Extension of Schiff's Gravitational Scaling Method to Compute the Precession of the Perihelion of Mercury

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A theoretical approach proposed by Schiff in 1960 to determine the angle of displacement of star images during solar eclipses has been extended to enable calculation of planetary trajectories around the Sun. A key assumption in Schiff's method is the concept of the gravitational scaling of physical units implied by Einstein's equivalence principle. In the present work, the same scaling procedure has been employed for the radial and perpendicular velocity components of an object moving at a different gravitational potential than the observer. Rather than follow Schiff's suggestion that an equation of motion for the planet be provided as input for the calculations, however, it has been assumed instead that the required information about the acceleration due to gravity can be obtained directly from a relativistic modification of Newton's inverse square law (ISL) with appropriate scaling for an observer on Earth. Numerical

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calculations of the relativistic contribution to the precession of the perihelion of planetary orbits around the Sun are carried out on this basis, and the results are found to agree with observed values to within experimental error (43".0033/cy calc. and 43".2±0.9/cy obs. for Mercury, for example; Einstein's value is 43".0076/cy), and to vary in the same manner with mean radius and eccentricity of orbit and gravitational mass of the Sun as is predicted by Einstein's general theory of relativity (GTR). (Style: Abstract text)

Keywords: gravitational scaling of units, planetary orbits, Schiff's method, Newton's inverse-square law, general relativity, trajectory calculation.

I. Introduction

The Universal Gravitation Theory of Newton stood unchallenged for over two centuries, and led to many spectacular advances in the understanding of astrophysical phenomena. After Einstein [1] introduced his special theory of relativity (STR), he quickly turned his attention to its extension for gravitational forces [2]. Three main applications were considered: the gravitational red shift, the displacement of star images by massive objects, and the precession of the perihelion of Mercury's orbit around the Sun. Early attempts to adapt Newton's inverse-square law (ISL) to the requirements of STR were only partially successful, as for example in Sommerfeld's [3] treatment of planetary motion. Einstein [4] ultimately succeeded in obtaining a quantitative description of all three of the above phenomena with his development of the general theory of relativity (GTR).

It is often said that GTR grew out of the failure to find a suitable means of combining Newton's ISL with STR [5]. In its final form GTR has been described as one of the greatest achievements of the human mind [6], and the elegance of its mathematical formulation has inspired generations of theoretical physicists. Nonetheless, over the years there have been a number of attempts to find alternative gravitational theories, but subsequent experimental tests have invariably come down on the side of GTR [7]. Indeed, the very success of GTR has led to the view, held by Einstein himself [8], that STR is a flawed theory, and that the ISL is only a good approximation whose validity does not extend beyond weak gravitational fields.

Yet both the latter theories have enjoyed remarkable success in a variety of applications, which at least raises the theoretical possibility that a viable synthesis may indeed exist, but has simply never been found. In the following report, this possibility will be explored in detail, beginning with consideration of Einstein's prediction of the gravitational red shift [2]. A computational method introduced by Schiff [9] to compute the angle of displacement of star images in the gravitational fields of the Sun and stars will prove quite useful in this discussion.

II. Gravitational Scaling of Units

The gravitational red shift can be explained without the full GTR apparatus, and is thus not really a test of the latter. Einstein [2] argued on the basis of his equivalence principle that the magnitude of a given light frequency increases by a factor of $1+\frac{gdh}{c^2}$ as the source is raised by a distance dh in a gravitational field of local magnitude g (c is the speed of light in free space). The derivation of this result assumes [2, 10] that the gravitational mass m_G of an object is equal to its inertial mass m_I (weak equivalence principle), and otherwise makes use of the ISL and the well-known result of STR for the energy E of an object,

$$E = m_I c^2 = \gamma(u)\mu c^2, \tag{1}$$

where μ is its proper mass, u is its speed and $\gamma(u)=\left(1-\frac{u^2}{c^2}\right)^{-0.5}$.

When an object falls between the above two potentials, respective local observers measure different energies, an effect which in classical physics is regarded as the conversion of gravitational potential energy $m_G g dh$ into an equal amount of kinetic energy. Einstein instead explained it as resulting from the change in the unit of energy as the distance from a gravitational source is varied [11]. He went on to argue [2] based on the Doppler effect [10] that the unit of light frequency ω changes in exactly the same proportion as the energy, which in turn is consistent with Planck's radiation law of quantum mechanics [12]. Terrestrial experiments by Pound and Rebka [13] have verified Einstein's result to an accuracy of 5%, and subsequent work has lowered the possible discrepancy to at most 1% [14].

Einstein extended this result to other temporal processes such as reaction rates (*jeder physikalische Prozess* [2]), concluding that the unit of time decreases with gravitational potential by the same factor as the energy increases. He also gave an argument [2, 10] indicating that the unit of distance measured parallel to the gravitational field increases by the same factor, while that for distances measured perpendicular to the field is unchanged. The above results are only valid for infinitesimal variations in gravitational potential, but it is a simple matter to eliminate this restriction by carrying out an appropriate integration between any two distances from the gravitational source. A convenient means of incorporating this extension into the theory is to define a factor A_p, such that

$$A_p = 1 + \int_{R_p}^{\infty} \frac{g dR}{c^2} = 1 + \frac{GM_s}{c^2 R_p}, \tag{2}$$

where G is the universal gravitation constant, M_s is the gravitational mass of the source, and R_p is the distance of the object from the source. Accordingly, the ratio of the radiative frequency/energy observed at R_o to that generated at R_p is $\frac{A_o}{A_p}$.

If one assumes that eq. (1) holds locally at both R_o and R_p, it follows from the energy conservation principle that for macroscopic bodies the exact ratio is $\frac{\gamma(u_o)}{\gamma(u_p)}$, where u_o and u_p are the respective speeds of the object measured locally as it falls (rises) between R_p and R_o. In other words, the exact definition of A_p must ensure that

$$\frac{\gamma(u_p)}{A_p} = \frac{\gamma(u_o)}{A_o} \tag{3}$$

as the object's distance from the gravitational source is varied (assuming that no other forces are present). One can conveniently summarize the above discussion by giving the exponent n of $S = \frac{A_o}{A_p}$ by which the units of various quantities vary with gravitational potential (Table 1).

Table 1. Gravitational scaling factors employed in the present study. For a given quantity X, X_p is its value when the object is at rest at the same gravitational potential as the observer P. The value given in the second column for each quantity is for an observer O who in general is located at a different gravitational potential than the object but is also at rest with respect to it. The corresponding power n of the scaling factor $S = \frac{A_o}{A_p}$ is given the third column [the A_p, A_o factors are defined in eq. (2)].

Quantity	Gravitational scaling factor	n
Energy (E)	$\left(\frac{A_o}{A_p}\right)E_p$	1
Frequency (ω)	$\left(\frac{A_o}{A_p}\right)\omega_p$	1
Time (Δt)	$\left(\frac{A_p}{A_o}\right)\Delta t_p$	−1
Distance (L)	L_p	0
Speed of light[a] (c)	$\left(\frac{A_o}{A_p}\right)c$	1
Radial velocity[b,c] (v^r)	$\left(\frac{A_o}{A_p}\right)^2 v_p^r$	2
Transverse velocity (v^t)	$\left(\frac{A_o}{A_p}\right)v_p^t$	1
Inertial mass (m_I)	$\left(\frac{A_p}{A_o}\right)(m_I)_p$	−1
Gravitational mass (m_G)	$(m_G)_p$	0
Acceleration due to gravity[b] (g)	$\left(\frac{A_p}{A_o}\right)^3 g(P)$	−3

[a] at the gravitational potential of observer P
[b] only used in Schiff's procedure for trajectory calculations. Note that observer O is located at infinity (A_o=1) in this case
[c] *instantaneous* radial velocity components are scaled the same as transverse; see text

III. Analysis of Schiff's Method: Scaling of Distance and Velocity

To consider further how these units vary with gravitational potential, it is instructive to analyse Schiff's method [9] for computing the angular displacement of the images of stars that occurs during solar eclipses. Following Einstein [10], it is assumed that the units of time and distance vary with gravitational potential. A Cartesian coordinate system is chosen with the Sun at its origin. A series of observers *co-moving with the Sun* are to measure the location and velocity of a light ray that starts at infinity and passes close to the Sun on its way to the Earth. One of these observers (*O*) is located at infinity, and it is his objective to map out the trajectory of the light ray employing his set of units. He is always in communication with that observer (*P*) who is momentarily at the same gravitational potential as the light ray. The latter is conveniently referred to as the local observer. Note that the identity of observer *P* is constantly changing from one period to another, however.

A key assumption in Schiff's approach is that the speed of the light ray observed by *P* is always equal to c and he finds that it is travelling in a straight line in all cases. For an infinitesimal period while the light is at his gravitational potential, *P* resolves its velocity into its radial (vl^r) and transverse (vl^t) components relative to the Sun. This information is eventually passed on to *O* at infinity [with A_o=1 according to eq. (2)], who then employs a scaling procedure along the lines discussed in the previous section to obtain the corresponding values for each of these velocity components from his perspective. For this purpose he also needs to know *P*'s momentary distance from the Sun, R_p. The latter is assumed to be constant throughout the current time interval. He employs this value to compute A_p according to eq. (2) and then obtains the gravitational scaling ratio

$S = \frac{A_o}{A_p} = \frac{1}{A_p}$ which he uses to determine his own values for the light velocity components by multiplying vl_t with S and vl_r with S^2 (see Table 1). This is sufficient information for *O* to compute the speed c' of the light ray in his units at *P*'s gravitational potential.

In order to compute the angle by which the light ray appears to be bent by the gravitational field of the Sun, more information is required, however. This is because the criterion employed to determine this angle ($d\Theta$), both by Einstein [10] and Schiff [9], is Huygens' principle:

$$d\Theta = \left(\frac{dc'}{dy}\right)\frac{dx}{c}, \tag{4}$$

where y is the lateral distance from the Sun and dx is a distance interval along the light path itself (the total angular displacement is obtained by integrating over dx). Schiff was able to obtain the various quantities in this equation analytically and his result is in perfect agreement with the GTR expression originally given by Einstein [4].

It is interesting to implement Schiff's approach by employing a finite time-step procedure. First of all, it is clear that *two* light beams are needed, separated by a small distance dy. More importantly, one has to have a definite location for each of them at every stage of the computation. The question thus arises: whose coordinates do we use, *O*'s or *P*'s? Because of the gravitational scaling, the trajectory calculated by *P* is not the same as that found by *O* using the above procedure. Moreover, since the location of *P* changes continuously throughout, including his position in the gravitational field of the Sun, this fact introduces some uncertainty as to how to correctly define the "local" trajectory of the light ray over its entire path.

Despite these questions, it seems clear that one should be able to devise a computational procedure that obtains the same result as Schiff for the angular displacement of star images. It was therefore decided to simply use *the same path for each of the local observers* for this purpose. This means that each light ray travels along a straight line and the local observers always agree on its current location, including its distance from the Sun. But the observer at infinity (*O*) must find a different trajectory according to the gravitational scaling procedure employed in Schiff's method. The deviation between the corresponding paths computed by *O* and *P*, respectively, becomes considerable when the light approaches quite closely to the Sun. The situation is illustrated in the schematic diagram shown in Fig. 1.

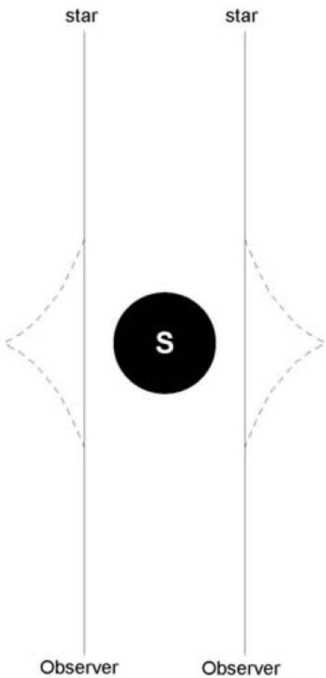

Fig. 1. Diagram illustrating the "pseudo-trajectory" inferred from the velocity vector computed for the observer on Earth in Schiff's procedure (see [9]). Note that on the initial approach the light appears to veer away from the Sun (convex trajectory) because the gravitational scaling reduces the magnitude of the radial component relative to the local straight-line path. This result demonstrates that the direction of the latter velocity is ignored in Schiff's method, which nonetheless obtains perfect agreement with Einstein's value for the angle of displacement of star images during solar eclipses because of its reliance on Huygens' principle to define this angle. General relativity employs the same definition for the displacement angle.

The trajectory computed by *O* using Schiff's method directly is not concave to the Sun, contrary to what is concluded in the original paper [9], but rather is Λ-shaped. It veers away from the Sun on its initial approach because the ratio of the radial and transverse components of the light velocity, $\frac{vl^r}{vl^t}$, observed by *P* is reduced upon scaling for O ($A_o=1<A_p$, $S<1$). This trend continues until the radial component vanishes as it passes the solar midpoint (see Fig. 1). Thereafter, the direction of the trajectory turns back toward the straight line assumed for the local observers because the radial component of the light velocity is now pointed away from the Sun. It is important to note, however, that this result actually has no bearing on the computation of the angle of displacement in Schiff's procedure because only the *magnitude* of the light velocity (i.e, the speed c') is required in eq. (4), not its direction.

Results that are consistent with Schiff's analytical value for Θ are obtained when one uses the *local trajectory exclusively* to obtain the required value of R_p at each point along the trajectory. The distance travelled along this trajectory from *O*'s perspective is then obtained by multiplying the elapsed time $dt(O)$ with his observed light speed $c'(O)$. Since the latter value is always less than *c*, this procedure leads to a decrease in the distance the light travels in a given time interval on *O*'s clock relative to what he would have found in the absence of a gravitational field. The effect increases as the lateral distance from the Sun is reduced. The result is that the light ray farther from the Sun arrives at *O*'s position on Earth sooner than the other, but that both travel along a perfectly straight-line trajectory. One can interpret this as a rotation of the wave front of the light that left the star, whereby the direction of the rotation is always *away from the Sun*, in agreement with observation (more details concerning the above numerical calculations may be found elsewhere [15]).

In the present context, however, the most important observation from the above discussion is that *radial distance is actually not scaled* in Schiff's procedure. Instead, the transverse and radial *velocity* components, vl^t and vl^r are scaled as S and S^2, respectively (they are smaller for O at infinity than for P near the Sun; see Table 1). The former result is consistent with the scaling of time, that is, $dt(O) = \frac{dt(P)}{S}$ if one assumes that the distance $l^t(O) = l^t(P)$. The latter is *at least not inconsistent* with the corresponding relation for radial distances. It simply indicates that O must scale *the radial velocity component* with an additional factor of S in order to obtain *timing* results that agree with experiment (note that the value of A_p changes with radial motion, unlike the case in a direction transverse to the field). If this is not done, a value for the total angular displacement is predicted that is only half of that observed, as shown explicitly in Schiff's work [9]. Indeed, this was the result obtained in Einstein's 1911 paper [10] that was later corrected by him on the basis of GTR [14]. As shown in Fig. 1, however, if one tries to rationalize this difference in scale factors for the two velocity components by claiming that radial distances observed by O are S times as large (recall that $S<1$) than for P, the conclusion is that O's observed trajectory is Λ-shaped, not always concave to the Sun as usually assumed. More importantly, basing O's measurement of R_p on this trajectory no longer produces a result for Θ that is consistent with experiment and GTR. In summary, Schiff's scaling procedure ultimately provides the ratio of the speeds of light observed by P and O, respectively. *The direction is always the same for both, however, namely in a straight line.* They also always agree on the current position of the light ray. The computed difference in speeds simply translates into *a distinction in the elapsed times* required for the light to traverse a given portion of this trajectory based on their respective clocks.

IV. Scaling of the Acceleration Due to Gravity

In the above calculations it is assumed that the speed of light is equal to c for a local observer. As Schiff pointed out in his work [9], the situation is not so simple when dealing with the motion of planets moving at variable speeds that are much smaller than *c*. In classical gravitational theory Newton's ISL is employed to compute *g*, the acceleration due to gravity, and the motion of the planets is then computed on this basis. As noted in the Introduction, this approach has been quite successful historically and it has been shown to give results of high accuracy in many applications. At the same time, it is clear that relativistic effects are not adequately accounted for in the classical theory and thus to further improve accuracy, this deficiency must be removed, at least to a very good approximation. If one attempts to exploit Schiff's method for this purpose, it is necessary to consider how the gravitational acceleration varies with both the state of motion of the object and its relative position in a gravitational field. This approach will be pursued below.

The first step is to assume that the ISL is directly applicable when the object (passive mass) is not moving relative to the observer and is located at the same gravitational potential. In this case the acceleration due to gravity is given as

$$a = g = \frac{GM_S}{R^2}, \tag{5}$$

in the radial direction, exactly as Newton stated in the 17th century [see eq. (2) for definitions of the symbols]. In Schiff's approach, however, an observer (P) who is co-moving with the Sun and is at the same gravitational potential as the object must provide information about its current location and velocity. It is therefore necessary to know how an object's acceleration varies with its velocity relative to the observer.

According to the special theory of relativity (STR), the transverse component of the acceleration vector is damped by a factor of $\gamma^{-2}=\left(1-\frac{u^2}{c^2}\right)$ when the object is moving at speed u relative to the observer [16]. This result is a consequence of Einsteinean time dilation, that is, because clocks run more slowly on the moving object, the local acceleration is larger by a factor of γ^2 than for a stationary observer. Ascoli [17] has argued that the same relationship holds for the radial component. His position is based on the Einstein velocity addition formula [18] for motion in a direction radial to a gravitational field [19], although it is clear that the same argument does not hold for motion in other directions. In a companion paper, [20] it is shown that a generally valid derivation of Ascoli's result can be given, however, by simply taking account of the way the variables in eq. (5) vary with u. Specifically, the distance r varies in direct proportion to γ, whereas the gravitational mass M_S is completely independent of u, so that g must vary as γ^{-2}.

On this basis it has been assumed in the computational procedure to be described below that the acceleration due to gravity for the stationary observer P at the same gravitational potential as the object is

$$g(P)=g(M)\gamma^{-2}(u_p), \tag{6}$$

where $g(M)$ at the location of the object is computed from the ISL in eq. (5), u_p is the speed of the object relative to P, and the acceleration is radial toward the active mass (Sun). Implicit in the above procedure is the assumption that the gravitational mass of the Sun is the same for all observers, independent of both their position in the gravitational field and their state of relative motion [20]. In other words, observer P employs the *same* values for both M_s and R in eqs. (5, 6) that O measures for these quantities at the gravitational potential of the Earth, consistent with Schiff's approach [9]. It is interesting to note that eq. (6) is also consistent with Schiff's assumption that light always travels at a constant velocity for a local observer. This follows from the fact that u_p=c in this case, so that the acceleration due to gravity observed for light is always equal to zero [17]. A more extensive discussion of this point is given elsewhere [15, 20, 21].

It is important to see that eq. (6) is consistent with Galileo's unicity principle (Eötvös experiment), since this only requires that the inertial and gravitational masses of all objects always be *in the same proportion* for any given observer. The proportionality constant is simply γ. One consequence of this relationship is that the gravitational mass of a photon (or any other system with null proper mass) is zero for all observers, even though its inertial mass ($\frac{h\nu}{c^2}$ in free space) varies with the relative speed of the observer to the light source (Doppler effect). This result is thus consistent with Newton's Third Law, since it indicates that a photon is incapable of exerting a gravitational force on any other object on this basis. Since photons always move with speed c for a local observer, according to Ascoli's result of STR [17] their local acceleration due to a gravitational field is also always zero. Thus, there is neither action nor reaction in this case.

A conceivable means of employing $g(P)$ to compute the trajectory of the object would be to simply use it to compute the change in velocity in a given time slice for the observer P at the local gravitational potential. The final velocity could then be scaled according to the scheme employed in Schiff's computation of the angular displacement of star images to obtain the corresponding result observed by O. This procedure does not give satisfactory results, however, and must be rejected. The latter scheme implicitly assumes that the values of the distance R_p and gravitational mass M_s that O uses to evaluate g with the ISL are the same as for P. There is precedence for this not being the case, however, namely in the scaling of the radial component of the velocity discussed in the preceding section. In that case the scaling is done in such a manner as if the radial distance itself needs to be scaled by a factor of $S = \frac{1}{A_p}$. If the analogous procedure is employed to evaluate $g(O)$, it would mean multiplying $g(P)$ in eq. (6) by a factor of S^{-2}.

Once one decides that R_p must be scaled differently in the ISL than in computing the actual location of the object, however, another possibility emerges, namely that M_s also needs to be scaled to obtain g (O) from $g(P)$. For this purpose it is instructive to consider how the inertial mass m_I scales with the gravitational potential of the observer [20]. This can be done by assuming that eq. (1) holds for all observers. Since both the energy E and the speed of light c scale as S, it therefore follows that m_I must scale as S^{-1}, the same as elapsed times (see Table 1). If one assumes that the gravitational mass M_s in eq. (6) must be scaled in the same manner, the result is

$$g(O) = A_p^3 g(P), \tag{7}$$

that is, *O* measures the acceleration due to gravity to be S^{-3} times *larger* than *P* (Table 1, $A_p > A_o = 1$). Altogether then, the value of *g* that *O* needs to use in his computation of the object's trajectory is [20]

$$g(O) = \gamma^{-2}(u_p) A_p^3 g(M), \tag{8}$$

where *g*(*M*) is the value obtained directly from the ISL in eq. (6). Employing this scaling procedure leads to results for planetary trajectories that are in quite good agreement with both experiment and GTR, similarly as for the angular displacement of star images in Schiff's original work [9].

V. Calculation of Planetary Orbits

A. Computational Procedure

The following procedure has been adopted to compute the trajectories of objects moving in a gravitational field based on the above considerations. It is assumed that the initial velocity $\boldsymbol{u_o}$ and position $\boldsymbol{P}$ of the object are known relative to a primary (stationary) observer *O* located at infinity (A_o=1). A coordinate system is adopted such that the Sun (in the general case, the gravitational source) is at the origin and it is assumed that *O* is co-moving with the Sun. The value of the scaling quantity A_p is calculated according to eq. (2), from which the key ratio $S = \frac{1}{A_p}$ is obtained. This allows *O* to compute the *local* velocity $\boldsymbol{u_p}$ measured by another observer (*P*) who is also co-moving with the Sun but is located at the same gravitational potential as the object (planet). Based on the discussion in the previous Section, *O* simply has to take into account the difference in clock rates for the two observers. Since *P*'s clock runs $\frac{A_p}{A_o} = A_p$ times *slower* than *O*'s (see Table 1), his value of the object's velocity is A_p times greater (A_p>1), i.e., $\boldsymbol{u_p}=A_p\boldsymbol{u_o}$. This conversion is only made to obtain an initial value for $\boldsymbol{u_p}$. In succeeding time cycles, the value of $\boldsymbol{u_p}$ obtained at the end of the previous cycle will be used for this purpose.

The next step is to compute the acceleration exerted on the object by the gravitational field of the source. To this end it is assumed that the ISL is valid for an observer *M* who is at the same gravitational potential and is at rest with respect to the object. The corresponding value used by *O* is given by eq. (8), i.e. by using the current value of $\boldsymbol{u_p}$ in conjunction with the ISL value at the location of the object, *g*(*M*).

The above information allows one to compute the change of velocity of the object over a small time interval Δ*t* in *O*'s system of standard units. To do this, however, he must use Schiff's procedure to convert $\boldsymbol{u_p}$ to the corresponding value in his units ($\boldsymbol{u_o}$), as indicated in Table 1. This means he must first resolve $\boldsymbol{u_p}$ into its transverse and radial components, $u_p{}^t$ and $u_p{}^r$, and then *divide* these values by A_p and A_p^2, respectively. It should be noted that this is not just the inverse of the scaling procedure used above to obtain the initial value of $\boldsymbol{u_p}$ from $\boldsymbol{u_o}$, in which case we would simply divide all components uniformly by A_p. The reason for making this distinction will be discussed below, but first let us compute the change in the object's velocity from *O*'s perspective as:

$$\Delta \boldsymbol{u_o} = \boldsymbol{g}(O)\,\Delta t(O), \tag{9}$$

with $\boldsymbol{g}$(O) radial to the gravitational field. The velocity at the end of the time interval is then obtained by vector addition employing the velocity addition rule of STR [18]. This is an important point since use of simple vector addition of $\Delta\boldsymbol{u_o}$ to the original value of $\boldsymbol{u_o}$ in each time cycle causes significant accumulation of error over a complete orbital period.

The final velocity $\boldsymbol{u'_o}$ is then scaled using Schiff's procedure (Table 1) to obtain the corresponding local value $\boldsymbol{u'_p}$, that is, by multiplying the radial component by A_p^2 and the transverse by A_p. The distance Δs_o travelled by the object in the current time cycle from O's perspective is computed by multiplying the average velocity $\boldsymbol{u_o^a} = \frac{(\boldsymbol{u_o} + \boldsymbol{u'_o})}{2}$ by $\Delta t(O)$. The direction taken is that of the average *local* velocity $\boldsymbol{u_p^a} = \frac{(\boldsymbol{u_p} + \boldsymbol{u'_p})}{2}$, however, not that of $\boldsymbol{u_o^a}$. Note that since there is no gravitational acceleration of light in Schiff's method for computing the angular displacement of star images [9], the magnitude of $\boldsymbol{u_p^a}$ is always equal to c in this case and its direction is constant as the light passes by the Sun. Taking the direction the light follows to be the same as that of $\boldsymbol{u_o^a}$ in that application leads to inaccuracies in both the trajectory and the displacement angle, as discussed in Sect. III. The final location of the object $\boldsymbol{P'}$ at the end of the cycle is thus computed as

$$\boldsymbol{P'} = \boldsymbol{P} + \left(\frac{\boldsymbol{u_p^a}}{u_p^a}\right)\Delta s_o \,. \tag{10}$$

It is important to see that all observers who are co-moving with O must measure exactly the same value for $\boldsymbol{P'}$ according to Table 1. They will only disagree on the amount of elapsed time for this portion of the object's trajectory because their respective clocks run at different rates depending on their position in the gravitational field [$\Delta t(\mathrm{P})=\dfrac{\Delta t(O)}{A_p}$]. In essence, *O*'s location at infinity makes him the ideal neutral observer. He and he alone can apply Schiff's scaling procedure to obtain the object's trajectory in his system of units (A_o=1), and this information can then be converted to the units of any other observer simply by knowing the latter's value of A_p.

In the specific computational approach adopted in the present work, there is another matter that needs to be clarified, however. Both $\boldsymbol{u_o}$ and $\boldsymbol{u_p}$ are continuous functions of time, but only one of them can remain the same in going from the end of one time cycle to the beginning of the next. This is because the distance of the object from the source is constantly changing, and therefore the value of the scaling parameter A_p generally varies between successive cycles. In view of the success of Schiff's approach to the calculation of the displacement of star images caused by the gravitational field of the Sun, in which case both the magnitude and the direction of the local light velocity $\boldsymbol{u_p}$ are held constant throughout, it seems preferable at the beginning of each cycle to set $\boldsymbol{u_p}$ equal to the value of $\boldsymbol{u'}_p$ at the end of the previous one, as already mentioned. In so doing, one must accept the fact that this choice generally precludes the existence of a similar equality between the corresponding values of $\boldsymbol{u_o}$ and $\boldsymbol{u'}_o$ for the primary observer in going from one time cycle to the next, but this is inconsequential because in the last analysis these quantities as defined are only an artefact of Schiff's method.

B. Results of the Calculations

The above procedure has been applied to the calculation of the relativistic contribution to the advancement angle of the perihelion of planetary orbits around the Sun. At the start of the calculation the position and velocity of the planet are taken from experiment (based on the observed values for the mean radius r and eccentricity e of a given orbit). The solar mass is taken to be 1.99×10^{30} kg and the mass of the planet is not required, consistent with the unicity principle. The time interval $\Delta t(O)$ for each cycle in the numerical procedure has been varied in all cases to insure that a proper degree of convergence is obtained for the calculated results (quadruple precision has been used in all computations).

The value of the precession angle Θ of the perihelion of Mercury's orbit around the Sun obtained from the present treatment is 43".0033/cy, in good agreement with both the currently accepted experimental value for this quantity of 43".2 ± 0".9/cy [22] and that computed by Einstein from GTR of 43".0076/cy [4, 23]. In the latter work he obtained a closed expression [23, 24] which indicates that the precession angle in general is proportional to M_s and inversely proportional to both r and $(1-e^2)$. Tests have therefore been carried out for different values of the latter three quantities, and very good agreement with the predictions of GTR has been found in all cases. Indeed, since the amount of computer time required increases with r, most of the tests carried out are for a hypothetical planet with one-thousandth of Mercury's radius and therefore a period of revolution around the Sun of only 240 s. When the solar mass is increased by a factor of 10.0, it is found that the value of Θ is 10.0012 times greater. If the mean radius is cut in half, Θ is found to increase by a factor of 1.9990. Similarly good agreement with GTR is obtained if the radius is changed by factors of 10 and 100. Finally, when e is changed from its experimental value of 0.2056 for Mercury to 0.10, the value of Θ is found to be 0.9677 times smaller, as compared to the predicted factor of 0.9674.

The A_p factors have been computed in the present treatment in two different ways: by means of eq. (2) in each time-step, or by making use of the proportionality relationship of eq. (3) after using eq. (2) to obtain an initial value only. The corresponding two values of Θ agree to within a factor of 1.000093, with that obtained with the latter definition being higher. This result thus clearly supports the conclusion that the whole concept of gravitational scaling is rooted in the conservation of energy principle (see Sect. II).

One can summarize the above results as follows. The present theoretical approach obtains results that are consistent with all known measurements of perihelion precession angles, *including those of Earth and Venus*. They are also in nearly quantitative agreement with the predictions of GTR for the same quantities. The procedure employed can be viewed as a generalization of Schiff's method [9] for computing the displacement angle of star images during solar eclipses (strictly speaking, what is actually calculated via Huygens' principle is the angle by which the wave front of the light rotates relative to its starting orientation at the star [15, 21]). Corresponding tests have been carried out with the present procedure for an object moving with local speed *c*, and excellent agreement with Schiff's (and therefore Einstein's) result has been obtained, including the dependence of this angle on the mass of the gravitational source and the distance of closest approach by the light. A more detailed discussion of these results for the displacement of star images is given elsewhere [15].

VI. Conclusion

For nearly an entire century physicists have held steadfastly to the belief that it is impossible to construct a viable gravitational theory based on Newton's ISL and Einstein's STR. The present work has investigated this position anew by assuming that all observers who are not in relative motion to one another must agree on measurements of distance and gravitational (but not inertial) mass independent of their respective positions in a gravitational field. On this basis, one can obtain accurate predictions of trajectories so long as one assumes further that the ISL must be applied *locally* to obtain the correct acceleration due to gravity. The latter result simply needs to be converted to the system of units of an observer located at infinity. There is an additional complication, however, due to the fact that the object is generally in relative motion to the primary observer O. Ascoli [17] has argued that this circumstance implies that the g-field calculated with the ISL must therefore be damped by a factor of $\gamma^{-2}\left(u_p\right)$, where u_p is the speed of the object relative to a local observer *P* at the same gravitational potential as the object but at rest with respect to *O*. A more general derivation of this result has been give above, however, and is discussed elsewhere [20].

The conversion of the local acceleration due to gravity to the units of the primary observer requires that one also know how the inertial mass of an object varies with distance to the gravitational source. In the present work, it is argued that this information can be deduced by assuming that Einstein's $E=mc^2$ relation is universally valid. It is shown in Sect. IV that this leads to the conclusion that inertial mass must scale in the opposite direction with distance from the source as both energy and the speed of light. In addition, it is assumed that the radial distance between masses needs to be scaled in the same way as energy to obtain the gravitational scaling factor for *g*. The acceleration due to gravity in the units of the stationary observer O (located at infinity in Schiff's procedure) is thus obtained from the corresponding value measured by the local observer P by multiplying this value by a factor of $\left(\frac{A_p}{A_o}\right)^3 = A_p^3$. The various components of velocity (transverse and radial) are scaled in the same manner as proposed by Schiff in his original work [9]. The present calculations have demonstrated that the relativistic contribution to the precession angle of the perihelion of planetary orbits can be accurately obtained on a quite general basis by applying this procedure for the gravitational scaling of physical units. There is also an interesting qualitative conclusion that can be drawn on this basis, namely that for strong fields a purely Newtonian description of the motion can greatly *underestimate* the effects of gravity on a given object by failing to adjust the local g-field by the above factor.

The trajectory calculations provide a justification for employing a coordinate system in Euclidean space in which all objects of the universe can be located uniquely. All observers *who are not in relative motion to one another* [20] must agree on this basis with regard to the instantaneous position of each of these objects. As long as one takes proper account of the fact that the units of time, velocity and acceleration vary with one's position in a gravitational field, it is then possible to carry out trajectory calculations exclusively in Euclidean space. The necessary adaptation can be accomplished by inserting a small number of statements in a comparatively simple computer program which otherwise treats planetary motion strictly on the basis of Newton's ISL.

The development of a comprehensive gravitational theory that relies on the local validity of the ISL inevitably raises questions about whether such forces can be transmitted instantaneously across long distances. Newton himself rejected such an interpretation in the strongest terms, but this did not keep him from using the ISL to solve longstanding problems in astrophysics. The fact remains, however, that the above computer program uses time intervals as small as 10^{-4} s to calculate the change in velocity of a planet caused by the Sun which is as much as 7×10^{10} m distant. It is a matter of opinion whether GTR succeeds in eliminating the need for "action at a distance" by introducing the concept of "curved space-time." The present work indicates that the units of physical quantities vary in a precisely predictable manner with the distance of a given location from the gravitational source, suggesting that something like a distance-dependent stationary field exists at all times and therefore does not need to be transmitted to have its effect on any object that is located at that point in space. It has demonstrated that, with proper attention to detail, it is possible to obtain a level of accuracy in trajectory calculations that is comparable to that of GTR by merging the ISL with STR through the gravitational scaling of the above physical units. This experience speaks for the validity of the assumptions that form the basis for arriving at this synthesis, and at least underscores the practicality of the ISL that Newton so skillfully exploited during his lifetime.

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