Towards the Unity of Classical Physics

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The unity of classical mechanics and electromagnetism is proposed to be established through putting both on equal footing. The special-relativistic equations of motion for the particles and fields, the Maxwell-Lorentz force, and the Yukawa potential are derived exploiting Newton's and Euler's (stationary-)state descriptions, Newton's decomposition of forces into body- and position-dependent factors, and Helmholtz's analysis of the relationships between forces and energies. For instance, the magnetic Maxwell-Lorentz force is a special case of the Lipschitz force being a general class of forces that leave the kinetic energy constant. Adding this force to Newton's force of gravity leads to self-standing fields representing the mediating agent of interaction sought by Newton. Thus, equal footing is realized through a foundation on common principles, but not through a reduction to mechanical models.

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Introduction

While classical mechanics and classical electromagnetism are often called 'the two pillars of classical physics', quantum mechanics and quantum electrodynamics are *not* called 'the two pillars of quantum physics'. The reason for this is, perhaps, that the former two are treated on rather *different* axiomatic footings, *viz*, Newton's Laws and Maxwell's equations, respectively. For the sake of the unity of (classical) physics, both, mechanics and electromagnetism, should be put on *equal* footing, however. This has been undertaken, *eg*, by Bopp (1962). The spirit of Bopp's principles of electromagnetism is relatively close to the principles of mechanics as laid down by Newton (*Principia*, 1687/1999), Euler (*Anleitung zur Naturlehre*, ca. 1750) and Helmholtz (1847, 1911). As a matter of fact, these authors have formulated the foundations of classical mechanics in such a general manner, that, *cum grano salis*, they apply *well beyond* their original scope, *eg*, to fluid mechanics (Euler, *Ibid.*), special theory of relativity (Suisky & Enders 2005), statistical mechanics (Enders 2004, 2007, 2008a), and even to quantum mechanics (Enders & Suisky 2005, Enders 2006). Due to this *conceptional* power, the most important ones of Bopp's principles can be traced to purely mechanical reasons (Enders 2008b).

In this contribution, I propose *a common foundation of point mechanics and of the microscopic Maxwell equations* exploiting not only the *Laws*, but also the *Definitions* in Newton's *Principia*, because the *Definitions* already contain – in a certain sense – the concepts of charge and field. Following Helmholtz, I will add to the force field considered by Newton a force field which leaves the kinetic energy unchanged. Both together constitute the self-standing agent of interaction Newton sought.

States and forces in Newton's *Principia*

For a comprehensive survey of Newton's mechanics, one had to start with his unpublished fragment *De gravitatione...* (Newton 1988/2007), where he discusses the occupation of space by bodies and related questions in more detail than in the *Principia* which was actually designed as *celestial* mechanics. For the foundation of Newton's force law and of the microscopic Maxwell equations, the *Principia* is sufficient, however, because those questions have been completed in Euler's work that I will exploit as well. The definitions and laws are quoted from the new translation by Cohen et al. (Newton 1999).

'Charges' and 'fields' in the *Definitions*

Definition 2 *Quantity of motion is a measure of motion that arises from the velocity and the quantity of matter jointly.*

This corresponds to Cartesian momentum (*cf* Newton 1999, p.95f.). In the following quotations, "motion" often means "quantity of motion".

Definition 6 *The absolute quantity of centripetal force is the measure of this force that is greater or less in proportion to the efficacy of the cause propagating it from a center through the surrounding regions.*
An example is the magnetic force, which is greater in one loadstone and less in another, in proportion to the bulk or potency of the loadstone.

The "absolute quantity" of a force thus means its source strength, such as gravitating mass, charge, pole strength, etc., while the dependence of interaction on distance will be embraced in the two following notions.

Definition 7 *The accelerative quantity of centripetal force is the measure of this force that is proportional to the velocity which it generates in a given time.*
One example is the potency of a loadstone, which, for a given loadstone, is greater at a smaller distance and less at a greater distance. Another example is gravity, which is greater in valleys and less on the peaks of high mountains ..., but which is everywhere the same at equal distances, because it equally accelerates all falling bodies (heavy or light, great or small), provided that the resistance of the air is removed.

Despite of the fact, that the potency of a loadstone refers to the location of the distant body upon which the loadstone acts, while in Definition 6, the potency is an intrinsic property of a loadstone, I will follow Newton and factorize the force between two bodies in body-specific and geometric components. The body-specific component is intrinsic and fixed for a given body, such as its charge and rest mass. The geometric component will be described by fields, which – within classical physics – represent the mediating agent assumed by Newton.

Definition 8 *The motive quantity of centripetal force is the measure of this force that is proportional to the motion which it generates in a given time.*

This is the notion of force occuring in Law 2 below.

Conservation and change of state in*Axioms, or the Laws of Motion*

Law 1 *Every body perseveres in its state of being at rest or moving uniformly straight forward, except insofar as it is compelled to change its state by forced impressed.*

Note that Newton's and Euler's notions of state deviates significantly from the Laplacian notion of state used nowadays, which changes even in straight uniform motion and needs the permanent action of inertia to do so (*cf* Weizsaecker 2002/2006). This difference has far reaching consequences for statistical as well as for quantum mechanics (Enders 2004, 2006, 2007, 2008a). Basing on this experience I will pay special attention to the notion of state in this paper, too. Actually, those notions correspond largely to the nowadays notion of stationary state; for this, I will call them 'stationary state', too (except in quotations).

Thus, Newton starts his axiomatics with the law of stationary-state conservation. In contrast to Definition 3, no force is evoked to maintain the current stationary state. The force of inertia occours only *during* the action of an external ("impressed") force and – like the impressed forces of Definition 4 – vanishes as soon as that action ceases.

Law 1 is most general, for the (Newtonian) state can also be described by other conserved quantities, such as kinetic or total energy. Consequently, *cum salo granis*, it applies even to quantum physics (Bohr 1913).

After the conservation of stationary states, Newton postulates the manner of change of stationary states.

Law 2 *A change in motion is proportional to the motive force impressed and takes place along the straight line in which that force is impressed.*

Together with Laws 1 and 3, Law 2 defines, (i), the momentum vector, $\vec{p}(t)$, as stationary-state variable and, (ii), motion as motion along trajectories. Correspondingly, the equations of stationary-state conservation and change read

$$\vec{p}(t) = const \tag{1}$$

and

$$d\vec{p}(t) \sim \vec{K}_{ext} dt \tag{2}$$

respectively.

Law 2 makes it difficult to generalize Newton's mechanics to other forms of motion, notably to that of quantum systems (*cf* Bohr 1913, Heisenberg 1925), because it fixes *axiomatically* the manner of change of stationary states.

Law 3 *To any action there is always an opposite and equal reaction; in other words, the actions of two bodies upon each other are always equal and always opposite in direction.*

I will exploit this law in the more general form of Hertz's (1910) interaction principle: If system A acts on system B, then, system B acts on system A, too. Again, it is favourable *not* to fix *axiomatically* how this principle is realized quantitatively.

States and forces after Euler

Euler's work on the foundations of mechanics has been largely forgotten, may be, because his central text *Anleitung zur Naturlehre* (ca. 1750) has been published as late as in 1862 only. Nevertheless, it has been praised by its unsurpassed stringency (Ueberweg 1924). In particular, Euler (1750a,b, 1768) has developed a unified concept of bodies and forces. There is only *one* type of bodies and only *one* type of forces. The conservation of stationary states is not due to a force, but due to the very nature of the bodies. The latter is given through their general properties: extension, movability, inertia and impenetrability (*cf* Newton, *De gravitatione...*). The impenetrability is the fundamental, "*essential*" property, from which the other three properties can even be derived.

The forces appear only due to competition for space occupation. Hence, the interaction between bodies is primarily that of elastic collisions, where forces are created in just that amount which is necessary to prevent the penetration of one body by another. As a consequence, the magnitude of the force (and thus of the action it performs) is *minimum*.

It is remarkable, that the elastic collision represents *the only genuine* classical-mechanical interaction, for it exhibits *no interaction constant*. This may be seen as another reason for the fact, that classical mechanics became that methodological proto-physics, on which all other physical disciplines are built.

Euler has tried to describe the planetary motion this way, too. Since he has not succeeded, I assume that this is principally impossible. For this, I will follow Newton's pragmatic approach and accept mathematical formulae as correctly describing observations even if the physics behind them is unclear.

States and axioms

Euler's work is the only one I'm aware of, where the notion of state plays that central role, that it actually assumes in all branches of physics. Euler's axiomatics for mechanics reduces Newton's one to the minimum, in that only the conservation of stationary states is postulated, while the change of stationary states is considered to be a problem to be solved according to the concrete situation. This allows for treating *non*-classical motions *without* losing contact to classical mechanics (Enders & Suisky 2005, Enders 2006, and below).

Like Newton, Euler follows the sequence

Law of conservation of stationary states – law of change of stationary states – law of motion

The existence of stationary states is postulated in the following axioms.

Axiom E0 *Every body is* EITHER *resting* OR *moving.*

This means, that the subsequent axioms E1 and E2 are not independent each of another; they exclude each another and, at once, they are in *harmony* each with another (Euler 1751).

Axiom E1 *Every body perseveres in its [stationary] state of being at rest, unless an external cause sets it in motion.*

Axiom E2 *Every body perseveres in its [stationary] state of straight uniform motion, unless an external cause forces it to change this state.*

The stationary-state variable is the velocity vector, $\vec{v}$ (the mass of a given body is always constant). Thus, the equations of stationary state read $\vec{v} = \vec{0}$ for the stationary state at rest, and $\vec{v} = const$ for the stationary state of straight uniform motion.

The equation of stationary-state change is *not postulated*, but *deduced* from the general properties of bodies as

$$d\vec{v}(t) = \frac{1}{m}\vec{K}_{ext}dt \tag{3}$$

(for more details, see Enders & Suisky 2005, Enders 2006).

Efficacy and energy conservation

Multiplying eq.(3) with $\vec{v}$ yields the equation of change of stationary states along the path interval, $d\vec{r}$, as

$$d\frac{1}{2}v^2 = \frac{1}{m}\vec{K}_{ext} \cdot d\vec{r} \tag{4}$$

where $\frac{1}{2}\vec{v}^2$ is another stationary-state variable. Here, Euler (1750b, § 75) has observed that the integral

$$\int_{\vec{r}_1}^{\vec{r}_2} \vec{K}_{ext} \cdot d\vec{r} \tag{5}$$

is independent of the path between $\vec{r}_1$ and $\vec{r}_2$, if the spatial distribution of the force field, $\vec{K}_{ext}(\vec{r})$, exhibits certain properties, *ie*, in modern notation, if it is a gradient field.

$$\vec{K}_{ext}(\vec{r}) = \nabla W(\vec{r}) \tag{6}$$

This distinguishes it from the integral $\int_{t_1}^{t_2} \vec{K}_{ext} dt$, and, thus, makes it to deserve an own name, "efficacy" ("Wirksamkeit").

In such cases, the change of efficacy equals the change of kinetic energy,

$$W(\vec{r}(t_2)) - W(\vec{r}(t_1)) = \frac{m}{2}v(t_2)^2 - \frac{m}{2}v(t_1)^2 \tag{7}$$

and there is a new constant of integration.

$$E = \frac{m}{2}v(t)^2 - W(\vec{r}(t)) = const \tag{8}$$

At Euler's time, however, there were, (i), a confusion of notions and, (ii), not enough evidence for the physical significance of E. The function $W(\vec{r})$ has been exploited by Gauss, Jacobi and other as "potential". However, the 'potential energy', $V = -W$, is superior to W as it directly represents the "available work storage" of a system (Helmholtz, 1911, §49). From this point of view, for any given system, the absolute minimum value of $V(\vec{r})$ equals zero.

Two generalizations of the Newton-Eulerian state descriptions

As an intermediate step, the power of Newton's and Euler's representations of classical mechanics is demonstrated by means of few simple examples.

The special-relativistic equation of stationary-state change

Euler makes – like Newton – implicitly the assumption that the amount of change of the stationary-state variable ($d\vec{v}$) is independent of its current value ($\vec{v}$). If this assumption is lifted, one arrives at the *special-relativistic* equation of stationary-state change (Suisky & Enders 2005),

$$d\frac{\vec{v}}{\sqrt{1 - v^2/v_{ref}^2}} = \frac{1}{m}\vec{K}_{ext}dt \tag{9}$$

where the reference velocity, v_{ref}, has been introduced for dimensional reasons (see also Milton & Schwinger 2006 for *both* the theoretical and experimental aspects of this change against Newton's formula).

This modification is necessary, if $\vec{K}_{ext}$ depends explicitly on $\vec{v}$ *as stationary-state variable*, as in the case of the magnetic Maxwell-Lorentz force, $q\vec{v} \times \vec{B}$. Here, at once, it makes the transformation properties of the mechnical equation of motion compatible with that of the field equations derived below (whence v_{ref} becomes the speed of light in vacuo, c_0). – For the derivation of the field equations, however, one can treat the bodies non-relativistically, *ie*, in the limit $v \ll v_{ref}$.

It should be noted that eq.(9) has been obtained for single bodies subject to external fields. It is thus not automatically justified to apply it to a system of interacting bodies (*cf* Dirac 1949, Stefanovich 2005/2007).

The Hamilton function as Newton-Eulerian stationary-state function

Definition *A Newton-Eulerian stationary-state function of a mechanical body or system is a function of the dynamic variables of the body or system,*
(i), which is time-independent as long as the body or system is free of external influences,
(ii), the change of which depends only on the external influences, not on its own current value.

For a body subject to an external force field, this function be $H(\vec{p}, \vec{r}, t)$. Then,

- if there are no further influences, $H(\vec{p}, \vec{r}, t) = H_0(\vec{p}, \vec{r}) = const$;
- its change depends only on the 'power of the external causes': $dH = (\partial H_{ext}/\partial t)dt$, where

$$H(\vec{p}, \vec{r}, t) = H_0(\vec{p}, \vec{r}) + H_{ext}(\vec{p}, \vec{r}, t) \tag{10}$$

All time-independent terms belong to H_0 and constitute the very system.

This means, that

$$dH = \frac{\partial H}{\partial \vec{p}} \cdot \frac{d\vec{p}}{dt} + \frac{\partial H}{\partial \vec{r}} \cdot \frac{d\vec{r}}{dt} + \frac{\partial H_{ext}}{\partial t}dt \stackrel{!}{=} \frac{\partial H_{ext}}{\partial t}dt \tag{11}$$

Hence,

$$\frac{\partial H}{\partial \vec{p}} : \frac{\partial H}{\partial \vec{r}} = -\frac{d\vec{r}}{dt} : \frac{d\vec{p}}{dt} \tag{12}$$

Comparing this with the Newtonian equation of motion, one obtains Hamilton's equations of motion, $H(\vec{p}, \vec{r}, t)$ being the Hamilton function (Enders & Suisky 2005, Enders 2006).

$$\frac{d\vec{r}}{dt} = \frac{\partial H}{\partial \vec{p}}; \quad \frac{d\vec{p}}{dt} = -\frac{\partial H}{\partial \vec{r}} \tag{13}$$

In the most simple case (8),

$$H = \frac{1}{2m} p^2 + V(\vec{r}) \tag{14}$$

equals the total energy, E.

It is noteworthy that in such cases the minimum value of total energy is fixed by the energy law (Helmholtz 1911), while that of the Lagrange function is not. For the absence of *perpetua mobile* implies the existence of a ground state for each closed system (*cf* Helmholtz 1847). A system can deliver energy to its environment only until it reaches the ground state, and only that amount is physically relevant.

On the force of inertia

In view of the ongoing discussion about the correct expression of the force of inertia in electromagnetism (see, *eg*, Coleman & van Vleck 1968, Stefanovich 2008, and references herein), I would like to show that the Newton-Eulerian state description can help to solve this issue.

Eqs.(9) and (13) suggest that Law 2 may be generalized such that the external force, $\vec{K}$, does primarily change not the kinetical momentum (2), but the canonical momentum.

$$\frac{d\vec{p}}{dt} = -\frac{\partial H}{\partial \vec{r}} \stackrel{?}{=} \vec{K} \tag{15}$$

This is not appropriate, however, as will be shown in what follows.

In order to come from the two first-order equations (13) to one second-order equation of motion like the Newtonian one, I make the Legendre transformation

$$\begin{aligned} L(\vec{v},\vec{r},t) &= \vec{v}\cdot\vec{p}(\vec{v},\vec{r},t) - H(\vec{p}(\vec{v},\vec{r},t),\vec{r},t) && (16)\\ \vec{v} &= \frac{\partial}{\partial\vec{p}}H(\vec{p}(\vec{v},\vec{r},t),\vec{r},t) = \frac{d\vec{r}}{dt} && (17)\\ \vec{p} &= \frac{\partial}{\partial\vec{v}}L(\vec{v}(\vec{p},\vec{r},t),\vec{r},t) && (18) \end{aligned}$$

This leads to Lagrange's equation of motion.

$$\frac{d}{dt}\frac{\partial L}{\partial\vec{v}} = \frac{\partial L}{\partial\vec{r}} \tag{19}$$

For the most simple Hamilton function (14),

$$L = \frac{m}{2}v^2 - V(\vec{r}) \tag{20}$$

Suppose now that the external force field depends also on time and on the velocity of the body under consideration: $\vec{K} = \vec{K}(\vec{v},\vec{r},t)$. One expects the representative, V, of $\vec{K}$ in L to depend on these variables, too: $V = V(\vec{v},\vec{r},t)$. Then, Lagrange's equation of motion (19) yields

$$m\frac{d\vec{v}}{dt} - \frac{d}{dt}\frac{\partial V}{\partial\vec{v}} = -\frac{\partial V}{\partial\vec{r}} \tag{21}$$

Therefore, the force changing primarily the velocity is no longer a gradient field, but equals

$$\vec{K}(\vec{v},\vec{r},t) = m\frac{d\vec{v}}{dt} = -\frac{\partial V}{\partial\vec{r}} + \frac{d}{dt}\frac{\partial V}{\partial\vec{v}} \tag{22}$$

The corresponding potential, $V(\vec{v},\vec{r},t)$, has been termed by Helmholtz "kinetic potential" (*cf* Helmholtz 1911, §76, Sommerfeld 2001, § 32B).

The canonical momentum now contains a potential part.

$$\vec{p} = \frac{\partial L}{\partial\vec{v}} = m\vec{v} - \frac{\partial V}{\partial\vec{v}} \tag{23}$$

As a consequence,

$$\frac{d\vec{p}}{dt} = \frac{d}{dt}\frac{\partial L}{\partial\vec{v}} = \frac{\partial L}{\partial\vec{r}} = -\frac{\partial V}{\partial\vec{r}} \neq \vec{K}(\vec{v},\vec{r},t) \tag{24}$$

of course. The question is, which expression is appropriate to describe external influences: the one which changes the canonical momentum, $\vec{p}$, or the one which changes the kinetic momentum, $m\vec{v}$, or another one, depending on the situation considered?

For a charged body in a static magnetic field, we have $v = const$, but $p \neq const$. At least here, the canonical momentum is *not* a Newton-Eulerian state variable, and the equations of stationary-state change contain dv, but not dp.

This is another example for the superiority of Euler's methodology, that the *change* of stationary states should be kept *outside* the axiomatics.

Relationships between forces and energies after Helmholtz

Helmholtz asked,

1. which forces constitute – together with the bodies they act upon – a conservative system?
2. which forces leave the kinetic energy of a body unchanged?

(1) Gradient forces – conservative systems

The answer to the first question reads, central forces between the bodies (Helmholtz 1847), or velocity-independent external gradient fields (Helmholtz 1911),

$$\vec{K}_{\text{grad}}(\vec{r}, t) = -\nabla V(\vec{r}, t) \tag{25}$$

For a single body, the kinetic energy equals

$$T = \frac{m}{2}\vec{v}^2 = \frac{1}{2m}\vec{p}^2 \tag{26}$$

and the Hamilton function,

$$H_{\text{grad}}(\vec{p}, \vec{r}, t) = \frac{1}{2m}\vec{p}^2 + V(\vec{r}, t) \tag{27}$$

is time-independent, if the potential energy is so, $V = V(\vec{r})$.

Application: Derivation of Newton's and Coulomb's force laws

Consider two point-like bodies interacting in the manner as described in Newton's *Definitions* above, *ie*, the force between them depends on intrinsic factors, $q_{1,2}$, and on their positions. Then,

$$m_1 \frac{d^2\vec{r}_1}{dt^2} = -q_1 \nabla V_{12}(\vec{r}_1, t) \qquad (28)$$

$$m_2 \frac{d^2\vec{r}_2}{dt^2} = -q_2 \nabla V_{21}(\vec{r}_2, t) \qquad (29)$$

where V_{ab} is the potential at the position of body a due to body b.

Now, by virtue of Helmholtz's decomposition theorem, $\nabla V(\vec{r}, t)$ is – up to a constant – uniquely determined by its sources, $\rho(\vec{r}, t)$, *ie*,

$$\nabla \cdot \nabla V(\vec{r}, t) = -\kappa\rho(\vec{r}, t) \qquad (30)$$

κ being a constant related to the units of measurement (a justification will be given below). Therefore,

$$\Delta V_{12}(\vec{r}_1, t) = -\kappa q_2 \delta(\vec{r}_1 - \vec{r}_2(t)) \qquad (31)$$

$$\Delta V_{21}(\vec{r}_2, t) = -\kappa q_1 \delta(\vec{r}_2 - \vec{r}_1(t)) \qquad (32)$$

This leads to the Newton-Coulombian force law

$$\vec{K}_{12} = -\vec{K}_{21} = \frac{\kappa}{4\pi} q_1 q_2 \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|^3} \qquad (33)$$

More generally, one can add a term μV to the Poisson equation (30), what leads to the Yukawa potential, see below.

(2) Lipschitz forces – static magnetic fields

The answer to Helmholtz's second question has been given by Lipschitz (1881) through the expression

$$\vec{K}_{\mathrm{Lip}}(t, \vec{r}, \vec{v}, \vec{a}, \ldots) = \vec{v} \times \vec{K}'(t, \vec{r}, \vec{v}, \vec{a}, \ldots) \qquad (34)$$

where $\vec{K}'(t, \vec{r}, \vec{v}, \vec{a}, \ldots)$ is a rather arbitrary function. Due to $\vec{K}_{\mathrm{Lip}} \cdot \vec{v} \equiv 0$, this 'Lipschitz force' deflects a body without changing its kinetic energy, and this *in*dependently of its current trajectory. The best known example for such a force is the magnetic Maxwell-Lorentz force.

Here, I stick to Lipschitz's original expression, in order to demonstrate the power of Helmholtz's approach. In a more independent treatment, one can exploit the existence of v_{ref} in eq.(9) for making the ansatz $\vec{K}_{\mathrm{Lip}} = \frac{\vec{v}}{v_{ref}} \times \vec{K}'$

(Heaviside units) expressing the relativistic nature of this force from the very beginning.

What do the Hamilton and Lagrange functions look like for a body subject to the Lipschitz force?

Since by definition the kinetic energy is unchanged, the Lagrange function reads

$$L_{\mathrm{Lip}}(\vec{v},\vec{r},t)=\frac{m}{2}\vec{v}^2-U_{\mathrm{Lip}}(\vec{v},\vec{r},t) \tag{35}$$

where U_{Lip} is the corresponding kinetic potential, *ie*,

$$\vec{K}_{\mathrm{Lip}}=\frac{d}{dt}\frac{\partial U_{\mathrm{Lip}}}{\partial\vec{v}}-\frac{\partial U_{\mathrm{Lip}}}{\partial\vec{r}} \tag{36}$$

Expanding $U_{\mathrm{Lip}}(\vec{v},\vec{r},t)$ in powers of $\vec{v}$,

$$U_{\mathrm{Lip}}(\vec{v},\vec{r},t)=U^{(0)}_{\mathrm{Lip}}(\vec{r},t)+\vec{U}^{(1)}_{\mathrm{Lip}}(\vec{r},t)\cdot\vec{v}+\frac{1}{2}\vec{v}\cdot\hat{U}^{(2)}_{\mathrm{Lip}}(\vec{r},t)\cdot\vec{v}+0(\vec{v}^3) \tag{37}$$

one obtains

$$\begin{aligned}\vec{K}_{\mathrm{Lip}}&=\vec{v}\times\vec{K}'(t,\vec{r},\vec{v},\vec{a},\ldots)\\&=-\nabla U^{(0)}_{\mathrm{Lip}}-\vec{v}\times\nabla\times\vec{U}^{(1)}_{\mathrm{Lip}}+\frac{\partial}{\partial t}\vec{U}^{(1)}_{\mathrm{Lip}}+\\&\qquad+\frac{d\vec{v}}{dt}\cdot\hat{U}^{(2)}_{\mathrm{Lip}}(\vec{r},t)+\vec{v}\cdot\frac{\partial}{\partial t}\hat{U}^{(2)}_{\mathrm{Lip}}+0(\vec{v}^2)\end{aligned} \tag{38}$$

This implies

$$\nabla U^{(0)}_{\mathrm{Lip}} = \frac{\partial}{\partial t}\vec{U}^{(1)}_{\mathrm{Lip}} \tag{39}$$

$$\hat{U}^{(2,3,\ldots)}_{\mathrm{Lip}} = \hat{0} \tag{40}$$

$$\vec{K}' = -\nabla\times\vec{U}^{(1)}_{\mathrm{Lip}} \tag{41}$$

Therefore, $\partial\vec{K}'/\partial t=\vec{0}$, *ie*, within Lagrange-Hamiltonian mechanics, $\vec{K}'$ is a static solenoidal field.

$$\vec{K}'(t,\vec{r},\vec{v},\vec{a},\ldots)=-\nabla\times\vec{U}^{(1)}_{\mathrm{Lip}}(\vec{r}) \tag{42}$$

This is compatible with the experimental fact, that a magnetic field without electrical field is a *static* one.

Extension of Helmholtz's analysis: Derivation of the microscopic Maxwell equations

Lipschitz plus $\vec{v}$-independent forces

In order to free us from this constraint, another, velocity-independent force, $\vec{K}_{\rm el}(\vec{r},t)$, be admitted, which allows for coping with $U^{(0)}_{\rm Lip}$, *ie*,

$$\vec{K}_{\rm el}(\vec{r},t)+\vec{K}_{\rm Lip}(t,\vec{r},\vec{v},\vec{a},\ldots)=\frac{d}{dt}\frac{\partial U_{\rm ML}(\vec{v},\vec{r},t)}{\partial\vec{v}}-\frac{\partial U_{\rm ML}(\vec{v},\vec{r},t)}{\partial\vec{r}} \tag{43}$$

where $U_{\rm ML}$ is the kinetic potential of this general Maxwell-Lorentz force. Expanding $U_{\rm ML}(\vec{v},\vec{r},t)$ in powers of $\vec{v}$ (again),

$$U_{\rm ML}(\vec{v},\vec{r},t)=U^{(0)}_{\rm ML}(\vec{r},t)+\vec{U}^{(1)}_{\rm ML}(\vec{r},t)\cdot\vec{v}+\frac{1}{2}\vec{v}\cdot\hat{U}^{(2)}_{\rm ML}(\vec{r},t)\cdot\vec{v}+0(\vec{v}^3) \tag{44}$$

one obtains

$$\begin{aligned}\vec{K}_{\rm el}(\vec{r},t)+\vec{v}\times\vec{K}'(t,\vec{r},\vec{v},\vec{a},\ldots)&\\ =-\nabla U^{(0)}_{\rm ML}-\vec{v}\times\nabla\times\vec{U}^{(1)}_{\rm ML}+\frac{\partial}{\partial t}\vec{U}^{(1)}_{\rm ML}+&\\ +\frac{d\vec{v}}{dt}\cdot\hat{U}^{(2)}_{\rm ML}(\vec{r},t)+\vec{v}\cdot\frac{\partial}{\partial t}\hat{U}^{(2)}_{\rm ML}+0(\vec{v}^2)&\end{aligned} \tag{45}$$

Obviously, $\hat{U}^{(2,3,\ldots)}_{\rm ML}=\hat{0}$ and

$$\vec{K}_{\rm el}(\vec{r},t) = -\nabla U^{(0)}_{\rm ML}+\frac{\partial}{\partial t}\vec{U}^{(1)}_{\rm ML} \tag{46}$$

$$\vec{K}'(t,\vec{r},\vec{v},\vec{a},\ldots) = \vec{K}'(\vec{r},t)=-\nabla\times\vec{U}^{(1)}_{\rm ML}(\vec{r},t) \tag{47}$$

The kinetic potential, $U_{\rm ML}$, is – despite of the charge – isomorphic with Schwarzschild's "electro-kinetic potential" (Schwarzschild 1903, Sommerfeld 2001, §32E), and the Lagrange function assumes the minimal-coupling form.

$$L(\vec{v},\vec{r},t)=\frac{m}{2}\vec{v}^2-\vec{v}\cdot\vec{U}^{(1)}_{\rm ML}(\vec{r},t)-U^{(0)}_{\rm ML}(\vec{r},t) \tag{48}$$

Here, the 6 force components ($\vec{K}_{\rm el}$, $\vec{K}_{\rm Lip}$) are expressed through only 4 potential components. As a consequence, there are two compatibility conditions.

$$\nabla\vec{K}'(t,\vec{r}) = 0 \tag{49}$$

$$\frac{\partial}{\partial t}\vec{K}'(t,\vec{r}) = -\nabla\times\vec{K}_{\rm el}(\vec{r},t) \tag{50}$$

Moreover, the potentials are not unique, for the gauge transformation

$$\vec{U}^{(1)}_{\rm ML} = \vec{U}^{(1)\prime}_{\rm ML} + \nabla\chi; \quad U^{(0)}_{\rm ML} = U^{(0)\prime}_{\rm ML} + \frac{\partial\chi}{\partial t} \tag{51}$$

leaves the forces unchanged. We note these well-known facts only to demonstrate that they are not specific to electromagnetism.

The kinetic momentum, $m\vec{v}$, differs from the canonical one as

$$\vec{p} = m\vec{v} - \vec{U}^{(1)}_{\rm ML} \tag{52}$$

The kinetic energy term in the Hamilton function,

$$H(\vec{p}, \vec{r}, t) = \frac{1}{2m}\left(\vec{p} + \vec{U}^{(1)}_{\rm ML}(\vec{r}, t)\right)^2 + U^{(0)}_{\rm ML}(\vec{r}, t) \tag{53}$$

equals numerically the force-free expression $\frac{m}{2}\vec{v}^2$, in agreement with the definition of $\vec{K}_{\rm Lip}(t, \vec{r})$.

In other words, the Maxwell-Lorentz force can be traced back to and even be derived by means of purely mechanical reasoning. In the next subsection, I will account for the charge.

Interacting bodies – the homogeneous Maxwell equations

Consider now the case that the forces $\vec{K}_{\rm el}$ and $\vec{K}_{\rm Lip}$ factorize into body-specific 'charges', $q_{\rm el,Lip}$, and geometric 'fields', $\vec{F}_{\rm el,Lip}$, as described in Newton's *Definitions* above.

$$\vec{K}_{\rm el} + \vec{K}_{\rm Lip} = q_{\rm el}\vec{F}_{\rm el}(\vec{r}, t) + \vec{v} \times q_{\rm Lip}\vec{F}_{\rm Lip}(\vec{r}, t) \tag{54}$$

Then, eqs.(38) become

$$-\nabla U^{(0)}_{\rm ML} - \vec{v} \times \nabla \times \vec{U}^{(1)}_{\rm ML} + \frac{\partial}{\partial t}\vec{U}^{(1)}_{\rm ML} = q_{\rm el}\vec{F}_{\rm el}(\vec{r}, t) + \vec{v} \times q_{\rm Lip}\vec{F}_{\rm Lip}(\vec{r}, t) \ldots) \tag{55}$$

Hence, both 'charges' are equal,

$$q_{\rm el} = q_{\rm Lip} = q \tag{56}$$

and the fields are of the form of the electrical field strength, $\vec{E}$, and magnetic induction, $\vec{B}$, where $U^{(0)}_{\rm ML} = \Phi$, $\vec{U}^{(1)}_{\rm ML} = -\vec{A}$.

$$\vec{F}_{\rm el}(\vec{r}, t) = -\frac{\partial}{\partial t}\vec{A}(\vec{r}, t) - \nabla\Phi(\vec{r}, t) = \vec{E}(\vec{r}, t) \tag{57a}$$

$$\vec{F}_{\rm Lip}(\vec{r}, t) = \nabla \times \vec{A}(\vec{r}, t) = \vec{B}(\vec{r}, t) \tag{57b}$$

Eqs. (49) become the two homogeneous Maxwell equations.

$$\nabla \vec{B}(\vec{r}, t) = 0 \tag{58a}$$

$$\nabla \times \vec{E}(\vec{r}, t) = -\frac{\partial}{\partial t}\vec{B}(\vec{r}, t) \tag{58b}$$

Vector potential, $\vec{A}(\vec{r}, t)$, and scalar potential, $\Phi(\vec{r}, t)$, represent the fields $\vec{E}$ and $\vec{B}$ in the Hamilton function as

$$H(\vec{p}, \vec{r}, t) = \frac{1}{2m}\left(\vec{p} - q\vec{A}(\vec{r}, t)\right)^2 + q\Phi(\vec{r}, t) \tag{59}$$

and in the Lagrange function as

$$L(\vec{v}, \vec{r}, t) = \frac{m}{2}\vec{v}^2 + q\vec{v} \cdot \vec{A}(\vec{r}, t) - q\Phi(\vec{r}, t) \tag{60}$$

So far, $\vec{E}(\vec{r}, t)$ and $\vec{B}(\vec{r}, t)$ are given external fields. If they are not given, one also needs $\nabla\vec{E}$ and $\nabla \times \vec{B}$ to calculate them. The determination of $\nabla\vec{E}$ and $\nabla \times \vec{B}$ in what follows will, at once, account for the back reaction of the bodies upon $\vec{E}$ and $\vec{B}$ as well as answer the question how charged bodies interact with another.

TPC symmetry of the field variables

In the absence of other information, I explore the TPC symmetry properties of all the variables in the Maxwell-Lorentz force equation

$$m\frac{d^2\vec{r}}{dt^2} = \vec{K}_{\mathrm{ML}} = q\vec{E}(\vec{r}, t) + q\vec{v} \times \vec{B}(\vec{r}, t) \tag{61}$$

Variable	Relation to $\vec{r}$, t, q	T	P	C
Position	$\vec{r}$	+	-	+
Time	t	-	+	+
Velocity	$\vec{v} = d\vec{r}/dt$	-	-	+
Inertial mass	m	+	+	+
Force	$\vec{K} = m\frac{d^2\vec{r}}{dt^2}$	+	-	+
Charge	q	+	+	-
Current density	$[\vec{j}] = [q]\,[v]$	-	-	-
Electrical field strength	$[\vec{E}] = [\vec{K}]/\,[q]$	+	-	-
Scalar potential	$[\Phi] = [\vec{E}][\vec{r}]$	+	+	-
$\nabla \cdot \vec{E}$	$[\nabla\vec{E}] = [\vec{r}][\vec{E}]$	+	+	-
Magnetic induction	$[\vec{B}] = [\vec{K}]/\,[q]\,[v]$	-	+	-
Vector potential	$[\vec{A}] = [\vec{B}]/\,[\vec{r}] = [\vec{E}][t]$	-	-	-
$\nabla \times \vec{B}$	$[\nabla \times \vec{B}] = [\vec{r}]\,[\vec{B}]$	-	-	-

Table 1: Symmetry properties of the point-mechanical variables and of the field variables in eq.(61). T = Time reversal: $t \to -t$, P= Parity: $\vec{r} \to -\vec{r}$, C = Charge conjugation: $q \to -q$. Only the inertial mass exhibits $TPC = +++$.

It is also understood that the meaning of the notion *inter*action implies that, in homogeneous and isotropic space, the locations and charges of all bodies enter the formulae in a *symmetrical* manner.

Yukawa potential and Gauss' law

Now, only q and Φ exhibit the same symmetry, as $\nabla\vec{E} and \Delta\Phi$, therefore,

$$-\Delta\Phi = \kappa_1\rho + \kappa_2\Phi \tag{62}$$

$\kappa_{1,2}$ are universal constants connecting the mechanical and electromagnetic units of measurement; they do not depend on space and time, because these were assumed to be independent of matter, and also not on the bodies, because their interaction properties are – by definition – given by their charges. Their numerical values are to be determined experimentally.

The potential of the $\vec{E}$-field acting upon body a is thus determined by the charge distribution of all other bodies.

$$-\Delta\Phi_a(\vec{r},t) - \kappa_2\Phi_a(\vec{r},t) = \kappa_1\sum_{b\neq a} q_b\delta(\vec{r} - \vec{r}_b(t)); \quad \vec{r} = \vec{r}_a \tag{63}$$

Here, the δ-function is a short-hand description of the positions of the point-like bodies such that their impenetrability is not violated and that there is no self-interaction. The r.h.s. is linear in the charges, q_b, because the forces between charged bodies are by definition bilinear in their charges, *cf* eq.(61).

For two bodies, the general solution to the (inhomogeneous) Helmholtz equation (63) reads

$$\Phi_{12} = \frac{\kappa_1}{4\pi}\frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|}\left\{ \begin{array}{cc} a_+ e^{\sqrt{-\kappa_2}|\vec{r}_1-\vec{r}_2|} + a_- e^{-\sqrt{-\kappa_2}|\vec{r}_1-\vec{r}_2|}; & \kappa_2 \leq 0 \\ s\sin\sqrt{\kappa_2}\,|\vec{r}_1 - \vec{r}_2| + c\cos\sqrt{\kappa_2}\,|\vec{r}_1 - \vec{r}_2|\,; & \kappa_2 \geq 0 \end{array} \right\} \tag{64}$$

$a_+ = 0$ for obvious reasons. Spatial oscillations have never been observed, thus, $s = c = 0$. It remains the Yukawa (1935) potential

$$\Phi_{12} = \frac{\kappa_1}{4\pi}\frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|} e^{-\sqrt{-\kappa_2}|\vec{r}_1-\vec{r}_2|} \tag{65}$$

For $\kappa_2 = 0$, this becomes the Newton-Coulomb potential, *cf* eq.(33). In this case, eq.(62) becomes essentially Gauss' law for the electrical field.

Ampere-Maxwell's flux law

Further, there are three expressions complying with the symmetry '$---$' of $\nabla \times \vec{B}$, *ie*,

$$\nabla \times \vec{B} = \kappa_3 \vec{j} + \kappa_4 \frac{\partial \vec{E}}{\partial t} + \kappa_5 \vec{A} \tag{66}$$

$\kappa_{3,4,5}$ are universal constants to be determined experimentally for the same reasons as $\kappa_{1,2}$ are.

Let's combine this equation with the induction law (58b) as

$$\nabla \times \nabla \times \vec{B} = -\Delta \vec{B} = \kappa_3 \nabla \times \vec{j} - \kappa_4 \frac{\partial^2 \vec{B}}{\partial t^2} + \kappa_5 \vec{B} \tag{67}$$

For electromagnetic waves, the experimental observations indicate $\kappa_5 = 0$ and $\kappa_4 = 1/c_0^2$.

Analogously one obtains

$$\kappa_3 \frac{\partial \vec{j}}{\partial t} + \kappa_4 \frac{\partial^2 \vec{E}}{\partial t^2} + \kappa_5 \frac{\partial \vec{A}}{\partial t} = \nabla \times \frac{\partial \vec{B}}{\partial t} = -\nabla \times \nabla \times \vec{E} \tag{68}$$

ie,

$$\kappa_3 \frac{\partial \vec{j}}{\partial t} + \frac{1}{c_0^2}\frac{\partial^2 \vec{E}}{\partial t^2} = \Delta \vec{E} - \kappa_1 \nabla \rho - \kappa_2 \nabla \Phi \tag{69}$$

For electromagnetic waves, the experimental observations indicate $\kappa_2 = 0$.

Finally, since for a given body, mass and charge are constant properties, the continuity equation holds true.

$$\nabla\vec{j}(\vec{x},t) + \frac{\partial}{\partial t}\rho(\vec{x},t) = 0 \tag{70}$$

Together with eq.(62), this implies $\kappa_3 = \kappa_1\kappa_4 = \kappa_1/c_0^2$. Thus, we arrive at the two inhomogeneous microscopic Maxwell equations ($\kappa_1 = 1/\varepsilon_0$).

$$\nabla\vec{E} = \frac{\rho}{\varepsilon_0} \tag{71}$$

$$\nabla\times\vec{B} = \frac{1}{\varepsilon_0 c_0^2}\vec{j} + \frac{1}{c_0^2}\frac{\partial\vec{E}}{\partial t} \tag{72}$$

Summary and Discussion

Although the Maxwell-Lorentz force and the microscopic Maxwell's equations originated from *non*-mechanical problems, they can be traced back to and even be derived entirely through purely mechanical reasoning basing on Newton's and Euler's original representations of classical point mechanics and exploiting Helmholtz's analysis of the relationships between forces and energies. The 'minimal coupling' is the only possible step from velocity-independent forces to Lipschitz forces, if the complete sets of independent dynamic variables are not to be enlarged, so that a Newton-Eulerian state description and a Hamiltonian description of motion still exist.

It should be interesting to extend Helmholtz's explorations to other stationary - state variables and conserved quantities, respectively, for instance, the (the modulus of) the angular momentum or even the Laplace-Runge-Lenz vector. For within Schrödinger's wave mechanics, one may ask which external influences leave $<\psi|\psi>$ and $<\psi|\hat{H}|\psi>$ unchanged? (Enders 2006, 2008a). This questions leads to gauge invariance and reveals several results of this paper, too. Of course, this cannot be exploited within this purely classical treatment.

Despite of the fundamental nature of Maxwell's equations, they leave various questions open. Consequently, the discussion about the foundations of electromagnetism is not finished. There are axiomatic approaches to pure electromagnetism without potentials, for instance, the premetric approach (Hehl & Obukhov 2003). Within the Wigner-Dirac relativistic dynamics

(Dirac 1949, Newton & Wigner 1949) with the Darwin-Breit Hamiltonian (Darwin 1920, Breit 1929), the interaction between charged particles is described by potentials which do *not* represent a self-standing electromagnetic field mediating the interaction (see, *eg*, Coleman & van Vleck 1968, and Stefanovich 2008 for a recent review). The microscopic Maxwell equations can also be derived through generalizing Coulomb's law along the rules of special relativity (Field 2004, 2005). Feynman has derived the microscopic Maxwell's equations through making the commutator between position and velocity non-vanishing (Dyson 1990, Tanimura 1993). It's truly surprising how far this modification of classical mechanics reaches, but it is unknown, whether there is a deeper mechanical reason for it.

The approach presented here yields a coherent derivation of the special-relativistic dynamics of both the particles and the fields. The application to the gravito-electromagnetic equations (Mashhoon 2003) should be straightforward. Methodologically, the approach by Newton, Euler and Helmholtz has the further advantage that the subject under investigation is defined *before* the mathematical formalism is developed. This keeps the latter physically clear.

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