Biquaternion formulation of relativistic tensor dynamics

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In this paper we show how relativistic tensor dynamics and relativistic electrodynamics can be formulated in a biquaternion tensor language. The treatment is restricted to mathematical physics, known facts as the Lorentz Force Law and the Lagrange Equation are presented in a relatively new formalism. The goal is to fuse anti-symmetric tensor dynamics, as used for example in relativistic electrodynamics, and symmetric tensor dynamics, as used for example in introductions to general relativity, into one single formalism: a specific kind of biquaternion tensor calculus.

Keywords: biquaternion, relativistic dynamics, Lorentz Force Law, Lagrange Equation

Introduction

We start by quoting Yefremov. *One can say that space-time model and kinematics of the Quaternionic Relativity are* *nowadays studied in enough details and can be used as an effective mathematical tool for calculation of many relativistic effects. But respective relativistic dynamic has not been yet formulated, there are no quaternionic field theory; Q-gravitation, electromagnetism, weak and strong interactions are still remote projects. However, there is a hope that it is only beginning of a long way, and the theory will mature. [1]*

We hope that the content of this paper will contribute to the project described by Yefremov.

Quaternions can be represented by the basis $(\mathbf{1}, \mathbf{I}, \mathbf{J}, \mathbf{K})$. This basis has the properties $\mathbf{II} = \mathbf{JJ} = \mathbf{KK} = -\mathbf{1}$; $\mathbf{11} = \mathbf{1}$; $\mathbf{1K} = \mathbf{K1} = \mathbf{K}$ for $\mathbf{I}, \mathbf{J}, \mathbf{K}$; $\mathbf{IJ} = -\mathbf{JI} = \mathbf{K}$; $\mathbf{JK} = -\mathbf{KJ} = \mathbf{I}$; $\mathbf{KI} = -\mathbf{IK} = \mathbf{J}$. A quaternion number in its summation representation is given by $A = a_0\mathbf{1} + a_1\mathbf{I} + a_2\mathbf{J} + a_3\mathbf{K}$, in which the a_μ are real numbers . Biquaternions or complex quaternions in their summation representation are given by

$$\begin{aligned} C = A + \mathbf{i}B = \\ (a_0 + \mathbf{i}b_0)\mathbf{1} + (a_1 + \mathbf{i}b_1)\mathbf{I} + (a_2 + \mathbf{i}b_2)\mathbf{J} + (a_3 + \mathbf{i}b_3)\mathbf{K} = \\ a_0\mathbf{1} + a_1\mathbf{I} + a_2\mathbf{J} + a_3\mathbf{K} + \mathbf{i}b_0\mathbf{1} + \mathbf{i}b_1\mathbf{I} + \mathbf{i}b_2\mathbf{J} + \mathbf{i}b_3\mathbf{K}, \end{aligned} \tag{1}$$

in which the $c_\mu = a_\mu + \mathbf{i}b_\mu$ are complex numbers and the a_μ and b_μ are real numbers. The complex conjugate of a biquaternion C is given by $\widetilde{C} = A - \mathbf{i}B$. The quaternion conjugate of a biquaternion is given by

$$\begin{aligned} C^\dagger = A^\dagger + \mathbf{i}B^\dagger = \\ (a_0 + \mathbf{i}b_0)\mathbf{1} - (a_1 + \mathbf{i}b_1)\mathbf{I} - (a_2 + \mathbf{i}b_2)\mathbf{J} - (a_3 + \mathbf{i}b_3)\mathbf{K}. \end{aligned} \tag{2}$$

In this paper we only use the complex conjugate of biquaternions.

Biquaternions or complex quaternions in their vector representation are given by

$$C_\mu = \begin{bmatrix} c_0\mathbf{1} \\ c_1\mathbf{I} \\ c_2\mathbf{J} \\ c_3\mathbf{K} \end{bmatrix}, \tag{3}$$

or by

$$C^\mu = [c_0\mathbf{1}, c_1\mathbf{I}, c_2\mathbf{J}, c_3\mathbf{K}] \tag{4}$$

We apply this to the space-time four vector of relativistic biquaternion 4-space R_μ as

$$R_\mu = \begin{bmatrix} \mathbf{i}ct\mathbf{1} \\ r_1\mathbf{I} \\ r_2\mathbf{J} \\ r_3\mathbf{K} \end{bmatrix} = \begin{bmatrix} \mathbf{i}r_0\mathbf{1} \\ r_1\mathbf{I} \\ r_2\mathbf{J} \\ r_3\mathbf{K} \end{bmatrix}. \tag{5}$$

The space-time distance s can be defined as $\widetilde{R}^\mu R^\mu$, or

$$\widetilde{R}^\mu R^\mu = [-\mathbf{i}ct\mathbf{1}, r_1\mathbf{I}, r_2\mathbf{J}, r_3\mathbf{K}]\,[\mathbf{i}ct\mathbf{1}, r_1\mathbf{I}, r_2\mathbf{J}, r_3\mathbf{K}], \tag{6}$$

giving

$$\widetilde{R}^\mu R^\mu = c^2t^2\mathbf{1} - r_1^2\mathbf{1} - r_2^2\mathbf{1} - r_3^2\mathbf{1} = (c^2t^2 - r_1^2 - r_2^2 - r_3^2)\mathbf{1}. \tag{7}$$

So we get $\widetilde{R}^\mu R^\mu = s\mathbf{1}$ with the usual

$$s = c^2t^2 - r_1^2 - r_2^2 - r_3^2 = r_0^2 - r_1^2 - r_2^2 - r_3^2 \tag{8}$$

providing us with a $(+,-,-,-)$ signature.

Adding the dynamic vectors

The basic definitions we use are quite common in the usual formulations of relativistic dynamics, see [2], [3]. We start with an observer who has a given three vector velocity as $\mathbf{v}$, a rest mass as m_0 and an inertial mass $m_i = \gamma m_0$, with the usual $\gamma = (\sqrt{1 - v^2/c^2})^{-1}$. We use the Latin suffixes as abbreviations for words, not for numbers. So m_i stands for inertial mass and U_p for potential energy. The Greek suffixes are used as indicating a summation over the numbers 0, 1, 2 and 3. So P_μ stands for a momentum four-vector with components $p_0 = \frac{1}{c}U_i$, p_1, p_2 and p_3. The momentum three-vector is written as $\mathbf{p}$ and has components p_1, p_2 and p_3.

We define the coordinate velocity four vector as

$$V_\mu = \frac{d}{dt}R_\mu = \begin{bmatrix} \mathbf{i}c\mathbf{1} \\ v_1\mathbf{I} \\ v_2\mathbf{J} \\ v_3\mathbf{K} \end{bmatrix} = \begin{bmatrix} \mathbf{i}v_0\mathbf{1} \\ v_1\mathbf{I} \\ v_2\mathbf{J} \\ v_3\mathbf{K} \end{bmatrix}. \tag{9}$$

The proper velocity four vector on the other hand will be defined using the proper time t_0, with $t = \gamma t_0$, as

$$U_\mu = \frac{d}{dt_0}R_\mu = \frac{d}{\frac{1}{\gamma}dt}R_\mu = \gamma V_\mu = \begin{bmatrix} \mathbf{i}\gamma c\mathbf{1} \\ \gamma v_1\mathbf{I} \\ \gamma v_2\mathbf{J} \\ \gamma v_3\mathbf{K} \end{bmatrix}. \tag{10}$$

The momentum four vector will be

$$P_\mu = m_i V_\mu = m_0 U_\mu. \tag{11}$$

We further define the rest mass density as

$$\rho_0 = \frac{dm_0}{dV_0}, \tag{12}$$

so with

$$dV = \frac{1}{\gamma} dV_0 \tag{13}$$

and the inertial mass density as

$$\rho_i = \frac{dm_i}{dV} \tag{14}$$

we get, in accordance with Arthur Haas' 1930 exposition on relativity ([4], p. 365),

$$\rho_i = \frac{dm_i}{dV} = \frac{d\gamma m_0}{\frac{1}{\gamma} dV_0} = \gamma^2 \rho_0. \tag{15}$$

The momentum density four vector will be defined as

$$G_\mu = \begin{bmatrix} \mathbf{i}\frac{1}{c}u_i\mathbf{1} \\ g_1\mathbf{I} \\ g_2\mathbf{J} \\ g_3\mathbf{K} \end{bmatrix} = \begin{bmatrix} \mathbf{i}g_0\mathbf{1} \\ g_1\mathbf{I} \\ g_2\mathbf{J} \\ g_3\mathbf{K} \end{bmatrix}, \tag{16}$$

in which we used the inertial energy density $u_i = \rho_i c^2$. For this momentum density four vector we have the variations

$$G_\mu = \frac{d}{dV} P_\mu = \frac{dm_i}{dV} V_\mu = \rho_i V_\mu = \gamma^2 \rho_0 V_\mu = \gamma \rho_0 U_\mu = \gamma G_\mu^{proper}. \tag{17}$$

The four vector partial derivative ∂_μ will be defined as

$$\partial_\mu = \begin{bmatrix} -\mathbf{i}\frac{1}{c}\partial_t \mathbf{1} \\ \nabla_1 \mathbf{I} \\ \nabla_2 \mathbf{J} \\ \nabla_3 \mathbf{K} \end{bmatrix} \equiv \frac{\partial}{\partial R_\mu}. \tag{18}$$

The electrodynamic potential four vector will be defined as

$$A_\mu = \begin{bmatrix} \mathbf{i}\frac{1}{c}\phi \mathbf{1} \\ A_1 \mathbf{I} \\ A_2 \mathbf{J} \\ A_3 \mathbf{K} \end{bmatrix} = \begin{bmatrix} \mathbf{i}A_0 \mathbf{1} \\ A_1 \mathbf{I} \\ A_2 \mathbf{J} \\ A_3 \mathbf{K} \end{bmatrix}. \tag{19}$$

The electric four current density will be given by

$$J_\mu = \begin{bmatrix} \mathbf{i}c\rho_e \mathbf{1} \\ J_1 \mathbf{I} \\ J_2 \mathbf{J} \\ J_3 \mathbf{K} \end{bmatrix} = \begin{bmatrix} \mathbf{i}J_0 \mathbf{1} \\ J_1 \mathbf{I} \\ J_2 \mathbf{J} \\ J_3 \mathbf{K} \end{bmatrix} = \rho_e V_\mu, \tag{20}$$

with ρ_e as the electric charge density.

Adding the dynamic vector products, scalars

The dynamic Lagrangian density $\mathcal{L}$ can be defined as

$$\mathcal{L} = -\widetilde{V}^\nu G^\nu = -(u_i - \mathbf{v}\cdot\mathbf{g})\mathbf{1} = -u_0\mathbf{1} \tag{21}$$

and the accompanying Lagrangian L as

$$L = -\widetilde{V}^\nu P^\nu = -(U_i - \mathbf{v}\cdot\mathbf{p})\mathbf{1} = -\frac{1}{\gamma}U_0\mathbf{1}, \tag{22}$$

with u_0 as the rest system inertial energy density and U_0 as the rest system inertial energy. The latter is the usual Lagrangian of a particle moving freely in empty space.

The Lagrangian density of a massless electric charge density current in an electrodynamic potential field can be defined as

$$\mathcal{L} = -\widetilde{J}^\nu A^\nu = -(\rho_e \phi - \mathbf{J} \cdot \mathbf{A})\mathbf{1}. \tag{23}$$

On the basis of the Lagrangian density we can define a four force density as

$$f_\mu \equiv \frac{\partial \mathcal{L}}{\partial R_\mu} = \partial_\mu \mathcal{L} = -\partial_\mu u_0. \tag{24}$$

In the special case of a static electric force field, and without the densities, the field energy is $U_0 = q\phi_0$ and the relativistic force reduces to the Coulomb Force

$$\mathbf{F} = -\nabla U_0 = -q\nabla\phi_0. \tag{25}$$

Using $\mathcal{L} = -\widetilde{V}^\nu G^\nu$ the relativistic four force density of Eq.(24) can be written as

$$f_\mu = -\partial_\mu \widetilde{V}^\nu G^\nu. \tag{26}$$

We can define the absolute time derivative $\frac{d}{dt}$ of a continuous, perfect fluid like, space/field quantity through

$$-V^\mu \widetilde{\partial}^\mu = -\widetilde{V}^\mu \partial^\mu = \mathbf{v} \cdot \nabla + \partial_t \mathbf{1} = \frac{d}{dt}\mathbf{1}. \tag{27}$$

Thus we can define the mechanic four force density as

$$f_\mu \equiv \frac{d}{dt} G_\mu = -(V^\nu \widetilde{\partial}^\nu) G_\mu = -V^\nu (\widetilde{\partial}^\nu G_\mu), \tag{28}$$

using the fact that biquaternion multiplication is associative.

Adding the dynamic vector products, tensors

The mechanical stress energy tensor, introduced by Max von Laue in 1911, was defined by him as ([5], [6], p.150)

$$T^{\nu}{}_{\mu} = \rho_0 U^{\nu} U_{\mu}. \tag{29}$$

Pauli gave the same definition in his standard work on relativity ([2], p. 117). With the vector and density definitions that we have given we get

$$T^{\nu}{}_{\mu} = \rho_0 U^{\nu} U_{\mu} = \gamma^2 \rho_0 V^{\nu} V_{\mu} = \rho_i V^{\nu} V_{\mu} = V^{\nu} \rho_i V_{\mu} = V^{\nu} G_{\mu}. \tag{30}$$

So the mechanical stress energy tensor can also be written as

$$T^{\nu}{}_{\mu} = V^{\nu} G_{\mu}. \tag{31}$$

In the exposition on relativity of Arthur Haas, the first definition $\rho_0 U^{\nu} U_{\mu}$ is described as the "Materie-tensor" of General Relativity, while $\rho_i V^{\nu} V_{\mu}$ is described as the "Materie-tensor" of Special Relativity ([4], p. 395 and p. 365).

Although the derivation seems to demonstrate an equivalence between the two formulations of equation (29) and equation (31), the difference between the two is fundamental. Equation (29) is symmetric by definition, while equation (31) can be asymmetric, because, as von Laue already remarked in 1911, V^{ν} and G_{μ} do not have to be parallel all the time ([5], [6] p. 167) This crucial difference between $\rho_0 U^{\nu} U_{\mu}$ and $V^{\nu} G_{\mu}$ was also discussed by de Broglie in connection with his analysis of electron spin ([7], p. 55). In our context, where we want to fuse the symmetric and antisymmetric formalism into one, we prefer the stress energy density tensor of equation (31), the one called the "Materie-tensor" of Special Relativity by Arthur Haas.

So the stress energy density tensor ${T^{\nu}}_{\mu}$ can be given as ${T^{\nu}}_{\mu} = \widetilde{V}^{\nu} G_{\mu}$ and gives

$$
{T^{\nu}}_{\mu} = [-\mathbf{i}v_0\mathbf{1}, v_1\mathbf{I}, v_2\mathbf{J}, v_3\mathbf{K}] \begin{bmatrix} \mathbf{i}g_0\mathbf{1} \\ g_1\mathbf{I} \\ g_2\mathbf{J} \\ g_3\mathbf{K} \end{bmatrix} =
$$

$$
\begin{bmatrix} v_0 g_0 \mathbf{1} & \mathbf{i}v_1 g_0 \mathbf{I} & \mathbf{i}v_2 g_0 \mathbf{J} & \mathbf{i}v_3 g_0 \mathbf{K} \\ -\mathbf{i}v_0 g_1 \mathbf{I} & -v_1 g_1 \mathbf{1} & -v_2 g_1 \mathbf{K} & v_3 g_1 \mathbf{J} \\ -\mathbf{i}v_0 g_2 \mathbf{J} & v_1 g_2 \mathbf{K} & -v_2 g_2 \mathbf{1} & -v_3 g_2 \mathbf{I} \\ -\mathbf{i}v_0 g_3 \mathbf{K} & -v_1 g_3 \mathbf{J} & v_2 g_3 \mathbf{I} & -v_3 g_3 \mathbf{1} \end{bmatrix} \tag{32}
$$

Its trace is $T^{\nu\nu} = \widetilde{V}^{\nu} G^{\nu} = -\mathcal{L}$.

In relativistic dynamics we have a usual force density definition through the four derivative of the stress energy density tensor

$$
\partial^{\nu} {T^{\nu}}_{\mu} = -f_{\mu} \tag{33}
$$

or

$$
\partial^{\nu} V^{\nu} G_{\mu} = -f_{\mu} \tag{34}
$$

We want to find out if these equations still hold in our biquaternion version of the four vectors, tensors and their products.

We calculate the left hand side and get for $\partial^{\nu} {T^{\nu}}_{\mu} = \partial^{\nu} \widetilde{V}^{\nu} G_{\mu}$:

$$\left[-\frac{\mathbf{i}}{c}\partial_t\mathbf{1}, \nabla_1\mathbf{I}, \nabla_2\mathbf{J}, \nabla_3\mathbf{K}\right]$$

$$\left[\begin{array}{cccc} v_0g_0\mathbf{1} & \mathbf{i}v_1g_0\mathbf{I} & \mathbf{i}v_2g_0\mathbf{J} & \mathbf{i}v_3g_0\mathbf{K} \\ -\mathbf{i}v_0g_1\mathbf{I} & -v_1g_1\mathbf{1} & -v_2g_1\mathbf{K} & v_3g_1\mathbf{J} \\ -\mathbf{i}v_0g_2\mathbf{J} & v_1g_2\mathbf{K} & -v_2g_2\mathbf{1} & -v_3g_2\mathbf{I} \\ -\mathbf{i}v_0g_3\mathbf{K} & -v_1g_3\mathbf{J} & v_2g_3\mathbf{I} & -v_3g_3\mathbf{1} \end{array}\right] \quad (35)$$

which equals

$$\left[\begin{array}{c} -\mathbf{i}\frac{1}{c}\partial_t v_0g_0\mathbf{1} - \mathbf{i}\nabla_1v_1g_0\mathbf{1} - \mathbf{i}\nabla_2v_2g_0\mathbf{1} - \mathbf{i}\nabla_3v_3g_0\mathbf{1} \\ -\frac{1}{c}\partial_t v_0g_1\mathbf{I} - \nabla_1v_1g_1\mathbf{I} - \nabla_2v_2g_1\mathbf{I} - \nabla_3v_3g_1\mathbf{I} \\ -\frac{1}{c}\partial_t v_0g_2\mathbf{J} - \nabla_1v_1g_2\mathbf{J} - \nabla_2v_2g_2\mathbf{J} - \nabla_3v_3g_2\mathbf{J} \\ -\frac{1}{c}\partial_t v_0g_3\mathbf{K} - \nabla_1v_1g_3\mathbf{K} - \nabla_2v_2g_3\mathbf{K} - \nabla_3v_3g_3\mathbf{K} \end{array}\right]$$

$$= -\left[\begin{array}{c} \mathbf{i}(\frac{1}{c}\partial_t v_0g_0 + \nabla_1v_1g_0 + \nabla_2v_2g_0 + \nabla_3v_3g_0)\mathbf{1} \\ (\frac{1}{c}\partial_t v_0g_1 + \nabla_1v_1g_1 + \nabla_2v_2g_1 + \nabla_3v_3g_1)\mathbf{I} \\ (\frac{1}{c}\partial_t v_0g_2 + \nabla_1v_1g_2 + \nabla_2v_2g_2 + \nabla_3v_3g_2)\mathbf{J} \\ (\frac{1}{c}\partial_t v_0g_3 + \nabla_1v_1g_3 + \nabla_2v_2g_3 + \nabla_3v_3g_3)\mathbf{K} \end{array}\right] \quad (36)$$

Using the chain rule this leads to

$$\partial^\nu T^\nu{}_\mu = - \begin{bmatrix} \mathbf{i}(\frac{1}{c}v_0\partial_t g_0 + v_1\nabla_1 g_0 + v_2\nabla_2 g_0 + v_3\nabla_3 g_0)\mathbf{1} \\ (\frac{1}{c}v_0\partial_t g_1 + v_1\nabla_1 g_1 + v_2\nabla_2 g_1 + v_3\nabla_3 g_1)\mathbf{I} \\ (\frac{1}{c}v_0\partial_t g_2 + v_1\nabla_1 g_2 + v_2\nabla_2 g_2 + v_3\nabla_3 g_2)\mathbf{J} \\ (\frac{1}{c}v_0\partial_t g_3 + v_1\nabla_1 g_3 + v_2\nabla_2 g_3 + v_3\nabla_3 g_3)\mathbf{K} \end{bmatrix} -$$

$$\begin{bmatrix} \mathbf{i}(\frac{1}{c}(\partial_t v_0)g_0 + (\nabla_1 v_1)g_0 + (\nabla_2 v_2)g_0 + (\nabla_3 v_3)g_0)\mathbf{1} \\ (\frac{1}{c}(\partial_t v_0 g_1) + (\nabla_1 v_1)g_1 + (\nabla_2 v_2)g_1 + (\nabla_3 v_3)g_1)\mathbf{I} \\ (\frac{1}{c}(\partial_t v_0 g_2) + (\nabla_1 v_1)g_2 + (\nabla_2 v_2)g_2 + (\nabla_3 v_3)g_2)\mathbf{J} \\ (\frac{1}{c}(\partial_t v_0 g_3) + (\nabla_1 v_1)g_3 + (\nabla_2 v_2)g_3 + (\nabla_3 v_3)g_3)\mathbf{K} \end{bmatrix}$$

$$= -(\frac{1}{c}v_0\partial_t + v_1\nabla_1 + v_2\nabla_2 + v_3\nabla_3) \begin{bmatrix} \mathbf{i}g_0\mathbf{1} \\ g_1\mathbf{I} \\ g_2\mathbf{J} \\ g_3\mathbf{K} \end{bmatrix}$$

$$-(\frac{1}{c}\partial_t v_0 + \nabla_1 v_1 + \nabla_2 v_2 + \nabla_3 v_3) \begin{bmatrix} \mathbf{i}g_0\mathbf{1} \\ g_1\mathbf{I} \\ g_2\mathbf{J} \\ g_3\mathbf{K} \end{bmatrix}$$

$$= (\widetilde{V}^\nu \partial^\nu)G_\mu + (\partial^\nu \widetilde{V}^\nu)G_\mu \quad (37)$$

This can be abbreviated to

$$\partial^\nu(\widetilde{V}^\nu G_\mu) = (\widetilde{V}^\nu \partial^\nu)G_\mu + (\partial^\nu \widetilde{V}^\nu)G_\mu \quad (38)$$

So

$$\partial^\nu T^\nu{}_\mu = (\widetilde{V}^\nu \partial^\nu)G_\mu + (\partial^\nu \widetilde{V}^\nu)G_\mu. \quad (39)$$

We have $\widetilde{V}^\nu \partial^\nu = -\frac{d}{dt}$ and if we assume the bare particle velocity continuity equation $\partial^\nu \widetilde{V}^\nu = 0$, then we get

$$\partial^\nu T^\nu{}_\mu = -\frac{d}{dt}G_\mu = -f_\mu. \quad (40)$$

Electrodynamic vector products

If we apply this to the case in which we have a purely electromagnetic four momentum density $G_\mu = \rho_e A_\mu$ then we have

$$\mathcal{L} = -\widetilde{V}^\nu G^\nu = -\widetilde{V}^\nu \rho_e A^\nu = -\widetilde{J}^\nu A^\nu, \tag{41}$$

and

$$T^\nu{}_\mu = \widetilde{V}^\nu G_\mu = \widetilde{J}^\nu A_\mu. \tag{42}$$

The relativistic force equation

$$\partial^\nu T^\nu{}_\mu = (\widetilde{V}^\nu \partial^\nu) G_\mu + (\partial^\nu \widetilde{V}^\nu) G_\mu. \tag{43}$$

can be given its electrodynamic expression as

$$\partial^\nu T^\nu{}_\mu = (\widetilde{J}^\nu \partial^\nu) A_\mu + (\partial^\nu \widetilde{J}^\nu) A_\mu. \tag{44}$$

If the charge density current continuity equation $\partial^\nu \widetilde{J}^\nu = 0$ can be applied, then this reduces to

$$\partial^\nu T^\nu{}_\mu = (\widetilde{J}^\nu \partial^\nu) A_\mu = (J^\nu \widetilde{\partial}^\nu) A_\mu = J^\nu (\widetilde{\partial}^\nu A_\mu). \tag{45}$$

The electrodynamic force field tensor $B^\nu{}_\nu$ is given by

$$B^\nu{}_\mu = \widetilde{\partial}^\nu A_\mu. \tag{46}$$

In detail this reads

$$B^\nu{}_\mu = \left[\mathbf{i}\frac{1}{c}\partial_t \mathbf{1}, \nabla_1 \mathbf{I}, \nabla_2 \mathbf{J}, \nabla_3 \mathbf{K}\right] \begin{bmatrix} \mathbf{i}\frac{1}{c}\phi\mathbf{1} \\ A_1\mathbf{I} \\ A_2\mathbf{J} \\ A_3\mathbf{K} \end{bmatrix} =$$

$$\begin{bmatrix} -\frac{1}{c^2}\partial_t\phi\mathbf{1} & \mathbf{i}\frac{1}{c}\nabla_1\phi\mathbf{I} & \mathbf{i}\frac{1}{c}\nabla_2\phi\mathbf{J} & \mathbf{i}\frac{1}{c}\nabla_3\phi\mathbf{K} \\ \mathbf{i}\frac{1}{c}\partial_t A_1\mathbf{I} & -\nabla_1 A_1\mathbf{1} & -\nabla_2 A_1\mathbf{K} & \nabla_3 A_1\mathbf{J} \\ \mathbf{i}\frac{1}{c}\partial_t A_2\mathbf{J} & \nabla_1 A_2\mathbf{K} & -\nabla_2 A_2\mathbf{1} & -\nabla_3 A_2\mathbf{I} \\ \mathbf{i}\frac{1}{c}\partial_t A_3\mathbf{K} & -\nabla_1 A_3\mathbf{J} & \nabla_2 A_3\mathbf{I} & -\nabla_3 A_3\mathbf{1} \end{bmatrix} \tag{47}$$

To see that this tensor leads to the usual EM force field biquaternion, we have to rearrange the tensor terms according to their biquaternionic affiliation, so arrange them according to the basis $(\mathbf{1}, \mathbf{I}, \mathbf{J}, \mathbf{K})$. This results in

$$\begin{bmatrix} (-\frac{1}{c^2}\partial_t\phi - \nabla_1 A_1 - \nabla_2 A_2 - \nabla_3 A_3)\mathbf{1} \\ (\nabla_2 A_3 - \nabla_3 A_2 + \mathbf{i}\frac{1}{c}\nabla_1\phi + \mathbf{i}\frac{1}{c}\partial_t A_1)\mathbf{I} \\ (\nabla_3 A_1 - \nabla_1 A_3 + \mathbf{i}\frac{1}{c}\nabla_2\phi + \mathbf{i}\frac{1}{c}\partial_t A_2)\mathbf{J} \\ (\nabla_1 A_2 - \nabla_2 A_1 + \mathbf{i}\frac{1}{c}\nabla_3\phi + \mathbf{i}\frac{1}{c}\partial_t A_3)\mathbf{K} \end{bmatrix} \tag{48}$$

This equals

$$F_\mu = \begin{bmatrix} \widetilde{\partial}^\nu A^\nu \mathbf{1} \\ (B_1 - \mathbf{i}\frac{1}{c}E_1)\mathbf{I} \\ (B_2 - \mathbf{i}\frac{1}{c}E_1)\mathbf{J} \\ (B_3 - \mathbf{i}\frac{1}{c}E_1)\mathbf{K} \end{bmatrix} = \begin{bmatrix} F_0\mathbf{1} \\ F_1\mathbf{I} \\ F_2\mathbf{J} \\ F_3\mathbf{K} \end{bmatrix}. \tag{49}$$

For this biquaternion to be the exact match with the standard EM force field, one has to add the Lorenz gauge condition $F_0 = \widetilde{\partial}^\nu A^\nu = 0$. (If $F_0 \neq 0$, then the usual biquaternion expressions for the Lorentz Force and the two inhomogenious Maxwell Equations contain extra terms. The biquaternion formalism demonstrated in this paper doesn't involve these extra terms.) The operation of rearranging the tensor terms according to their biquaternion affiliation is external to the mathematical physics of this paper. We try to develop a biquaternion version of relativistic tensordynamics. The above operation destroys the tensor arrangement of the terms involved. It is alien to the system we try to develop in this context. It may be a useful operation in others areas though, for example in quantum physics.

The electrodynamic force field tensor ${B_\mu}^\nu$ can also be given by

$$ {B_\mu}^\nu = \widetilde{\partial}_\mu A^\nu. \tag{50} $$

This leads to the same EM force field biquaternion.

The combination of Eq.(45) and Eq.(46) leads to

$$ \partial^\nu {T^\nu}_\mu = J^\nu {B^\nu}_\mu, \tag{51} $$

which is valid if charge is conserved so if $\partial^\nu \widetilde{J}^\nu = 0$.

We can write Eq.(45) also as

$$ \partial^\nu {T^\nu}_\mu = (\widetilde{J}^\nu \partial^\nu) A_\mu = \rho_e(\widetilde{V}^\nu \partial^\nu) A_\mu = -\rho \frac{d}{dt} A_\mu. \tag{52} $$

The two EM force expression we gave in this and the previous sections based on $f_\mu = -\partial^\nu {T^\nu}_\mu$ and $f_\mu = \partial_\mu \mathcal{L}$ do not result in the well known Lorentz Force. But we can establish a relationship between these force expressions and the Lorentz Force.

The Lorentz Force Law

The relativistic Lorentz Force Law in its density form is given by the expression

$$ f_\mu = J^\nu(\widetilde{\partial}^\nu A_\mu) - (\partial_\mu \widetilde{A}^\nu) J^\nu, \tag{53} $$

or

$$ f_\mu = J^\nu {B^\nu}_\mu - (\partial_\mu \widetilde{A}^\nu) J^\nu. \tag{54} $$

This expression matches, qua terms involved, the standard relativistic Lorentz Force Law. It doesn't have the problem of the extra terms that are usually present in biquaternion versions of the Lorentz Force Law.

If charge is conserved we also have

$$f_\mu = \partial^\nu T^\nu{}_\mu - (\partial_\mu \widetilde{A}^\nu) J^\nu \tag{55}$$

as an equivalent equation. If we examen this last part $(\partial_\mu \widetilde{A}^\nu) J^\nu$ in more detail, an interesting relation arises. We begin with the equation

$$-\partial_\mu \mathcal{L} = \partial_\mu \widetilde{J}^\nu A^\nu. \tag{56}$$

Now clearly we have $\widetilde{J}^\nu A^\nu = \widetilde{A}^\nu J^\nu = u_0 \mathbf{1}$ as a Lorentz invariant scalar. Together with the chain rule this leads to

$$\partial_\mu \widetilde{J}^\nu A^\nu = (\partial_\mu \widetilde{J}^\nu) A^\nu + (\partial_\mu \widetilde{A}^\nu) J^\nu. \tag{57}$$

This equation is crucial for what is to come next, the connection of a Lagrange Equation to the Lorentz Force Law. So we have to prove it in detail, provide an exact proof, specially because biquaternion multiplication in general is non-commutative. We start the proof with $\partial_\mu \widetilde{A}^\nu$:

$$\partial_\mu \widetilde{A}^\nu = \begin{bmatrix} -\mathbf{i}\frac{1}{c}\partial_t \mathbf{1} \\ \nabla_1 \mathbf{I} \\ \nabla_2 \mathbf{J} \\ \nabla_3 \mathbf{K} \end{bmatrix} [-\mathbf{i} A_0 \mathbf{1}, A_1 \mathbf{I}, A_2 \mathbf{J}, A_3 \mathbf{K}] =$$

$$\begin{bmatrix} -\frac{1}{c}\partial_t A_0 \mathbf{1} & -\mathbf{i}\frac{1}{c}\partial_t A_1 \mathbf{I} & -\mathbf{i}\frac{1}{c}\partial_t A_2 \mathbf{J} & -\mathbf{i}\frac{1}{c}\partial_t A_3 \mathbf{K} \\ -\mathbf{i}\nabla_1 A_0 \mathbf{I} & -\nabla_1 A_1 \mathbf{1} & \nabla_1 A_2 \mathbf{K} & -\nabla_1 A_3 \mathbf{J} \\ -\mathbf{i}\nabla_2 A_0 \mathbf{J} & -\nabla_2 A_1 \mathbf{K} & -\nabla_2 A_2 \mathbf{1} & \nabla_2 A_3 \mathbf{I} \\ -\mathbf{i}\nabla_3 A_0 \mathbf{K} & \nabla_3 A_1 \mathbf{J} & -\nabla_3 A_2 \mathbf{I} & -\nabla_3 A_3 \mathbf{1} \end{bmatrix} \tag{58}$$

In the next step we calculate $(\partial_\mu \widetilde{A}^\nu) J^\nu$ and use the fact that

$(\nabla_3 A_1)J_2 = J_2(\nabla_3 A_1)$:

$$(\partial_\mu \widetilde{A}^\nu)J^\nu =$$

$$\begin{bmatrix} -\frac{1}{c}\partial_t A_0 \mathbf{1} & -\mathbf{i}\frac{1}{c}\partial_t A_1 \mathbf{I} & -\mathbf{i}\frac{1}{c}\partial_t A_2 \mathbf{J} & -\mathbf{i}\frac{1}{c}\partial_t A_3 \mathbf{K} \\ -\mathbf{i}\nabla_1 A_0 \mathbf{I} & -\nabla_1 A_1 \mathbf{1} & \nabla_1 A_2 \mathbf{K} & -\nabla_1 A_3 \mathbf{J} \\ -\mathbf{i}\nabla_2 A_0 \mathbf{J} & -\nabla_2 A_1 \mathbf{K} & -\nabla_2 A_2 \mathbf{1} & \nabla_2 A_3 \mathbf{I} \\ -\mathbf{i}\nabla_3 A_0 \mathbf{K} & \nabla_3 A_1 \mathbf{J} & -\nabla_3 A_2 \mathbf{I} & -\nabla_3 A_3 \mathbf{1} \end{bmatrix}$$

$$[\mathbf{i}J_0\mathbf{1}, J_1\mathbf{I}, J_2\mathbf{J}, J_3\mathbf{K}] =$$

$$\begin{bmatrix} (-\mathbf{i}\frac{1}{c}J_0\partial_t A_0 + \mathbf{i}\frac{1}{c}J_1\partial_t A_1 + \mathbf{i}\frac{1}{c}J_2\partial_t A_2 + \mathbf{i}\frac{1}{c}J_3\partial_t A_3)\mathbf{1} \\ (J_0\nabla_1 A_0 - J_1\nabla_1 A_1 - J_2\nabla_1 A_2 - J_3\nabla_1 A_3)\mathbf{I} \\ (J_0\nabla_2 A_0 - J_1\nabla_2 A_1 - J_2\nabla_2 A_2 - J_3\nabla_2 A_3)\mathbf{J} \\ (J_0\nabla_3 A_0 - J_1\nabla_3 A_1 - J_2\nabla_3 A_2 - J_3\nabla_3 A_3)\mathbf{K} \end{bmatrix} \quad (59)$$

Now clearly $(\partial_\mu \widetilde{A}^\nu)J^\nu$ and $(\partial_\mu \widetilde{J}^\nu)A^\nu$ behave identical, only J and A have changed places, so

$$(\partial_\mu \widetilde{J}^\nu)A^\nu =$$

$$\begin{bmatrix} (-\mathbf{i}\frac{1}{c}A_0\partial_t J_0 + \mathbf{i}\frac{1}{c}A_1\partial_t J_1 + \mathbf{i}\frac{1}{c}A_2\partial_t J_2 + \mathbf{i}\frac{1}{c}A_3\partial_t J_3)\mathbf{1} \\ (A_0\nabla_1 J_0 - A_1\nabla_1 J_1 - A_2\nabla_1 J_2 - A_3\nabla_1 J_3)\mathbf{I} \\ (A_0\nabla_2 J_0 - A_1\nabla_2 J_1 - A_2\nabla_2 J_2 - A_3\nabla_2 J_3)\mathbf{J} \\ (A_0\nabla_3 J_0 - A_1\nabla_3 J_1 - A_2\nabla_3 J_2 - A_3\nabla_3 J_3)\mathbf{K} \end{bmatrix} \quad (60)$$

If we add them and use the inverse of the chain rule we get

$$(\partial_\mu \widetilde{A}^\nu)J^\nu + (\partial_\mu \widetilde{J}^\nu)A^\nu =$$

$$\begin{bmatrix} -\mathbf{i}\frac{1}{c}(\partial_t J_0 A_0 - \partial_t J_1 A_1 - \partial_t J_2 A_2 - \partial_t J_3 A_3)\mathbf{1} \\ (\nabla_1 J_0 A_0 - \nabla_1 J_1 A_1 - \nabla_1 J_2 A_2 - \nabla_1 J_3 A_3)\mathbf{I} \\ (\nabla_2 J_0 A_0 - \nabla_2 J_1 A_1 - \nabla_2 J_2 A_2 - \nabla_2 J_3 A_3)\mathbf{J} \\ (\nabla_3 J_0 A_0 - \nabla_3 J_1 A_1 - \nabla_3 J_2 A_2 - \nabla_3 J_3 A_3)\mathbf{K} \end{bmatrix} =$$

$$\begin{bmatrix} -\mathbf{i}\frac{1}{c}\mathbf{1} \\ \nabla_1 \mathbf{I} \\ \nabla_2 \mathbf{J} \\ \nabla_3 \mathbf{K} \end{bmatrix} (J_0 A_0 - J_1 A_1 - J_2 A_2 - J_3 A_3) = \partial_\mu (\widetilde{J}^\nu A^\nu) \tag{61}$$

Thus we have given the exact proof of the statement

$$\partial_\mu \widetilde{J}^\nu A^\nu = (\partial_\mu \widetilde{J}^\nu)A^\nu + (\partial_\mu \widetilde{A}^\nu)J^\nu. \tag{62}$$

So we get

$$-\partial_\mu \mathcal{L} = \partial_\mu \widetilde{J}^\nu A^\nu = (\partial_\mu \widetilde{J}^\nu)A^\nu + (\partial_\mu \widetilde{A}^\nu)J^\nu. \tag{63}$$

We now have two force equations, $f_\mu^L = \partial_\mu \mathcal{L} = -\partial_\mu u_0$ and $f_\mu^T = -\partial^\nu T^\nu{}_\mu = \frac{d}{dt}G_\mu$. We combine them into a force equation that represents the difference between these two forces:

$$f_\mu = -f_\mu^T + f_\mu^L = \partial^\nu T^\nu{}_\mu + \partial_\mu \mathcal{L}. \tag{64}$$

For the purely electromagnetic case this can be written as

$$f_\mu = \partial^\nu \widetilde{J}^\nu A_\mu - \partial_\mu \widetilde{J}^\nu A^\nu \tag{65}$$

and leads to

$$f_\mu = (\widetilde{J}^\nu \partial^\nu)A_\mu + (\partial^\nu \widetilde{J}^\nu)A_\mu - (\partial_\mu \widetilde{J}^\nu)A^\nu - (\partial_\mu \widetilde{A}^\nu)J^\nu. \tag{66}$$

If we have $\partial^\nu \widetilde{J}^\nu = 0\mathbf{1}$ and $\partial_\mu \widetilde{J}^\nu = \mathbf{0}$ then this general force equation reduces to the Lorentz Force Law

$$f_\mu = (\widetilde{J}^\nu \partial^\nu) A_\mu - (\partial_\mu \widetilde{A}^\nu) J^\nu. \tag{67}$$

This of course also happens if $\partial^\nu \widetilde{J}^\nu = \partial_\mu \widetilde{J}^\nu$, so if the RHS of this equation has zero non-diagonal terms.

The Lagrangian Equation

If the difference between f^T_μ and f^L_μ is zero, we get the interesting equation

$$-\partial^\nu T^\nu{}_\mu = \partial_\mu \mathcal{L}. \tag{68}$$

For the situation where $\partial^\mu \widetilde{V}^\nu = 0$ we already proven the statement

$$\partial^\nu T^\nu{}_\mu = -\frac{d}{dt} G_\mu, \tag{69}$$

so we get

$$\frac{d}{dt} G_\mu = \partial_\mu \mathcal{L}, \tag{70}$$

which equals

$$\frac{d}{dt} G_\mu = \frac{\partial \mathcal{L}}{\partial R_\mu}. \tag{71}$$

We will prove that

$$G_\mu = -\frac{\partial \widetilde{V}^\nu G^\nu}{\partial V_\mu} = \frac{\partial \mathcal{L}}{\partial V_\mu}, \tag{72}$$

see the Appendix for the proof and its limitations.

Combined with the forgoing equation, this leads us to

$$\frac{d}{dt} (\frac{\partial \mathcal{L}}{\partial V_\mu}) = \frac{\partial \mathcal{L}}{\partial R_\mu} \tag{73}$$

as equivalent to

$$-\partial^\nu T^\nu{}_\mu = \partial_\mu \mathcal{L}. \tag{74}$$

A canonical Lagrangian density

If we choose a canonical Lagrangian density as

$$\mathcal{L} = -\widetilde{V}^\nu G^\nu + \widetilde{J}^\nu A^\nu = \mathbf{v} \cdot \mathbf{g} - \mathbf{J} \cdot \mathbf{A} - u_i + \rho_e \phi, \tag{75}$$

and an accompanying stress energy density tensor

$$T^\nu{}_\mu = \widetilde{V}^\nu G_\mu - \widetilde{J}^\nu A_\mu, \tag{76}$$

then our force equation $f^T_\mu = f^L_\mu$ can be split in an inertial LHS and an EM RHS

$$(-f^T_\mu + f^L_\mu)_{inertial} = -(-f^T_\mu + f^L_\mu)_{EM}. \tag{77}$$

For situations were $(f^L_\mu)_{inertial} = -\partial_\mu u_0 = 0$ this results in

$$f^{inertial}_\mu = f^{Lorentz}_\mu. \tag{78}$$

as

$$\frac{d}{dt} G_\mu = J^\nu (\widetilde{\partial}^\nu A_\mu) - (\partial_\mu \widetilde{A}^\nu) J^\nu. \tag{79}$$

Maxwell's inhomogeneous equations

We end with the formulation of the two inhomogeneous equations of the set of four Maxwell Equations, as they can be expressed in our terminology. They read

$$\partial^\nu \widetilde{\partial}^\nu A_\mu - \partial_\mu \widetilde{\partial}^\nu A^\nu = \mu_0 J_\mu. \tag{80}$$

As with the Lorenz Force Law, this expression matches the standard relativistic inhomogeneous Maxwell Equations, it doesn't contain extra terms as can be the case with the usual biquaternion formulation of Maxwell's Equations.

The previous equation can be written as

$$(-\nabla^2 + \frac{1}{c^2}\frac{d^2}{dt^2})A_\mu - \partial_\mu(-\partial_t\phi - \nabla\cdot\mathbf{A}) = \mu_0 J_\mu, \qquad (81)$$

so as the difference between a wave like part and the divergence of the Lorenz gauge part.

Conclusions

We have presented a specific kind of biquaternion relativistic tensor dynamics. We formulated the general force equation

$$\partial^\nu T^\nu{}_\mu + \partial_\mu \mathcal{L} = 0. \qquad (82)$$

The stress energy density tensor of a massive moving charged particle in a potential field was formulated as $T^\nu{}_\mu = \widetilde{V}^\nu G_\nu + \widetilde{J}^\nu A_\nu$ with an accompanying Lagrangian density $\mathcal{L}$ as its trace $\mathcal{L} = T^{\nu\nu}$. Under curtain continuity conditions for the four current and the four velocity, this leads to the Lorentz Force Law and to the usual equations of relativistic tensor dynamics. One advantage of our specific kind of biquaternion formalism is that it is very akin to the standard relativistic space-time language and that it lacks the extra terms that usually arise in biquaternionic electrodynamics. Our formalism contains the results of both symmetric and anti-symmetric relativistic tensor dynamics. Curiously, our Lorentz Force Law in terms of the potentials and currents is not anti-symmetric, nor is it symmetric. This non-symmetric property of Eq.(53) was then related to the general force equation Eq.(82).

Appendix

We want to proof that, under curtain conditions, we have

$$\frac{\partial \mathcal{L}}{\partial V_\mu} = -\frac{\partial}{\partial V_\mu}\widetilde{V}^\nu G^\nu = G_\nu. \tag{83}$$

The chain rule as we have used and shown before gives a first hunch. The chain rule leads us to

$$\frac{\partial}{\partial V_\mu}\widetilde{V}^\nu G^\nu = (\frac{\partial}{\partial V_\mu}\widetilde{V}^\nu)G^\nu + (\frac{\partial}{\partial V_\mu}\widetilde{G}^\nu)V^\nu. \tag{84}$$

As before, we cannot assume this, because it uses commutativity, so we have to prove it.

We start the proof with $\frac{\partial}{\partial V_\mu}\widetilde{V}^\nu$:

$$\frac{\partial}{\partial V_\mu}\widetilde{V}^\nu = \begin{bmatrix} -\mathbf{i}\frac{\partial}{\partial v_0}\mathbf{1} \\ \frac{\partial}{\partial v_1}\mathbf{I} \\ \frac{\partial}{\partial v_2}\mathbf{J} \\ \frac{\partial}{\partial v_3}\mathbf{K} \end{bmatrix} [-\mathbf{i}v_0\mathbf{1}, v_1\mathbf{I}, v_2\mathbf{J}, v_3\mathbf{K}] =$$

$$\begin{bmatrix} -\frac{\partial}{\partial v_0}v_0\mathbf{1} & -\mathbf{i}\frac{\partial}{\partial v_0}v_1\mathbf{I} & -\mathbf{i}\frac{\partial}{\partial v_0}v_2\mathbf{J} & -\mathbf{i}\frac{\partial}{\partial v_0}v_3\mathbf{K} \\ -\mathbf{i}\frac{\partial}{\partial v_1}v_0\mathbf{I} & -\frac{\partial}{\partial v_1}v_1\mathbf{1} & \frac{\partial}{\partial v_1}v_2\mathbf{K} & -\frac{\partial}{\partial v_1}v_3\mathbf{J} \\ -\mathbf{i}\frac{\partial}{\partial v_2}v_0\mathbf{J} & -\frac{\partial}{\partial v_2}v_1\mathbf{K} & -\frac{\partial}{\partial v_2}v_2\mathbf{1} & \frac{\partial}{\partial v_2}v_3\mathbf{I} \\ -\mathbf{i}\frac{\partial}{\partial v_3}v_0\mathbf{K} & \frac{\partial}{\partial v_3}v_1\mathbf{J} & -\frac{\partial}{\partial v_3}v_2\mathbf{I} & -\frac{\partial}{\partial v_3}v_3\mathbf{1} \end{bmatrix}. \tag{85}$$

Now we use the property of the orthogonal basis, so $\frac{\partial}{\partial v_\mu} v_\nu = \delta_{\mu\nu}$:

$$\frac{\partial}{\partial V_\mu}\widetilde{V}^\nu = \begin{bmatrix} -1\mathbf{1} & 0\mathbf{I} & 0\mathbf{J} & 0\mathbf{K} \\ 0\mathbf{I} & -1\mathbf{1} & 0\mathbf{K} & 0\mathbf{J} \\ 0\mathbf{J} & 0\mathbf{K} & -1\mathbf{1} & 0\mathbf{I} \\ 0\mathbf{K} & 0\mathbf{J} & 0\mathbf{I} & -1\mathbf{1} \end{bmatrix}. \quad (86)$$

Then we multiply G^ν with the result, giving

$$(\frac{\partial}{\partial V_\mu}\widetilde{V}^\nu)G^\nu = \begin{bmatrix} -1\mathbf{1} & 0\mathbf{I} & 0\mathbf{J} & 0\mathbf{K} \\ 0\mathbf{I} & -1\mathbf{1} & 0\mathbf{K} & 0\mathbf{J} \\ 0\mathbf{J} & 0\mathbf{K} & -1\mathbf{1} & 0\mathbf{I} \\ 0\mathbf{K} & 0\mathbf{J} & 0\mathbf{I} & -1\mathbf{1} \end{bmatrix} [\mathbf{i}g_0\mathbf{1}, g_1\mathbf{I}, g_2\mathbf{J}, g_3\mathbf{K}] = \begin{bmatrix} -\mathbf{i}g_0\mathbf{1} \\ -g_1\mathbf{I} \\ -g_2\mathbf{J} \\ -g_3\mathbf{K} \end{bmatrix} = -G_\mu. \quad (87)$$

The result of this part is

$$-(\frac{\partial}{\partial V_\mu}\widetilde{V}^\nu)G^\nu = G_\mu. \quad (88)$$

For the second part,

$$-(\frac{\partial}{\partial V_\mu}\widetilde{G}^\nu)V^\nu, \quad (89)$$

we have two options. The first is the easiest, assuming particle velocity and particle momentum to be independent properties, which makes this part zero and gives us the end result

$$\frac{\partial \mathcal{L}}{\partial V_\mu} = -\frac{\partial}{\partial V_\mu} \widetilde{V}^\nu G^\nu = -(\frac{\partial}{\partial V_\mu} \widetilde{V}^\nu) G^\nu = G_\mu. \tag{90}$$

In the case that $\mathcal{L} = \widetilde{J}^\nu A^\nu$ this assumption is allowed.

The second option is that particle velocity and particle momentum are mutually dependent through the relation $G^\nu = \rho_i V^\nu$, with ρ_i as the inertial mass density. In that case we have to go back to the original equation. If we assume a velocity independent mass density this gives

$$\frac{\partial \mathcal{L}}{\partial V_\mu} = -\frac{\partial}{\partial V_\mu} \widetilde{V}^\nu G^\nu = -\rho \frac{\partial}{\partial V_\mu} (\widetilde{V}^\nu V^\nu) = \\ -\rho \frac{\partial}{\partial V_\mu} (v_0^2 - v_1^2 - v_2^2 - v_3^2) = 2G_\mu. \tag{91}$$

The last situation is assumed in relativistic gravity, where the stress energy tensor is given by $\rho_i \widetilde{U}_\nu U^\nu$. In that situation could be tempted to choose the Lagrangian as $\mathcal{L} = \frac{1}{2}\rho_i \widetilde{U}^\nu U^\nu$ in order to preserve the outcome

$$\frac{\partial \mathcal{L}}{\partial V_\mu} = G_\nu. \tag{92}$$

This is done for example by Synge in his book on relativity ([8], page 394).

But that is outside our scope. So we have to restrict the use of

$$\frac{\partial \mathcal{L}}{\partial V_\mu} = -\frac{\partial}{\partial V_\mu} \widetilde{V}^\nu G^\nu = G_\nu \tag{93}$$

to the situations in which V_μ and G_μ are independent of each other.

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