The Global Positioning System and the Lorentz Transformation

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The Global Positioning System has been hailed as one of the great triumphs of relativity theory. It is based on the principle that the rates of atomic clocks are affected in a well-defined manner by changes in their relative velocity to the Earth and their position in a gravitational field. It also is consistent with the modern definition of the meter as the distance light travels in free space in c^{-1} seconds. Two experiments carried out on GPS satellites are considered that provide further crucial tests of Einstein's special theory of relativity (STR), in particular of its prediction of the non-simultaneity of physical events and Fitzgerald-Lorentz contraction (FLC) of objects in relative motion to the observer. Each of them leads to a contradiction of the Lorentz space-time transformation that is generally considered to be the fundamental basis of STR. The results are consistent with the relativistic velocity transformation, however, and also do not stand in contradiction to the Lorentz invariance of the electromagnetic field, both of which have a clear experimental basis in contrast to the FLC and non-simultaneity. They also do not violate either of Einstein's two fundamental postulates, Galileo's relativity principle and the constancy of the speed of light in free space. Based on these arguments, it is concluded that the laws of physics are indeed the same in all inertial systems, but that the fundamental units in which they are expressed vary with both the state of motion of the observer and his position in a gravitational field.

Keywords: absolute simultaneity, objectivity of measurement, relativistic velocity transformation, alternative Lorentz transformation, isotropic length expansion

I. Introduction

The Global Positioning System (GPS) is based on a well-defined principle, namely that the ratio of the rates of two atomic (or other natural) clocks in relative motion remains constant so long as they do not change either their respective positions in a gravitational field or their relative speed to one another. One knows from relativity theory how this ratio changes when the above condition is not met. This makes it possible to adjust (pre-correct) the frequency of an atomic clock before it is sent into orbit on a satellite so that its rate will be exactly the same as one left behind on the Earth's surface. The fact that GPS works quite accurately in everyday applications is a strong verification of the latter principle. The question that will be considered below is whether this experience with GPS is perfectly consistent with Einstein's special theory of relativity (STR) or instead underscores the need to alter one or more of its fundamental assumptions.

II. GPS Experiments to Test Simultaneity and Length Contraction

Let us assume at the outset that there are two clocks on the satellite, one (S) with its natural frequency and the other (Q) adjusted so that it runs at exactly the same rate as a third clock (E) on the Earth's surface. To simplify matters it is helpful in the present discussion to eliminate gravitational effects on the clock rates. We can do this by assuming that all three clocks are at the same gravitational potential, for example. Under these conditions, the GPS clock (Q) runs R times faster ($R > 0$) than S at all times but at exactly the same rate as E. Two light pulses are emitted in opposite directions on the satellite. According to clock S they arrive at their respective detectors at exactly the same time T. In other words, these two events are simultaneous for an observer on the satellite. We can use the Lorentz transformation (LT) of STR to predict what the situation is from the standpoint of an observer on the Earth. The pertinent equation is:

$$t = \gamma \left(t' + \frac{ux'}{c^2} \right), \tag{1}$$

where t and t' are the measured times on the Earth and on the satellite, respectively, x' is the position of a given detector as measured on the satellite, and $\gamma = \left(1 - \frac{v^2}{c^2} \right)^{-0.5}$ (v is the speed of the satellite relative to the observer). Since the value of x' is different for the two detectors, it follows unequivocally from the LT that these two events are not simultaneous for the observer on Earth. This is an example of the non-simultaneity principle of STR enunciated by Einstein in his original paper [1].

The observer on Earth has employed his local clock (E) to carry out the above timing measurements. We know, however, that the GPS clock Q on the satellite runs at exactly the same rate as clock E at all times, so we can just as well use Q for this purpose. But it runs $R = \gamma$ times faster than clock S at all times. Based on Q, therefore, the two light pulses arrive at their detectors at time $t = RT$ in both cases, that is, simultaneously. The same must be true even if we do not neglect the effects of gravity on clock rates. This result stands in clear contradiction to the above prediction of STR. There are two ways to resolve it. One must either deny the principle of constant clock-rate ratios, which is the experimental basis of the GPS method, or else deny the validity of eq. (1) of the LT, which is one of the most basic components of STR. In fact, this is not a real choice. The evidence from GPS is overwhelming, whereas the non-simultaneity principle of Einstein's original theory has never been verified experimentally [2]. The goal must be to alter relativity theory in such a way as to maintain consistency with all known experimental findings, including all inferences from GPS applications, but without relying further on the LT. In particular, this result calls into question any and all predictions based on the LT that have thus far never been subjected to experimental verification.

This brings us to a second fundamental experiment based on the GPS clock-rate ratio principle. An object is placed on the satellite while it is still on the Earth's surface. Its length is measured to be *L* m. This is done be using the modern definition of the meter as the distance traveled by light in free space in c^{-1} s. The measurement is made using both clocks *E* and *S*, and there is perfect agreement between them since they are not yet moving relative to one another, that is, both clocks find that $\frac{L}{c}$ s elapse as the light traverses the object. Next clock *S* is put into orbit, causing its rate to decrease by a factor of *R* as before (again excluding gravitational effects so as to concentrate on the role of relative motion on the clock rates). Clock *Q* has been pre-corrected so that its rate onboard the satellite is again exactly the same as clock E left behind on the Earth. The length of the object is then measured anew on the satellite. In accordance with the relativity principle (RP), the same result (L m) is obtained as before, that is, an elapsed time of $\frac{L}{c}$ s is registered on clock S for the light to traverse the object.

According to STR, however, an observer on Earth will measure a different value from his perspective because the satellite is now moving relative to him. This prediction again follows from another equation of the LT:

$$x' = \gamma(x - ut)\,. \qquad (2)$$

The Fitzgerald-Lorentz contraction effect (FLC) is derived by setting $x' = L$ m, that is, the value measured on the satellite with clock *S*, and $t = 0$ in eq. (2). The latter choice is made because the positions of the two ends of the object must be measured simultaneously on clock E to obtain a valid result. The conclusion from the FLC is thus that $x = \frac{L}{\gamma}$ m and therefore that the length of the object is γ times smaller after it has been sent into orbit on the satellite ($\gamma > 0$). This result has never been checked experimentally, however, and there has been some skepticism that the above condition of $t = 0$ can actually be achieved in practice. The measurement can be carried out with the help of the GPS technology, however. Clock *Q* on the satellite is known to run at exactly the same rate as clock *E* on the Earth, and thus the measurement can actually be carried out locally. As noted above, clock *Q* runs $R = \gamma$ times faster than clock *S*. This means that it will record the elapsed time for the light to traverse the object on the satellite to be $\frac{RL}{c}$ s, where again $R > 0$ when neglecting gravitational effects on the clock rates, from which one is forced to conclude that the same value for this elapsed time must be measured on clock *E* located on the Earth. This procedure therefore circumvents any difficulties with simultaneous measurement of both ends of the object and also eliminates any need to send light signals back and forth to the satellite in order to obtain the desired result. The speed of light is equal to c on the Earth, however, so the length of the object is therefore determined unequivocally to be *RL* m. Since $R > 0$, the conclusion is therefore that the length of the object has *increased*, not decreased, as a result of its being shot away from the Earth on the GPS satellite. There is another feature of the FLC that has not yet been mentioned, however. The predicted contraction is anisotropic according to the theory. The value of $\frac{L}{\gamma}$ m derived above is only obtained when the object is measured along the x axis, that is, in a direction parallel to that of the satellite's motion relative to the Earth, in which case it is the minimum value. Since the length measurement in the GPS procedure is based entirely on the elapsed time measured on clock *Q*, it is clear that no variation in the dimensions of the object can occur merely because of a change in its orientation. *In short, this experiment proves that objects expand on the satellite, not contract, and that the effect is perfectly isotropic*. Consistent with the RP, however, the observer on the satellite doesn't notice such changes in the dimensions of the objects located there anymore than he is able to detect a slowing down in local natural clock rates. Most importantly, the main conclusion from this experiment is that the predictions of the LT are also not fulfilled for length measurements.

III. Basing Relativity Theory on the Velocity Transformation

The GPS experiments discussed in the previous section show that STR in its present form is in need of modification. Despite the failure of the LT to successfully describe the results of these experiments, it cannot be overlooked that many other aspects of Einstein's theory have proven to be perfectly consistent with measurements of high-speed phenomena. Clearly, any new version of relativity theory must preserve the strong points of STR while removing its deficiencies. To proceed further it is therefore important to review the great successes of the existing theory and only consider modifications that do not in any way come into conflict with any experimental findings that have hitherto been found to be in harmony with it.

The first point to stress is that GPS in entirely consistent with the relativistic velocity transformation (VT). The main purpose of the latter is to successfully account for the experimental fact that the speed of light in free space is the same in all rest frames (as above, the following discussion excludes consideration of gravitational effects). The equations of the VT are given below:

$$v'_x = \eta \left(v_x - u \right) \tag{3a}$$

$$v'_y = \frac{\eta v_y}{\gamma} \tag{3b}$$

$$v'_z = \frac{\eta v_z}{\gamma}, \tag{3c}$$

with $\eta = \left(1 - \frac{u v_x}{c^2}\right)^{-1}$ (v_x, v_y, v_z and v'_x, v'_y, v'_z are the respective components of the velocity of an object for observers in two different rest frames moving with speed u relative to each other along the *x* axis). One can use eq. (3a), for example, to show that a light pulse moving along the *x* axis ($v_x = c$) relative to one of the observers will have the same relative speed ($v'_x = c$) for the other. The VT has been verified experimentally by means of the aberration of starlight at the zenith and also Fizeau's observations of "light drag" in transparent media [3]. It might be thought that the same experiments serve as verification of the LT since it can be used to derive the VT [i.e. by dividing x' in eq. (2) by t' from the inverse of eq. (1) to obtain v'_x in eq. (3a)]. Lorentz [4] pointed out as early as 1899, however, that the condition of constancy of the speed of light only allows the relativistic space-time transformation to be defined within a common factor in each of its equations. This is evident from the definition of velocity as the ratio of distance moved to time elapsed. His remarks apply in particular to the relativistic transformation of Maxwell's equations [5] for which multiplying all space-time derivative operators with a common factor has no effect on the desired invariance property. In short, the LT can be eliminated from the theory and still remain consistent with Einstein's second postulate of STR as well as with the required invariance of the electromagnetic field equations.

As mentioned at the end of the last section, the GPS clock-rate ratio principle is also consistent with Einstein's first postulate, Galileo's RP. The laws of physics are no less valid on the satellite than on the Earth's surface despite the fact that the natural clock rates in the two rest frames are not the same. The symmetry implied by the LT is contradicted by the experience with GPS, however. Clock *S* on the satellite actually runs slower than clock *E* on the Earth and hence, the space-time Lorentz invariance condition implied by the LT is also contradicted by these experiments. Two clocks cannot both be running slower than the other. Clock *Q* on the satellite always runs at the same rate as clock E on the Earth but at all times faster by a factor of *R* than the natural clock *S* in the same rest frame. One can summarize the situation succinctly by stating that the *unit* of time is simply different in the two rest frames. The same holds true for the units of all other physical quantities. There is a *uniform scaling* of these properties that prevents the observer from detecting any changes in his local environment from one inertial system to another. When he looks outside his "window" and carries out measurements of objects in other rest frames, however, the situation is different. He notices, for example, that the time it takes for the Earth to make a full rotation about its axis is different on the satellite than on the ground provided he bases his measurements on natural clocks in each case. Given this state of affairs, it is helpful to restate the RP: *The laws of physics are the same in every inertial system but the units in which they are expressed vary in accordance with the observer's state of motion and also with his position in a gravitational field*.

One therefore needs a different space-time transformation than the LT but one that is valid in all inertial systems independent of what units are used in a given case. This new set of equations must satisfy the principle of simultaneity of events required by the first of the GPS experiments discussed in Sect. II. In other words, one needs the equation, $t = t'$, instead of eq. (1) of the LT, but with the proviso that *the unit of time employed is the same in both inertial systems*. Values measured on clock E on the Earth's surface need to be compared with those based on clock *Q* on the satellite, for example, not the natural clock *S*. This condition is not satisfied by the LT, which is the primary reason why the latter must be discarded as a possible candidate for the space-time transformation in relativity theory. It can be satisfied quite easily [6], however, by multiplying each of the VT eqs. (3a-c) with $t = t'$. The result is:

$$x' = \eta\left(x - ut\right) \tag{4a}$$

$$y' = \frac{\eta y}{\gamma} \tag{4b}$$

$$z' = \frac{\eta z}{\gamma}, \tag{4c}$$

$$t' = t. \tag{4d}$$

In what follows we will refer to this set of equations as the alternative Lorentz transformation (ALT). It is important to note that the units of x', y', z' are the same as for *x*, *y*, *z* in these equations, similarly as for t and t'. It is possible to change units by using a simple scaling procedure. For example, if the results obtained by the observer on Earth are to be converted over to the natural units employed on the satellite in the GPS procedure, it is necessary to divide all distance and timing results by the factor *R* introduced in Sect. II. This is because clock *S* there runs slower than clock *E* on the ground by this ratio. It is not necessary to change the value of the relative speed *u*, however. It is the same for both observers due to the fact that it is a ratio of distance to time. The same holds true for η and γ in these equations, since they are exclusively functions of speeds.

Moreover, the equations themselves are left unchanged by such a scaling procedure because of their linear nature. This is an essential feature of the relativistic space-time transformation, since it must be valid in all inertial systems. This requirement puts a clear restriction on the way in which the spatial and temporal variables are scaled. Ultimately, this is determined by the fact that the speed of light is the same for all observers, since this forces one to use the same scale factor for distances as for time in going from one rest frame to another. This also means that there is an extra step needed in inverting the ALT of eqs. (4a-d). The usual procedure is simply to exchange the primed and unprimed variables and change the sign of the relative speed u. This leaves the units unchanged, however. In order to convert the results measured by one of the observers into those measured by the other, it is also necessary to scale them so as to reflect the differences in their natural units. In a sense, this is not really necessary if the two observers agree on a common set of units. It is essential, however, if the goal is to predict what each of them will measure from their perspective when each uses his own natural units. This procedure insures that events occur simultaneously for both observers, even though their clocks run at different rates. Their measured clock readings simply have to be converted over to a common set of units, which is precisely what occurs in the GPS procedure through the use of clock Q on the satellite which has been pre-corrected to run at the same rate as clock E on the Earth's surface.

The above relationships are perhaps easier to master when one bases them directly on the VT of eqs. (3a-c) since all observers use the same unit of speed (when gravitational effects are excluded from consideration) and thus must agree on all numerical values of an object's velocity $\boldsymbol{v}$ relative to a given origin. Distance and direction traveled can always be obtained as the product of the elapsed time on the observer's clock with $\boldsymbol{v}$. The corresponding results measured by another observer can then be obtained by suitable conversion between their respective natural units. Simultaneity of events is assured by this procedure and the results are still perfectly consistent with both of Einstein's postulates. His belief that events might not be simultaneous for observers in relative motion was based [1] on his lack of understanding the practical consequences of clocks running at different rates, i.e. using different units of time.

The derivation of the ALT given above emphasizes that one needs an additional condition to completely specify this space-time transformation than just the constancy of the speed of light in all inertial systems, as Lorentz has pointed out [4]. A separate condition is needed for the energy-momentum ($E-\boldsymbol{p}$) transformation than for the ALT, however, one that does require Lorentz invariance. In this case it is necessary for the increase in energy of an object as it is accelerated to agree with the classical definition of kinetic energy in the low-velocity limit. The pertinent equations bear a close similarity to eqs. (1,2) of the LT:

$$E = \gamma\left(E' + up'_x\right) \tag{5a}$$

$$p_x = \gamma\left(p'_x + \frac{uE'}{c^2}\right), \tag{5b}$$

and do satisfy the Lorentz invariance condition unlike the ALT. For low relative velocities, $\gamma = 1 + \frac{u^2}{2c^2}$. By definition, $p'_x = 0$ in the object's rest frame, and its kinetic energy K as measured by a stationary observer is $\frac{\mu u^2}{2}$, where μ is its rest mass. In this limit therefore

$$K = E - E' = (\gamma - 1)E' = \left(\frac{u^2}{2c^2}\right)\mu c^2 = \frac{\mu u^2}{2} \tag{6}$$

as required since $E' = \mu c^2$. The point is that just because eqs. (5a,b) and the electromagnetic field equations are Lorentz invariant and agree fully with experimental observations in no way demands that the same be true for the corresponding space-time transformation. As we have seen in Sect. II, the opposite is true. Only by giving up the Lorentz invariance condition can one arrive at a suitable relativistic space-time transformation that is consistent with the clock-rate ratio principle established by the GPS methodology.

Einstein [1] first predicted the slowing down of clocks in relative motion on the basis of the LT, however, and so it is important to consider this success of his theory in the context of its other failures.

Obviously, one can't make a similar prediction based on eq. (4d) of the ALT, that is, the condition of simultaneity. The original prediction is obtained by setting $x' = 0$ in eq. (1) of the LT, which gives $t = \gamma t'$. The fact is that clocks in motion do not always run γ times slower than the observer's clock, however. The experiments carried out with circumnavigating airplanes [7] showed instead that this result is only obtained when one uses a reference clock located on the Earth's polar axis. One can obtain the ratio of clock rates for observers on different airplanes by first calculating their respective γ values relative to this reference clock and then computing the ratio of these two quantities. Hafele and Keating [7] rationalized this result by singling out the polar reference clock as the only one at rest in a truly inertial system. If true, this conclusion would greatly diminish the range of application for relativity theory since it would mean it could only be directly applied for an observer who is at rest in an inertial system. No such restriction is in fact necessary in applying either the VT or the ALT. One simply has to know the clock-rate ratio for any two observers. This defines the conversion factors between their respective sets of natural physical units and hence of the ratios of their measured values for any conceivable property. A more detailed discussion of the way in which the units of physical quantities vary with their state of motion has been given elsewhere [8]. The VT can be applied on an instantaneous basis independent of whether either the observer or the object of the measurement is accelerating at that moment in time. Hence, even in this context the LT does not give a true picture of the experimental situation.

The fact that clock rates vary when they are either accelerated or change their position in a gravitational field clearly needs to be taken into account when making timing comparisons. The situation is made somewhat easier in GPS technology by insuring that the clocks on the satellites maintain nearly constant velocity and altitude relative to the

Earth. When this is not the case, it is still possible to maintain synchronization by systematically changing the clock rates on an instantaneous basis by applying the same formulas that are used to pre-correct the GPS clocks. This procedure was in fact used in the Hafele-Keating experiments [7] to estimate the final timing results for the circumnavigating clocks upon returning to their original starting point. Whether aging is based on natural clock rates or those adjusted in the above manner is an open question, but one that cannot be answered definitively without the aid of independent experiments (Twin Paradox). In a real sense, these results point out a deficiency in using atomic clocks to measure time on a general basis, however, namely that their rates change in a rather complicated way over time. No matter how much “time-slippage” occurs between atomic clocks in relative motion, it is always possible to keep them synchronized on the basis of astrophysical observations. If one’s atomic clock speeds up, for example, this will be evident by using it to measure the time it takes for a planet to rotate about its axis. Such a procedure cannot be justified in Einstein’s original theory (STR) since it rejects the principle of simultaneity of events. The GPS technology, on the other hand, would not be possible if this principle were not operative.

VI. Conclusion

The success of the GPS technology rests on the principle that the ratio of the rates of atomic clocks does not change as long as they remain in constant relative motion to one another and do not change their respective positions in a gravitational field. This experimental result therefore stands in direct contradiction to Einstein’s prediction on the basis of the Lorentz transformation (LT) of the non-simultaneity of events for observers in relative motion. If two times are equal for one of the clocks, they must also be equal for the other. Only their respective clock readings will differ for each of the events. A useful way to describe this state of affairs is to say that the unit of time is simply different for the two clocks in their respective rest frames. The clock-rate ratio principle also shows that the Fitzgerald-Lorentz contraction effect (FLC) predicted by the LT is inconsistent with observation. The lengths of measuring rods must *increase* in strict proportion to the periods of atomic clocks in order for the speed of light to remain the same for observers in different inertial systems. The amount of the increase is independent of orientation, so isotropic expansion of objects upon acceleration is observed rather than the anisotropic contraction predicted by the LT.

In order to remove the above contradictions from relativity theory it is necessary to discard the LT and take account of the fact that the units of all physical quantities vary with the state of motion of the observer. The laws of physics remain the same in all inertial systems, in accordance with Einstein's first postulate, but the numerical values of quantities measured by different observers in relative motion differ in an easily predictable manner as a result. The relativistic velocity transformation (VT) retains its validity in this formulation because it can be derived from an alternative space-time transformation (ALT) that assumes that events are simultaneous for all observers. The unit of velocity is the same in every inertial system (at the same gravitational potential), so observers always agree on the numerical values of these quantities. The ratio of their respective clock rates can then be used as a conversion factor to successfully predict differences in their respective measured elapsed times and distances for the motion of a given object. One of the consequences of using a different space-time transformation in relativity theory is that it no longer satisfies the Lorentz invariance condition. This has no effect on the Lorentz invariance of the electromagnetic field and the energy-momentum transformation, however, because a different condition is required to completely specify the equations in this case. Thus, that part of Einstein's theory is unaffected by incorporating the principle of simultaneity of events to it. This is as it must be, given the overwhelming evidence from experiment that his predictions on this score are perfectly valid.

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