Fibonacci and Continued Fractions

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The Fibonacci sequence is used as a “hook” to direct interest toward generalizations.

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1. Introduction

Among simplest possible second-order difference equations is that descriptive of the famous Fibonacci sequence. Despite its rudimentary nature, this difference equation has many lessons to teach, some of which generalize to more interesting cases that cast new light on continued fractions at the definitional level … and on their generalizations.

2. The Fibonacci Difference Equation

In this section I shall focus attention on the classical Fibonacci sequence 1,1,2,3,5,8,13, …, said to date from AD 1202. The sequence can be expressed as a difference equation, $C_{n+2} - C_{n+1} - kC_n = 0$, with $C_0 = 0$, $C_1 = 1$, and with $k = 1$. ($k = -1$ can also be used, but it leads to an uninteresting repetitive sequence.) This will serve as a special case introductory to the more general cases of linear homogeneous difference equations of order higher than the first. Despite its simplicity, the special case will be found to embody all the main qualitative features that characterize the general cases. Employing the ratio $R_n = C_{n+1} / C_n$, $n = 1, 2, \cdots$, we can write the difference equation in a nonlinear form involving only two subscripts, hence lending itself to direct iteration,

$$R_{n+1} R_n - R_n - k = 0 . \tag{1}$$

There are two ways in which Eq. (1) can be expressed as an iteration, both of which give rise to formal continued-fraction-like objects having complementary convergence properties. We shall study this complementarity by addressing the two cases separately.

Case I. Isolating R_n, we can rewrite Eq. (1) as the iteration,

$$R_n = \frac{k}{-1 + R_{n+1}} , \tag{2a}$$

which expands into the "continued fraction (c.f.) with remainder,"

$$R_n = \cfrac{k}{-1 + \cfrac{k}{-1 + \ddots + \cfrac{k}{-1 + r}}} , \tag{2b}$$

where in the limit as $n \to \infty$ we shall suppose that a condition of asymptotic uniformity or stability, $R_{n+1} \to R_n \to r$, is satisfied. Note that there is no mathematically legitimate way of getting rid of some form of the remainder, a term non-vanishing "at infinity." In the same limit and under the same condition Eq. (1) goes to the quadratic

$$r^2 - r - k = 0 \tag{3}$$

(known as the "characteristic equation" of the difference equation). Suppose $k = 1$, the Fibonacci value. The two roots of (3) are $r_1 = \left(1 + \sqrt{5}\right)/2 = 1.618\cdots$, known as the "golden ratio," and $r_2 = \left(1 - \sqrt{5}\right)/2 = -0.618\cdots$. This second (negative) root gets less publicity, but is mathematically no less interesting. In magnitude it is seen to be the reciprocal of the first root; *i.e.*, $r_1 r_2 = -1$. There are simple geometrical interpretations of these roots: The first represents the ratio of the long side to the short side of the "golden rectangle," whereas the magnitude of the second root represents the ratio of the short to the long side of the same rectangle. The golden rectangle is supposed, according to a widely-honored (in the breach) view, to represent the rectangular shape most pleasing to the artistic human eye. There is no evidence that this ancient aesthetic prejudice has much to do with contemporary designs, as of movie or TV screens, photo formats, etc.

No matter how many stages (or "partial quotients") the c.f. possesses, we see with a bit of algebra, by putting $r = r_1$ in Eq. (2), that we always obtain identically an exact value of the c.f., $R_n = r_1$ for all n. But this result is deceptive. The iteration is in fact *numerically unstable*, so that if in actual calculations our starting value used for the remainder departs in even the most remote decimal place from the exact (irrational) remainder value r_1 the iteration "blows up" – its error increasing through repetition of the calculation.

You can verify this by doing the iteration with a pocket calculator. Interestingly, no matter what remainders (differing from the exact r_1) are used, including random (real) numbers, the iteration does not ultimately diverge or jump about, but eventually converges stably to the *other root*, $r = r_2$. (Try it. Start with any small departure, greater than the least-count sensitivity of your calculator, from $r = r_1$, constantly repeating the iteration of Eq. (2a), watch the progress of the "blow-up," and observe the final convergence toward r_2.) The convergence is not fast, but it is inexorable. The second root thus acts as a "strange attractor." This is poor terminology, since there is nothing particularly strange about it. It is simply a feature of the iteration used.

By contrast, the conventional definition of a continued fraction [1] forces it to be single-valued. By that definition the remainder *r* is arbitrarily *set equal to zero* (or infinity) at every stage, and hence differs from r_1. Because of this departure from r_1, owing to the above consideration, the process must converge by "strange attraction" to the dominant negative value r_2 – and this is in fact the conventional textbook "value" of the above c.f., evaluated with a sequence of zero remainders. Thus we could avoid a negative "value" for the c.f., Eq. (2), only by the employment of an "exceptional remainder sequence," $r = r_n = \{r_1, r_1, r_1, \cdots\}$, these exact $r = r_1$ remainder values being introduced at each stage of the limiting process (if any) for all *n* greater than some n_0. (Established c.f. theory [1] does not recognize the existence of exceptional remainder sequences. Hence there is no way such theory can induce convergence of Eq. (2) to the golden ratio.) Otherwise convergence is always to the dominant root, r_2. For $k = -1$ similar considerations apply, but with conjugate complex roots,

$r_{1,2} = \left(1 \pm i\sqrt{3}\right)/2$, of Eq. (3), which cause the c.f. with remainder, Eq. (2), to converge to either of the two values in the complex plane – one iterative process (convergent to the dominant root or attractor) being stable, the other unstable.

Case II. Isolating R_{n+1}, we can write Eq. (1) as the alternative iteration

$$R_{n+1} = 1 + \frac{k}{R_n}. \tag{4a}$$

This develops into the c.f. with remainder (or "terminated" c.f. – the terminology is open to negotiation … I have referred [2] to the subject as "terminal summation")

$$R_{n+1} = 1 + \cfrac{k}{1 + \cfrac{k}{1 + \ddots + \cfrac{k}{1 + k/R_1}}} \to r, \tag{4b}$$

where r is the same remainder as in Case I, because the iterations in both cases possess the *same characteristic equation*, Eq. (3), of the original difference equation, Eq. (1). The iteration (4) has convergence properties opposite from (complementary to) those of the previous iteration, Eq. (2). That is, for $k = 1$ it converges stably to the golden ratio r_1, the dominant root acting as "strange attractor," almost regardless of what (real) number is inserted for the starting value R_1; whereas it can be made to converge to the other (negative) root, r_2, only by employing precisely the initial value $R_1 = r_2$. These stability properties are again easily verified with a pocket calculator. As before, the case $k = -1$ prescribes a pair of conjugate complex "values" of the c.f., $r_{1,2} = \left(1 \pm i\sqrt{3}\right)/2$, with remainder, but with stability properties opposite (as to root dominance) from those of Case I.

The important thing to be noted here is the two-valuedness of the process. Such two-valuedness is not a recognized feature of "continued fractions" in the established literature of that specialty [1], because the subject has evolved historically under the unquestioned dominance of a universally-accepted definition that *imposes* single-valuedness through arbitrarily discarding those (in general finite) remainders whose presence is formally called for by the difference equation. Such a definition evidently impoverishes mathematics by preventing the recognition (or establishment) of a formal *equivalence* between c.f.'s and difference equations. That it inhibits the making of conceptual inter-connections among diverse mathematical specialties is in fact the best-known criterion by which a *bad definition* can be recognized in mathematics. (A definition has no way to shout its badness louder!)

How has this come about? One can only speculate that the earliest workers wished to draw a direct analogy between continued fractions and the already well-established topic of *infinite series*. The latter, being associated with a first-order difference equation, lent itself naturally to a definition that assured at most single-valuedness. But continued fractions, associated with a second-order difference equation having a quadratic (double-rooted) characteristic equation, are of an altogether different mathematical species. The direct analogy with series does not work. A Procrustean definition that imposes single-valuedness cripples the subject of continued fractions for life. That is what happened factually in history. To this day the subject, as practiced by mathematical professionals [1], remains crippled at the definitional level. I have discussed this thoroughly elsewhere [2]. There is no secret about it. The indefinite persistence of the problem could justly be seen as one of the major scandals of modern mathematics. It can in part be attributed to the faddism that views classical analysis as *passé*, hence as the pasture to which second-rate mathematical minds are put out.

3. Infinite Matrix Products

Once the 2-valuedness of c.f.'s has been recognized, implying their *equivalence* to second-order linear difference equations, another interconnection is readily made, this time to infinite products of 2×2 matrices. (This was brought out by the seminal work [3] of L. M. Milne-Thomson.) You will have no difficulty, apart from some pesky algebra, in verifying that the Case-I Fibonacci c.f. of Eq. (2) can be represented by

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \propto \begin{pmatrix} -1 & 1 \\ k & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ k & 0 \end{pmatrix} \cdots \begin{pmatrix} -1 & 1 \\ k & 0 \end{pmatrix} \begin{pmatrix} 1 \\ R_n \end{pmatrix}, \qquad (5)$$

the value of the c.f. being calculated as the limiting value as $n\to\infty$ of the *ratio* R_1 of matrix entries (lower entry C_2 divided by upper entry C_1), it being understood that $R_n \to r$, the same remainder value discussed above, which possesses what we have termed a stable and an unstable root.

Similarly, in Case II the Eq. (4b) c.f. value is represented by $R_{n+1} = C_{n+2}/C_{n+1} \to r$, via a product of *n* square matrices acting on a 2-entry column vector (operand),

$$\begin{pmatrix} C_{n+2} \\ kC_{n+1} \end{pmatrix} \propto \begin{pmatrix} R_{n+1} \\ k \end{pmatrix} \propto \begin{pmatrix} 1 & 1 \\ k & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ k & 0 \end{pmatrix} \cdots \begin{pmatrix} 1 & 1 \\ k & 0 \end{pmatrix} \begin{pmatrix} R_1 \\ k \end{pmatrix}, (6) \qquad (6)$$

wherein, as before, the stability properties of the roots are opposite from those of Case I. We use $\propto$ signs in the foregoing, rather than

= signs, because the basic difference equation to which all these processes are equivalent is linear and homogeneous; hence it determines its coefficients only within a constant multiplier, so that only ratios of those coefficients are numerically determinate. Note that in this case, by application of Eq. (4a), the c.f. value $R_{n+2} \to r$ is one plus the quotient of the lower entry k divided by the upper entry R_{n+1} of the 2-entry column vector obtained by performing all multiplications indicated in Eq. (6).

The important thing to recognize here is the mutual *equivalence* among second-order linear homogeneous difference equations, iterations, c.f.'s, and infinite 2×2 matrix products such as (5) or (6). In effect, all these so different-looking formulations amount to different notational disguises of the same "mathematical object." Value-wise, they all embody an essential "two-ness," hallmarked by the algebraic two-ness of the roots of the *quadratic* characteristic equation of the difference equation; and all have identical stability properties regarding convergence to the roots of the latter key equation.

4. Generalizations

If you have been successfully hooked by the Fibonacci example, your mind will be full of questions, such as, to begin with, what happens in the more useful case where the coefficients in the difference equation are functions of n instead of constants? Proceeding thus to the more general (second-order, homogeneous) difference equation $C_{n+2} + b_n C_{n+1} - a_n C_n = 0, \quad n = 1, 2, \cdots,$ and defining the ratio $R_n = C_{n+1} / C_n$, one finds at once a generalization of Case I through isolation of R_n in the form of the iteration

$$R_n = \frac{a_n}{b_n + R_{n+1}}, \tag{7a}$$

which expands to the general form of “c.f. with remainder,”

$$R_1 = \frac{C_2}{C_1} = \cfrac{a_1}{b_1 + \cfrac{a_2}{b_2 + \ddots + \cfrac{a_n}{b_n + R_{n+1}}}}. \tag{7b}$$

The characteristic equation deriving from $R_{n+1}R_n + b_nR_n - a_n = 0$ is the quadratic $r^2 + b_nr - a_n \approx 0$, where $r = r(n)$ and only the asymptotically dominant term as $n \to \infty$ need be considered. We can think of the c.f. (7b) as equivalent to an arbitrary difference equation with “boundary conditions at infinity.” Again, under the proviso that the coefficients a_n, b_n possess asymptotic expansions as $n \to \infty$, two roots of the characteristic equation are in general obtained, only one of which yields a stable iteration. It is often useful to express r itself as an asymptotic expansion, with coefficients determined successively through expressing all terms of $r_{n+1}r_n + b_nr_n - a_n \approx 0$ by their asymptotic expansions in a common basis. This can in general be done unambiguously, once the leading term has been identified by root selection with reference to the quadratic characteristic equation. (Examples have been given elsewhere [2].) Eq. (7) corresponds to our previous Case I [isolation of R_n as in (7a) and (2a)].

The corresponding generalization in Case II (isolation of R_{n+1}) leads to the iteration,

$$R_{n+1} = -b_n + \frac{a_n}{R_n}, \tag{8a}$$

which develops into

$$R_{n+1} = -b_n + \cfrac{a_n}{-b_{n-1} + \cfrac{a_{n-1}}{-b_{n-2} + \ddots + \cfrac{a_2}{-b_1 + \cfrac{a_1}{R_1}}}}. \tag{8b}$$

We see that, once hooked by Fibonacci, the student is inexorably drawn into a related field, that of asymptotics. Applying what is readily learned about that new topic, one will begin to entertain not only rough ideas of iterative "stability" and "instability" but also subtler questions relating to rates of convergence. This will lead on to theorems about speeding convergence with asymptotic approximations to process remainders. (You will have to devise these theorems for yourself, or consult [2], because they do not appear in conventional c.f. texts – which *define* c.f. remainders out of existence at the outset.) In Case I [isolation of R_n, leading to the iteration $R_n = a_n/(b_n + R_{n+1}) \to r$] a 2×2 matrix product equivalent to Eq. (7) is

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \propto P_1 P_2 \cdots P_n \begin{pmatrix} 1 \\ R_{n+1} \end{pmatrix}, \quad \text{where } P_i = \begin{pmatrix} b_i & 1 \\ a_i & 0 \end{pmatrix}, \tag{9}$$

the c.f. value being given by $R_1 = C_2 / C_1$, the ratio of the calculated matrix entries, lower divided by upper.

Similarly, the Case II iteration and c.f. of Eq. (8) can be represented in matrix product form by

$$\begin{pmatrix} C_{n+2} \\ a_{n+1} C_{n+1} \end{pmatrix} \propto \begin{pmatrix} R_{n+1} \\ a_{n+1} \end{pmatrix} \propto P_n \cdots P_2 P_1 \begin{pmatrix} R_1 \\ a_1 \end{pmatrix}, \text{ where } P_i = \begin{pmatrix} -b_i & 1 \\ a_{i+1} & 0 \end{pmatrix}. \tag{10}$$

In view of (8a), the coefficient quotient $R_{n+2} \to r$, representing the value of the process (10), may be calculated as $-b_{n+1}$ plus the quotient obtained by dividing the lower entry a_{n+1} of the calculated 2-entry column vector in (10) by the upper entry R_{n+1}. This Case-II formalism is equivalent to a linear homogeneous difference equation with boundary conditions at *finite n*-values. Note the reversed orders of sequential multiplication of the square matrices in Cases I and II. The convergence stability properties of the two cases are reversed, as previously discussed.

For many practical purposes such matrix product representations of the basic difference equation offer more efficient ways of computing c.f. "values" than do the c.f. representations. Thus it is often desirable to extend a calculation by one or more additional stages, for example in order to verify process convergence. To do this with a c.f. such as Eq. (7) or (9) requires "rolling up" the whole process from the bottom – in effect recalculating all stages previously computed. But in the matrix product calculation it is only necessary to store in computer memory the last 2×2 square matrix obtained in the n^{th} stage of calculation and to reuse it at the $(n+1)^{\text{st}}$ stage. In fact Eq. (10) does not require even this, since extra "stages" are added simply by left multiplication by a square matrix.

Down through the ages physicists have tended to shrink from "three-term recurrence relations," as they call the linear homogeneous second-order difference equations we have been dealing with here. They feel defeated if they can find no ingenious substitution or change of variable that will reduce their problem to a two-term recurrence (simple iteration). But we see here that in fact, with the help of continued fractions or matrix products, two- and three-term recurrences become very little different in numerical computational difficulty. But it is certainly true that if Schroedinger's radial wave equation for the H-atom could not be reduced to a two-term recurrence, implying that only numerical results could be obtained, the pedagogy of elementary quantum mechanics would suffer a severe blow.

At this point you should be thirsting for further generalizations. Having discovered that a second-order linear homogeneous difference equation can be represented by (and considered "equivalent" to) an infinite product of 2×2 matrices, you might speculate that a third-order linear homogeneous difference equation can be represented by a product of 3×3 matrices, and so on. You would be right. Also, you might guess that, whereas a quadratic algebraic "characteristic equation," with up to two distinct roots in the complex plane, governs the bi-valuedness of continued fractions and their corresponding matrix products, the third-order difference equation should generate a cubic characteristic equation with up to three distinct roots and accompanying tri-valuedness. Again you would be right. Unfortunately, the generalization beyond second order leaves continued fractions *per se* behind, because they are notationally limited to representation on two-dimensional paper. We have to be grateful for their "equivalence" to matrix products unaffected by that limitation.

To get the ball rolling on such further generalizations, we might take note here of the m^{th}-order linear homogeneous difference equation,

$$C_{n+m} + j_n C_{n+m-1} + \cdots + b_n C_{n+1} - a_n C_n = 0\,, \quad n = 1, 2, \cdots \tag{11}$$

with $R_n = C_{n+1}/C_n$ and the Case-I type of development appropriate to boundary conditions at infinity, namely,

$$\begin{pmatrix} C_1 \\ C_2 \\ \vdots \\ C_m \end{pmatrix} \propto P_1 P_2 \cdots P_n \begin{pmatrix} 1 \\ R_{n+1} \\ R_{n+1} R_{n+2} \\ \vdots \\ \prod_{j=1}^{m-1} R_{n+j} \end{pmatrix}, \text{where } P_i = \begin{pmatrix} b_i & c_i & \cdots & i_i & j_i & 1 \\ a_i & 0 & \cdots & 0 & 0 & 0 \\ 0 & a_i & \ddots & 0 & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_i & 0 & 0 \\ 0 & 0 & \cdots & 0 & a_i & 0 \end{pmatrix}, \quad (12)$$

for $i = 1, 2, \cdots, n$, P_i being an $m \times m$ matrix. As usual, the remainder terms are very much in evidence, and not at all ignorable – either in ordinary continued fractions or in these so-called "generalized continued fractions." The characteristic equation associated with this difference equation (can you write it down?) possesses up to *m* distinct roots in the complex plane, and the form (12) therefore can converge to as many as *m* different "values." Instead of two cases, we have *m* cases. The other forms appropriate to different types of boundary conditions or mixed finite and infinite b.c.'s have never been worked out, as far as I know.

This brings us to the frontier of generalization in this specialized subject area. Lest young readers proceed unwarily, I leave them with a word to the wise: It is not enough for instant success in this world that your work be valid, needed, and significant; it must also be trendy … and the direction in which I have pointed you here, being that of "classical analysis," is directly opposite to trends among pure mathematicians [6] of the 20th and doubtless the subsequent century – all such trends being steadily downhill as far as usefulness to mankind is concerned.

Finally, to make partial amends for blowing against the wind, let me mention what *is* trendy [4,5] about Fibonacci in the modern world: If the constant *k* of the first part of our discussion is chosen at random on the set $\{1, -1\}$, or some similar set, new properties emerge and the non-classical subject is introduced of *difference equations with stochastically determined coefficients*. Since there is little practical use for a theory of such objects discernable at present, it naturally becomes an instant vogue among pure mathematicians, who – following the philosophy of Hardy [6] – shun utility like the plague. But the new departure is full of wonderful hooks for catching the computer-conscious, so there may be compensations.

I have relegated to an appendix the proof of one of the more surprising assertions made here – *viz.*, that, if a c.f. such as that of Eq. (7) possesses a conventionally-defined "value" (*i.e.*, defined for a fixed remainder sequence $R_{n+1} = \{0, 0, 0, \cdots\}$ or $\{\infty, \infty, \infty, \cdots\}$), then almost any sequence of remainders, including random numbers, will cause eventual convergence to that conventional value (*i.e.*, the same one associated with the dominant root of the characteristic equation). Other proofs for c.f.'s with remainders, in particular bearing on convergence rates, can be found elsewhere [2].

References

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Appendix

Following Wall [1], I shall use the notation

$$F_n(w_n) = \cfrac{a_1}{b_1 + \cfrac{a_1}{b_2 + \ddots + \cfrac{a_n}{b_n + w_n}}} \quad . \tag{A.1}$$

According to a well-known result of conventional c.f. theory (Wall, p. 15),

$$F_n(w_n) = \frac{A_{n-1}w_n + A_n}{B_{n-1}w_n + B_n}, \tag{A.2}$$

where the *A*'s and *B*'s are computed for $n = 1, 2, \cdots$ from the difference equations

$$A_{n+1} = b_{n+1}A_n + a_{n+1}A_{n-1}, \; B_{n+1} = b_{n+1}B_n + a_{n+1}B_{n-1}, \tag{A.3}$$

with initial conditions $A_0 = 0, B_0 = 1, A_1 = a_1, B_1 = b_1$. Let us denote by L the conventional "value" of the c.f. process symbolized by Eq. (A.1), inherited from the Cauchy definition for the "value" of an infinite series. (This traditional definition, which imposes single-valuedness by fiat, as remarked in the text, is plausible for an infinite series, which is equivalent to a *first-order* inhomogeneous difference equation; but it is not plausible for a c.f. process, viewed as equivalent to a *second-order* difference equation.) We have then by definition

$$\lim_{n\to\infty} F_n(0) = \lim_{n\to\infty} F_n(\infty) = \lim_{n\to\infty} \frac{A_n}{B_n} = L\,. \tag{A.4}$$

Our result, which we state as a lemma, is

Lemma. Given that $\lim_{n\to\infty}(A_n / B_n) = L$, *and given any sequence of real numbers* w_n, $n = 1, 2, \cdots$, *such that*

$$\lim_{n\to\infty}\left(\frac{\dfrac{A_n}{B_n}-\dfrac{A_{n-1}}{B_{n-1}}}{1+w_n\dfrac{B_{n-1}}{B_n}}\right)\equiv V \tag{A.5}$$

then $\lim_{n\to\infty} F_n(w_n)=L+V$ *, where* $F_n(w_n)$ *is defined by Eq.* (A.1).

Proof: The stated result follows immediately from applying $\lim_{n\to\infty}$ to the formal identity

$$F_n(w_n)=\frac{A_n+A_{n-1}w_n}{B_n+B_{n-1}w_n}=\frac{A_{n-1}}{B_{n-1}}+\left(\frac{\dfrac{A_n}{B_n}-\dfrac{A_{n-1}}{B_{n-1}}}{1+w_n\dfrac{B_{n-1}}{B_n}}\right),\ n=1,2,\cdots. \tag{A.6}$$

Although the lemma is trivial, it has far-reaching implications. If the conventional value L of the c.f. exists, this means that $\lim_{n\to\infty}(A_n/B_n)$ exists, hence that

$$\left(\frac{A_n}{B_n}-\frac{A_{n-1}}{B_{n-1}}\right)\to 0 \text{ as } n\to\infty .$$

Consequently the numerator of V [defined by Eq. (A.5)] must vanish in the limit. For most choices – including random number selections – of the remainder sequence $w_1, w_2, \cdots$, the denominator of V will not vanish. Hence almost all $\{w_n\}$ sequences will result in $V=0$ and will thus prescribe convergence to the conventional c.f. value L. Thus there is nothing magic about the remainder sequence $\{0,0,0,\ldots\}$ demanded by the conventional definition. This all says nothing about speed of convergence. In general it is easy (by asymptotics, as mentioned in the text) to find remainder sequences that will markedly speed convergence. The "exceptional sequences" of remainders $\{ w_n \}$ that cause departure of the c.f. value from L are those for which the V denominator approaches zero at least as fast as the numerator, as $n \to \infty$. The Fibonacci example establishes that such sequences exist.