and amongst its prophetic achievements was its successful prediction of the existence of antiparticles. In its bare form, the Dirac Equation provided us with an impressive and accurate description of the Electron hence it being referred in most of the literature as the "Dirac Equation for the Electron". It also accounts well for quarks and other spin-1/2 particles, although in some of the cases, there is need for slight modifications while in others it fails - for example, Dirac's Equation in its bare and modified form can not account for the Neutron's gyromagnetic ratio; the Neutron, which as the Electron, is a spin-1/2 particle. The Neutron is a composite particle and for this reason, it is thought that the

Dirac Equation can not describe it [Neutron] because it [the Dirac Equation] does not describe composite particles but fundamental particles.

The first taste of glory of the Dirac Equation was it being able to account for the gyromagnetic ratio of the Electron, that is $g = 2$, which can not be accounted for using non-relativistic QM. For several years after it's discovery, most physicists believed that it described the Proton and the Neutron as-well, which are both spin-1/2 particles. In simple terms, it was thought or presumed that the Dirac Equation was a universal equation for spin-1/2 particles.

However, beginning with the experiments of Stern and Frisch in 1933, the magnetic moments of these particles were found to disagree significantly with the predictions of the Dirac Equation. The Proton was found to have a gyromagnetic ratio $g_p = 5.58$ which is 2.79 times larger than that predicted by the Dirac Equation. The Neutron, which is electrically neutral spin-1/2 particle was found to have a gyromagnetic ratio $g_n = -3.83$.

These "anomalous magnetic moments" of the Neutron and Proton which are clearly not confirmatory to the Dirac Theory have been taken to be experimental indication that these particles are not fundamental particles. In the case of the Neutron, yes it is clearly not a fundamental particle since it does decay into a Proton, Electron and Neutrino, that is, $\mathrm{n} \longrightarrow \mathrm{p} + \mathrm{e}^- + \nu$. If the Dirac Equation is a universal equation for fundamental fermion particles, then any fundamental fermion particle must conform to this equation. Simple, any spin-1/2 particle that can not be described by it, must therefore not be a fundamental particle of nature. By definition a fundamental particle is a particle known to have no sub-structure, that is, it can not be broken down into smaller particles thus will not decay into anything else.

From the Standard Model, we know that the Proton and Neutron are composed of quarks thus are not fundamental particles. The question is, is this the reason why these particle's gyromagnetic ratio is different from that predicted by the bare Dirac Equation? Prevailing wisdom suggests that anomalous gyromagnetic ratio arise because the particles under question are not fundamental particles. We undertook our own initiative in the reading Nyambuya (2008) to try an address this problem and we believe we are on the right path of finding a new version of the Dirac Equation that is applicable to curved spaces and at the same time this equation will hold for all spin-$s/2$ particles.

From this reading (Nyambuya 2008) we suggested that the gyromagnetic ratio differs from the expected $g = 2$ because particles do have a finite size and that spacetime is curved. In this theory, for a particle of finite spatial size and mass, the anomalous gyromagnetic ratio arises from the interaction of spin with the Lorentz force in a curved spacetime. The derived relation for the anomalous magnetic moment and the particle size is similar to that deduced by Brodsky & Drell (1980) and experimentally verified by Dehmelt (1989). Brodsky & Drell (1980) proposed that fermions do have a sub-structure and this gives rise to the anomalous gyromagnetic ratio which varies as the spatial size and inverse to the mass.

As an expository and instructive exercise, in the next section we shall go through the standard derivation of the Dirac Equation. In §(2), again as an expository and instructive exercise we go through the derivation of the historic fit of the Dirac Equation, that is how it theoretically accounted so well for the gyromagnetic ratio of the Electron. In §(4), we do another expository and instructive exercise that is central to this reading, we give a theoretical reason why the Dirac Equation is said to describe spin-1/2 particles. In §(3), we derive our main result and then seek its justification in §(5). In §(6) we show that the derived general spin Dirac equation gives the same gyromagnetic ratio as the original Dirac equation. In §(7) we give the modified energy equation resulting from the modification that we have made to the Dirac Equation and there argue that this found equation is consistent with the STR. Lastly, in §(2) we give a general discussion.

# 1 Dirac's Derivation

Suppose we have a particle of rest mass $m_0$, momentum $p$, and energy $E$, Albert Einstein, from his 1905 reading on the STR, derived the basic equation:

$$E^2 = p^2c^2 + m_0^2c^4, \tag{1}$$

which later formed the basis of the Klein-Gordon Theory upon which the Dirac Theory is founded. This equation can be written in the matrix form:

$$m_0^2c^2 = \begin{pmatrix} E/c \\ p_x \\ p_y \\ p_z \end{pmatrix}^T \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} E/c \\ p_x \\ p_y \\ p_z \end{pmatrix}, \tag{2}$$

where the $4 \times 4$ matrix sandwiched between the row and column vectors:

$$[\eta_{\mu\nu}] = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{3}$$

is the flat spacetime Minkowski metric and the superscript T in the left hand side column vector represents the transpose operation. Using the already established canonical quantisation procedures, Klein and Gordon, proposed the Klein-Gordon equation:

$$\Box\Psi = \left(\frac{m_0 c}{\hbar}\right)^2 \Psi, \tag{4}$$

which describes a spin-0 quantum mechanical scalar particle whose wave-function is $\Psi$ and $\Box$ is:

$$\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2, \tag{5}$$

where $\nabla^2$ is the usual Laplacian. This equation allows for negative probabilities and as already stated, Dirac was not satisfied with the Klein-Gordon Theory. He noted that the Klein-Gordon equation is second order differential equation and his suspicion was that the origin of the negative probability solutions may have something to do with this very fact. He was right!

He sought an equation linear in both the time and spatial derivatives that would upon "squaring" reproduce the Klein-Gordon equation. The equation he found was:

$$[i\hbar\gamma^\mu\partial_\mu - m_0 c]\,\psi = 0, \tag{6}$$

where:

$$\gamma^0 = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} \mathbf{0} & \sigma^i \\ -\sigma^i & \mathbf{0} \end{pmatrix} \tag{7}$$

are the $4 \times 4$ Dirac gamma matrices ($\mathbf{I}$ and $\mathbf{0}$ are the $2 \times 2$ identity and null matrices respectively) and $\psi$ is the four component Dirac wave-function. Throughout this reading, the Greek indices will be understood to mean $\mu, \nu, ... = 0, 1, 2, 3$ and lower case English alphabet indices $i, j, k... = 1, 2, 3$.

# 2 Dirac Gyromagnetic Ratio

It is shown here how the Dirac Equation discussed in the previous section accounts well for the gyromagnetic ratio of the Electron – for this reason, it [Dirac Equation] is said to account very well for the Electron. This discussion follows closely that of Zee (2003).

We immense the Electron inside an ambient electromagnetic field $A^{ex}_\mu$ and this is mathematically expressed by transforming the partial derivatives: $\partial_\mu \longmapsto D_\mu = \partial_\mu - ieA^{ex}_\mu$. Making this replacement into (6) results in this equation reducing to:

$$\left[i\hbar\gamma^\mu D_\mu - m_0 c\right]\psi = 0. \tag{8}$$

Now acting on this equation with: $(i\hbar\gamma^\mu D_\mu + m_0 c)$, we obtain:

$$\left(\gamma^\mu\gamma^\nu D_\mu D_\nu + m_0^2 c^2/\hbar^2\right)\psi = 0. \tag{9}$$

We have: $\gamma^\mu\gamma^\nu D_\mu D_\nu = \frac{1}{2}\left(\{\gamma^\mu, \gamma^\nu\} + [\gamma^\mu, \gamma^\nu]\right) D_\mu D_\nu = D_\mu D^\mu - i\sigma^{\mu\nu} D_\mu D_\nu$, and: $i\sigma^{\mu\nu} D_\mu D_\nu = (i/2)\sigma^{\mu\nu}\left[D_\mu, D_\nu\right] = (e/2)\sigma^{\mu\nu} F^{ex}_{\mu\nu}$, where: $F^{ex}_{\mu\nu}$, is the electromagnetic field tensor of the applied external field. The above calculations reduce to:

$$\left(D_\mu D^\mu - \frac{e}{2}\sigma^{\mu\nu} F^{ex}_{\mu\nu} + \frac{m_0^2 c^2}{\hbar^2}\right)\psi = 0. \tag{10}$$

Now consider a weak constant magnetic field in the $z - axis$ such that $\vec{\mathbf{A}} = -(1/2)\vec{\mathbf{r}} \times \vec{\mathbf{B}}$ where $\mathbf{B} = (\vec{0}, 0, B)$ so that $F_{12} = B$. Neglecting second order terms one is lead to:

$$\begin{aligned}(D_i)^2 &= (\partial_i)^2 - e(\partial_i A_i^{ex} + A_i^{ex}\partial_i) + O(A_{ex,i}^2) \\ &= (\partial_i)^2 - eB(x^1\partial_2 - x^2\partial_1) + O(A_{ex,i}^2) \quad , \\ &= \vec{\nabla}^2 - e\vec{\mathbf{B}}\cdot\vec{\mathbf{L}} + O(A_{ex,i}^2)\end{aligned} \tag{11}$$

where: $\vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}}$, is the orbital angular momentum operator which means that the orbital angular momentum generates orbital magnetic moment that interacts with the magnetic field. Now, if we write the Dirac four component wave-function as $\psi = \begin{pmatrix} \Phi \\ \chi \end{pmatrix}$, one finds that in the non-relativistic limit the component $\chi$ dominates. Thus: $e\sigma^{\mu\nu}F_{\mu\nu}^{ex}/2$, acting on: $\Phi$, effectively equals: $\frac{e}{2}\sigma^3(F_{12}^{ex} - F_{21}^{ex}) = 2e\vec{\mathbf{B}}\cdot\vec{\mathbf{S}}$, since: $\vec{\mathbf{S}} = (\vec{\boldsymbol{\sigma}}/2)$. Now writing: $\Phi = e^{-im_0t}\Psi$, where: $\Psi$ oscillates much more slowly than: $e^{im_0t}$, so that: $(\partial_0^2 + m_0^2c^2/\hbar^2)e^{-im_0c^2t/\hbar}\Psi \simeq e^{-im_0c^2t/\hbar}\left[-(2im_0c/\hbar)\partial_0\Psi\right]$. Putting all the bits and pieces together, one is lead to:

$$\left[\frac{\hbar^2}{2m_0}\vec{\nabla}^2 + \mu_B\vec{\mathbf{B}}\cdot(\vec{\mathbf{L}} + 2\vec{\mathbf{S}})\right]\Psi = -i\hbar\frac{\partial\Psi}{\partial t}, \tag{12}$$

and this equation, above and below embodies the historic fit of the Dirac Equation in that it automatically tells us that the gyromagnetic ratio of the Electron is 2 – this is deduced from the factor 2 in the spin term. However as already explained, precise measurements put this value slightly above $g = 2$ and this discrepancy in observations and theory caused the theorist to go back to the drawing board to seek harmony with observations. The reading Nyambuya (2008) is an attempt to give an alternative (to the existing explanation which makes use of Feynman diagrams) explanation to this phenomena.

## 3 Spin of the Dirac Spinor

In this section we shall ask the obvious question, "How do we know the Dirac Equation describes the Electron?" From a practical point of view we know the gyromagnetic ratio predicted by the Dirac Equation is very close to that measured in the laboratory. From a theoretical perspective (see *e.g.* Halzen & Martin 1984, or any good book on relativistic QM), we know that the Dirac Equation describes a spin-1/2 particle because the Dirac Hamiltonian, given:

$$\boldsymbol{\mathcal{H}}_D = -i\hbar c\alpha^k\partial_k + \beta m_0c^2, \tag{13}$$

where $\alpha^k := \gamma^0\gamma^k$ and $\beta := \gamma^0$; commutes with the angular momentum operator: $\mathbf{J} = \mathbf{L} + \mathbf{S}$, that is:

$$[\boldsymbol{\mathcal{H}}_D, \mathbf{J}] = 0, \tag{14}$$

where the orbital angular momentum: $\mathbf{L} = -i\hbar\vec{\mathbf{r}} \times \vec{\nabla}$, and:

$$\mathbf{S} = \frac{1}{2}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix} \text{ where } \boldsymbol{\sigma} = \sigma_1\mathbf{i} + \sigma_2\mathbf{j} + \sigma_3\mathbf{k}. \tag{15}$$

$\mathbf{S}$, is the spin-operator and the vectors $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$, are the usual orthogonal vectors defining the $3D$ space. The eigen value of the spin-operator (15) is 1/2 hence the Dirac particle is said to have spin-1/2. According to a theorem of QM, any observable corresponding to an operator that commutes with the Hamiltonian is a conserved quantity hence thus the total angular momentum $\mathbf{J}$ is a conserved quantity. From this it makes sense to say because the total angular momentum of a spin-1/2 particle, which in this case is given by $\mathbf{J}$ commutes with the Dirac Hamiltonian, then the spin of a Dirac particle must be 1/2.

From the above arguments, it is pristine clear that if one can find the total angular momentum for a general spin particle $\mathbf{J}_g$ and also a corresponding Hamiltonian for this particle $\mathcal{H}_g$ such that $[\mathcal{H}_g, \mathbf{J}_g] = 0$, then from the Schrödinger formulation, the equation $\mathcal{H}_g\psi = E\psi$ where $E$ is total energy of the particle; describes a general spin particle. Our task in the next section is to successfully seek this generalized Hamiltonian and total angular momentum for a general spin particle, hence thus, we shall write down a Dirac Equation for a general spin particle.

# 4 Modification

We can modify the Dirac Equation to describe in general, a particle of spin $s/2$ where $s = \pm1, \pm2, \pm3, \ldots$ *etc.* To do this, we note that if we make the transformation: $\vec{\nabla} \longmapsto s\vec{\nabla}$, the new Dirac Hamiltonian: $\boldsymbol{\mathcal{H}}_D(s) = -is\hbar c\alpha^k\partial_k + \beta m_0 c^2$, will commute with the total angular momentum operator: $\mathbf{J}(s) = \mathbf{L}(s) + \mathbf{S}(s)$, where: $\mathbf{L}(s) = -is\hbar\vec{\mathbf{r}} \times \vec{\nabla}$, and:

$$\mathbf{S}(s) = \frac{s}{2}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}. \tag{16}$$

By the same theoretical argument leading us to making the conclusion that the Dirac Equation, whose Hamiltonian $\boldsymbol{\mathcal{H}}_D$ commutes with $\mathbf{J}$ describes a particle of spin-1/2, we are lead directly to conclude that the modified Dirac Hamiltonian $\boldsymbol{\mathcal{H}}_D(s)$ which commutes with the total angular momentum operator $\mathbf{J}(s)$ describes a particle of spin $s/2$. Instead of writing: $\boldsymbol{\mathcal{H}}_D(s) = -is\hbar c\alpha^k\partial_k + \beta m_0 c^2$, let us write: $\boldsymbol{\mathcal{H}}_D(s) = -i\hbar c\gamma_s^k\partial_k + \beta m_0 c^2$, where: $\gamma_s^k = s\gamma^k$. For uniformity, we shall also give $\gamma^0$ the same notation, that is: $\gamma_s^0 = \gamma^0$. The modification: $\gamma^\mu \longmapsto \gamma_s^\mu$, leads to equation (6) describing any spin-particle, that is:

$$\left[i\hbar\gamma_s^\mu\partial_\mu - m_0 c\right]\psi = 0. \tag{17}$$

This is the equation we sought, now we need to justify it. This equation is just the same as the Dirac Equation with the important difference that the Dirac matrices $\gamma^\mu$ have now been replaced by $\gamma_s^\mu$.

# 5 Justification

As a way of justification, we shall prove here that $[\mathcal{H}_D(s), \mathbf{J}(s)] = 0$. The entirety of the present reading hinges on this very result $[\mathcal{H}_D(s), \mathbf{J}(s)] = 0$, hence it is most logical that this result be proved beyond any reasonable doubt and this be done much to the satisfaction of the reader. We begin: $[\mathcal{H}_D(s), \mathbf{J}(s)] = [\mathcal{H}_D(s), \mathbf{J}_x(s) + \mathbf{J}_y(s) + \mathbf{J}_z(s)]$ since $\mathbf{J}(s) = J_x(s)\mathbf{i} + J_y(s)\mathbf{j} + J_z(s)\mathbf{k}$, and from this it follows that $[\mathcal{H}_D(s), \mathbf{J}(s)]$ can be split into the three components *i.e.*: $[\mathcal{H}_D(s), J_x(s)]\,\mathbf{i} + [\mathcal{H}_D(s), J_y(s)]\,\mathbf{j} + [\mathcal{H}_D(s), J_z(s)]\,\mathbf{k}$. Now if $[\mathcal{H}_D(s), \mathbf{J}(s)] = 0$, then each of the components must equal zero, that is: $[\mathcal{H}_D(s), J_x(s)] = 0$, $[\mathcal{H}_D(s), J_y(s)] = 0$, $[\mathcal{H}_D(s), J_z(s)] = 0$. If just one of the component is zero, the rest are also zero, hence thus $[\mathcal{H}_D(s), \mathbf{J}(s)] = 0$. We shall prove for the $x$-component, that is: $[\mathcal{H}_D(s), J_x(s)] = 0$. We know that:

$$\mathbf{J} = -is\hbar \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ \partial_x & \partial_y & \partial_z \end{vmatrix} = \tag{18}$$

$$-is\hbar \begin{vmatrix} y & z \\ \partial_y & \partial_z \end{vmatrix} \mathbf{i} + is\hbar \begin{vmatrix} x & z \\ \partial_x & \partial_z \end{vmatrix} \mathbf{j} - is\hbar \begin{vmatrix} x & y \\ \partial_x & \partial_y \end{vmatrix} \mathbf{k},$$

and from this we pluck out the $\mathbf{i}^{th}$-component of $\mathbf{J}$, hence it follows that:

$$J_x(s) = is\hbar \begin{pmatrix} \boldsymbol{I}(y\partial_x - z\partial_y) & 0 \\ 0 & \boldsymbol{I}(y\partial_x - z\partial_y) \end{pmatrix} + \frac{s\hbar}{2} \begin{pmatrix} \boldsymbol{\sigma}_x & 0 \\ 0 & \boldsymbol{\sigma}_x \end{pmatrix}, \tag{19}$$

where $\boldsymbol{I}$ is the $2 \times 2$ identity matrix. From the above, it follows that:

$$J_x(s) = s\hbar \begin{pmatrix} i\boldsymbol{I}(y\partial_x - z\partial_y) + \boldsymbol{\sigma}_x/2 & 0 \\ 0 & i\boldsymbol{I}(y\partial_x - z\partial_y) + \boldsymbol{\sigma}_x/2 \end{pmatrix}. \tag{20}$$

Now, $[\mathcal{H}_D(s), J_x(s)] = \left[-i\hbar c\gamma_s^k \partial_k + \beta m_0 c^2, J_x(s)\right]$ and from this we have $[\mathcal{H}_D(s), J_x(s)] = \left[-i\hbar c\gamma_s^k \partial_k, J_x(s)\right] + \left[\beta m_0 c^2, J_x(s)\right]$. We shall compute the term $\left[\beta m_0 c^2, J_x(s)\right] = m_0 c^2 \left[\beta, J_x(s)\right]$, that is:

$$[\beta, J_x(s)] = s\hbar \left\{ \begin{pmatrix} \boldsymbol{I} & 0 \\ 0 & -\boldsymbol{I} \end{pmatrix} \begin{pmatrix} i\boldsymbol{I}(y\partial_x - z\partial_y) + \boldsymbol{\sigma}_x/2 & 0 \\ 0 & i\boldsymbol{I}(y\partial_x - z\partial_y) + \boldsymbol{\sigma}_x/2 \end{pmatrix} - \right.$$

$$\left. \begin{pmatrix} i\boldsymbol{I}(y\partial_x - z\partial_y) + \boldsymbol{\sigma}_x/2 & 0 \\ 0 & i\boldsymbol{I}(y\partial_x - z\partial_y) + \boldsymbol{\sigma}_x/2 \end{pmatrix} \begin{pmatrix} \boldsymbol{I} & 0 \\ 0 & -\boldsymbol{I} \end{pmatrix} \right\}. \tag{21}$$

Clearly the above is identically equal to zero *i.e.* $[\beta, J_x(s)] = 0$, hence thus: $[\mathcal{H}_D(s), J_x(s)] = -i\hbar c \left[\gamma_s^k \partial_k, J_x(s)\right]$. Thus if we can proof or show that $\left[\gamma_s^k \partial_k, J_x(s)\right] = 0$, this, according to arguments already presented; automatically implies $[\mathcal{H}_D(s), \mathbf{J}(s)] = 0$. Now the term $\left[\gamma_s^k \partial_k, J_x(s)\right]$ is equal to:

$$s\hbar\left\{\begin{pmatrix} \mathbf{0} & \boldsymbol{\sigma}^k\partial_k \\ -\boldsymbol{\sigma}^k\partial_k & \mathbf{0}\end{pmatrix}\begin{pmatrix} i\boldsymbol{I}(y\partial_x - z\partial_y)+\boldsymbol{\sigma}_x/2 & 0 \\ 0 & i\boldsymbol{I}(y\partial_x - z\partial_y)+\boldsymbol{\sigma}_x/2\end{pmatrix} - \begin{pmatrix} i\boldsymbol{I}(y\partial_x - z\partial_y)+\boldsymbol{\sigma}_x/2 & 0 \\ 0 & i\boldsymbol{I}(y\partial_x - z\partial_y)+\boldsymbol{\sigma}_x/2\end{pmatrix}\begin{pmatrix} \mathbf{0} & \boldsymbol{\sigma}^k\partial_k \\ -\boldsymbol{\sigma}^k\partial_k & \mathbf{0}\end{pmatrix}\right\}, \quad (22)$$

where $k = x, y, z$, which is also the same as $k = 1, 2, 3$. Now, dropping the factor $s\hbar$, meaning to say the new expression is: $\left[\gamma_s^k\partial_k, J_x(s)\right]/s\hbar$. The reduced expression of this new expression $\left[\gamma_s^k\partial_k, J_x(s)\right]/s\hbar$ , is:

$$\begin{matrix}\begin{pmatrix} \mathbf{0} & i\boldsymbol{\sigma}^k\partial_k\left[\boldsymbol{I}(y\partial_x - z\partial_y)+\boldsymbol{\sigma}_x/2\right] \\ -i\boldsymbol{\sigma}^k\partial_k\left[\boldsymbol{I}(y\partial_x - z\partial_y)+\boldsymbol{\sigma}_x/2\right] & \mathbf{0}\end{pmatrix} - \\ \begin{pmatrix} \mathbf{0} & i\left[\boldsymbol{I}(y\partial_x - z\partial_y)+\boldsymbol{\sigma}_x/2\right]\boldsymbol{\sigma}^k\partial_k \\ -i\left[\boldsymbol{I}(y\partial_x - z\partial_y)+\boldsymbol{\sigma}_x/2\right]\boldsymbol{\sigma}^k\partial_k & \mathbf{0}\end{pmatrix}\end{matrix}. \quad (23)$$

The expression in the matrix on the left handside of the minus sign: $\boldsymbol{\sigma}^k\partial_k[\boldsymbol{I}(y\partial_x - z\partial_y)+\boldsymbol{\sigma}_x/2] = \boldsymbol{\sigma}^k\left[\boldsymbol{I}(y\partial_x - z\partial_y)+\boldsymbol{\sigma}_x/2\right]\partial_k$, and for expression in the matrix on the right handside of the minus sign: $[\boldsymbol{I}(y\partial_x - z\partial_y)+\boldsymbol{\sigma}_x/2]\boldsymbol{\sigma}^k\partial_k = [\boldsymbol{I}(y\partial_x - z\partial_y)\boldsymbol{\sigma}^k+\boldsymbol{\sigma}_x\boldsymbol{\sigma}^k/2]\partial_k$ hence, the above reduces to:

$$\begin{matrix}\begin{pmatrix} \mathbf{0} & i\left[\boldsymbol{\sigma}^k(y\partial_x - z\partial_y)+\boldsymbol{\sigma}^k\boldsymbol{\sigma}_x/2\right]\partial_k \\ -i\left[\boldsymbol{\sigma}^k(y\partial_x - z\partial_y)+\boldsymbol{\sigma}^k\boldsymbol{\sigma}_x/2\right]\partial_k & \mathbf{0}\end{pmatrix} - \\ \begin{pmatrix} \mathbf{0} & i\left[(y\partial_x - z\partial_y)\boldsymbol{\sigma}^k+\boldsymbol{\sigma}_x\boldsymbol{\sigma}^k/2\right]\partial_k \\ -i\left[(y\partial_x - z\partial_y)\boldsymbol{\sigma}^k+\boldsymbol{\sigma}_x\boldsymbol{\sigma}^k/2\right]\partial_k & \mathbf{0}\end{pmatrix},\end{matrix} \quad (24)$$

and $\boldsymbol{I}(y\partial_x - z\partial_y)\boldsymbol{\sigma}^k = \boldsymbol{\sigma}^k(y\partial_x - z\partial_y)$ and $\boldsymbol{\sigma}_x\boldsymbol{\sigma}^k = \boldsymbol{I}$, hence, the final and clear expression emerging from all these calculations is:

$$\begin{pmatrix} \mathbf{0} & i\left[\boldsymbol{\sigma}^k(y\partial_x - z\partial_y)\partial_k + \boldsymbol{I}\partial_k/2\right] \\ -i\left[\boldsymbol{\sigma}^k(y\partial_x - z\partial_y)\partial_k + \boldsymbol{I}\partial_k/2\right] & \mathbf{0} \end{pmatrix} - \begin{pmatrix} \mathbf{0} & i\left[\boldsymbol{\sigma}^k(y\partial_x - z\partial_y)\partial_k + \boldsymbol{I}\partial_k/2\right] \\ -i\left[\boldsymbol{\sigma}^k(y\partial_x - z\partial_y)\partial_k + \boldsymbol{I}\partial_k/2\right] & \mathbf{0} \end{pmatrix} \equiv 0 \tag{25}$$

**This completes the *proof* . . . , QED.**

Hence thus $[\mathcal{H}_D(s), \mathbf{J}(s)] = 0$, the meaning of which is that $\mathcal{H}_D(s)$ is the Hamiltonian of the particle whose total angular momentum is $\mathbf{J}(s)$.

# 6 Implied Gyromagnetic Ratio

What is the gyromagnetic ratio expected from this equation (17)? To answer this, we simple redo the exercise in §(2). We immense this particle in an electromagnetic field $A_\mu^{ex}$ where upon the derivatives transform as: $\partial_\mu \longmapsto D_\mu = \partial_\mu - ieA_\mu^{ex}$. Making this replacement in equation (17) results in: $[i\hbar\gamma_s^\mu D_\mu - m_0c]\,\psi = 0$. Now acting on this equation from the left with: $[i\hbar\gamma_s^\mu\partial_\mu - m_0c]^\dagger$, one obtains the equation: $\left(\gamma_s^\mu\gamma_s^\nu D_\mu D_\nu + m_0^2c^2/\hbar^2\right)\psi = 0$. From this, we obtain after some rearranging:

$$\gamma_s^\mu\gamma_s^\nu D_\mu D_\nu = \frac{1}{2}\left(\{\gamma_s^\mu, \gamma_s^\nu\} + [\gamma_s^\mu, \gamma_s^\nu]\right) D_\mu D_\nu = \eta_{\mu\nu}^{(s)} D^\mu D^\nu - i\sigma_{(s)}^{\mu\nu} D_\mu D_\nu, \tag{26}$$

where:

$$\eta_{\mu\nu}^s = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -s^2 & 0 & 0 \\ 0 & 0 & -s^2 & 0 \\ 0 & 0 & 0 & -s^2 \end{pmatrix}. \tag{27}$$

The relations: $\eta_{\mu\nu}^s = \{\gamma_s^\mu, \gamma_s^\nu\}$, and: $\sigma_s^{\mu\nu} = [\gamma_s^\mu, \gamma_s^\nu]$, hold and from this flows: $i\sigma_s^{\mu\nu} D_\mu D_\nu = (i/2)\sigma_s^{\mu\nu}[D_\mu, D_\nu] = (e/2)\sigma_s^{\mu\nu} F_{\mu\nu}^{ex}$, where: $F_{\mu\nu}^{ex}$, is the electromagnetic field tensor of the applied external field. The above calculations reduce to:

$$\left(\eta^s_{\mu\nu}D^\mu D^\nu - \frac{e}{2}\sigma^{\mu\nu}_s F^{ex}_{\mu\nu} + \frac{m_0^2c^2}{\hbar^2}\right)\psi = 0. \tag{28}$$

We have: $\eta^s_{\mu\nu}D^\mu D^\nu = s^2 D_i D^i + s^2 D_0 D^0$ and in the non-relativistic limit: $s^2 D_i D^i = s^2\vec{\nabla}^2 - es\vec{\mathbf{B}}\cdot\vec{\mathbf{L}}(s) + s^2 O(A^2_{ex,i})$ and $s^2 D_0 D^0 \simeq s^2\partial_0\partial^0$. Now as in §(2), if we write the Dirac four component wave-function as $\psi = \begin{pmatrix}\Phi & \chi\end{pmatrix}^T$, one finds that in the non-relativistic limit the component $\chi$ dominates. Thus: $e\sigma^{\mu\nu}_s F^{ex}_{\mu\nu}/2$, acting on: $\Phi$, is effectively equals: $e\sigma^3_s(F^{ex}_{12} - F^{ex}_{21})/2 = 2e\vec{\mathbf{B}}\cdot\vec{\mathbf{S}}(s)$, since: $\vec{\mathbf{S}}(s) = (s\vec{\boldsymbol{\sigma}}/2)$.

Now writing: $\Phi = e^{-im_0t}\Psi$, where: $\Psi$ oscillates much more slowly than: $e^{im_0t}$, so that: $s^2(\partial_0^2 + m_0^2c^2/\hbar^2)e^{-im_0c^2t/\hbar}\Psi \simeq s^2 e^{-im_0c^2t/\hbar}\left[-(2im_0c/\hbar)\partial_0\Psi\right]$. Putting all the bits and pieces together, one is lead to:

$$\left(\frac{\hbar^2}{2m_0}\vec{\nabla}^2 + s^{-1}\mu_B\vec{\mathbf{B}}\cdot\left[\vec{\mathbf{L}}(s) + 2\vec{\mathbf{S}}(s)\right]\right)\Psi = -i\hbar\frac{\partial\Psi}{\partial t}, \tag{29}$$

Less the factor $s^{-1}$, this equation is exactly the same as the original (12) because the spin factor $s$ cancels out since $\vec{\mathbf{L}}(s) = s\vec{\mathbf{L}}$ and $\vec{\mathbf{S}}(s) = s\vec{\mathbf{S}}$. What this effectively means is that, an Electron of a higher or lower spin will – *contrary to what one would expect*; have the same gyromagnetic ratio $g = 2$. While this is contrary to what one expects, true is the fact that no Electron in a higher or lower spin state has ever been observed – expectations do not translate to reality.

# 7 New Energy-Momentum Equation

The modification made to the gamma-matrices results in the energy equation being modified, so that the new energy equation for a spin-$s/2$ particle is given by:

$$E^2 = s^2p^2c^2 + m_0^2c^4. \tag{30}$$

Having modified the energy-momentum dispersion relation in this manner, the first and most natural question that pops-up is – relative to reality, what does this new energy-momentum equation mean? Does it mean one can change the spin of an Electron by accelerating it to higher energies? If this is possible, it would be catastrophic for matter because for ordinary matter, it would mean Electrons, Neutrons and Protons would become Bosons and we all know what would happen – matter would collapse as the Pauli exclusion principle that is responsible for the stability of matter would no longer hold since the fermions have made a transition to Bosons. Fortunately, this equation does not imply this.

What we should realize is that, yes an Electron in an appropriate energy state will, according to (30) transmute to a higher spin-state; the energy to elevate the Electron to higher spin-states does not come from the Electron's *4*-momentum, but from the vacuum itself. One will have to excite the vacuum to its next or higher energy state. From the metric given in (27), it must be clear that $s$ is not a property of the particle but a property of the vacuum. Thus, an Electron in a vacuum that is in the spin-state $s = 1$ will never enter into a higher spin-state no matter the energy applied to it. To change the spin of the particle, one will have to excite the *vacuum* to a different energy state.

Now given that $E = \gamma m_0 c^2$ where $\gamma$ is the relativistic factor and the definition $m_0(s) = m_0/s$, we can write $E(s) = \gamma m_0(s) c^2 = E/s^2$, given all this, and then dividing equation (30) by $s^2$, one is lead to:

$$E^2(s) = p^2c^2 + m_0^2(s)c^4. \tag{31}$$

Thus this theory advanced here is exactly the same as the STR with the exception that the rest mass has been rescaled by the spin factor $s$. The aforesaid means this theory does not violate the first or the second postulates of the STR, hence thus it is fully consistent with the STR. A clear suggestion emanating from this is that an Electron or any particle for that matter, will have its rest mass given by the relation $m_0(s) = m_0/s$.

# 8 Discussion

From the vantage point of unity, simplicity and beauty, it is logical and most natural for one to wonder why there should be different equations to describe particles of different spins?! For example, the Klein-Gordon equation describes spin-0 particles, while the Dirac Equation describes spin-1/2, and the Rarita-Schwinger Equation (Rarita & Schwinger 1947) describes spin-3/2. Does it mean we have to look for another equation to describe spin-2 particles, and then spin-5/2 particles *etc*? This does not look beautiful, simple, or at the very least suggest a *Unification of the Natural Laws*. Beauty of a theory is not a physical principle but, one thing is clear to the searching mind – *i.e.*, a theory that possesses beauty, appeals to the mind; that it is most bound to have something to do with physical reality if it naturally submits itself to the test of experience – *e.g.* Newtonian Mechanics, Maxwell's Electromagnetic Theory, Einstein's STR, the Dirac Equation *etc* – the list is long. Equipped with simple principles, the champions of these theories (Newton, Maxwell, Einstein, Dirac, *etc*) beautifully explained the workings of the Universe in an elegant fashion.

One of the reasons why this modification needs to be considered is the congruent predictions with Supersymmetry Theory (see *e.g.* Weinburg 1999; Cooper *et al.* 1995; Wess & Bagger 1992) namely that, under an appropriate excitation of a vacuum, a fermion can transmute to a Boson and *vise-versa*. The current efforts of unifying all the forces of nature includes Supersymmetry. Supersymmetry is a theory of a perfect sort of symmetry that relates Fermions to Bosons, where each Fermion has a super-symmetric partner whose spin differs by half a unit of spin and *vise-versa*.

At present, according to our current understanding, Bosons and Fermions are distinct particles – just as electromagnetism and gravitation seem to be distinct forces existing with no clearly evident relationship to one another. Supersymmetry is the only theory known that forges such an intimate relationship between Fermions and Bosons. Given the present, that, if one takes the vacuum from state $s = 1 \longmapsto 2 \longmapsto \ldots$ *etc*, Fermions and Bosons will switch from Fermion to Boson and *vise-versa*, it means our theory shares a common ground with Supersymmetry. If supersymmetric particles are found, Supersymmetry may claim their existence proves this theory right but now we have the present to also consider.

In closing, allow me to say; without any justification we have chosen $s$ to take integral values. Since $s$ is associated with spin, clearly, one can derive from QM that $s$ takes integral values.

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# References

[1] Cooper F., Khare A. & Sukhatme U., 1995, *Supersymmetry in Quantum Mechanics*, *Phys. Rep.*, Vol. **251**, pp. 267-85: arXiv:hep-th/9405029.

[2] Dirac P. A. M, 1928a, *The Quantum Theory of the Electron*, *Proc. R. Soc.* (London) A, Vol. **117**, pp. 610-612.

[3] Dirac P. A. M, 1928b, *The Quantum Theory of the Electron Part II*, *Proc. R. Soc.* (London) A, Vol. **118**, pp. 351-361.

[4] Nyambuya G. G., 2007, *On a Unified Field Theory – Paper I - Gravitational, Electromagnetic, Weak & the Strong Force*, *Apeiron Journal*, Vol. **14**, Issue No 4., pp. 115-161. Visit: http://redshift.vif.com.

[5] Nyambuya G. G., 2008a, *New Curved Spacetime Dirac Equations*, *Foundations of Physics*, Vol. **38** No. 7, pp. 665-677. *preprint* arXiv:0709.0936.

[6] Nyambuya G. G., 2008b, *Curved Spacetime Dirac Equation Paper II,*: preprint arXiv:0711.4232.

[7] Rarita W. & Schwinger J., 1941, *On a Theory of Particles with Half-Integral Spin*, *Phys. Rev.* **60**, pp. 61.

[8] Weinberg S., 1999, *The Quantum Theory of Fields*, Vol. **3**: Supersymmetry, *Cambridge Uni. Press*: ISBN 0-521-66000-9.

[9] Wess J. & Bagger J., 1992, *Supersymmetry & Supergravity*, *Princeton Uni. Press*: ISBN 0-691-02530-4.

[10] Zee A., 2003, *Quantum Field Theory in a Nutshell*, *Princeton Univ. Press,* pp. 177178.