

Mathematical Constraints on Gauge in Maxwellian Electrodynamics

E. Comay*

School of Physics and Astronomy

Raymond and Beverly Sackler Faculty of Exact
Sciences

Tel Aviv University, Tel Aviv 69978, Israel

The structure of classical electrodynamics based on the variational principle together with causality and space-time homogeneity is analyzed. It is proved that in this case the 4-potentials are defined uniquely. On the other hand, the approach where Maxwell equations and the Lorentz law of force are regarded as cornerstones of the theory allows gauge transformations. For this reason, the two theories are *not equivalent*. A simple example substantiates this conclusion. Quantum physics is linked to the variational principle and it is proved that the same result holds for it. The compatibility of this conclusion with gauge invariance of the Lagrangian density is explained. Several alternative possibilities that may follow this work are pointed out.

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1. Introduction

One may regard the equations of motion of a physical system as the fundamental elements of a theory. Thus, the equations of motion can be used for deriving useful formulas that describe properties of the system. However, it is now recognized that other principles play a more profound role. Using this approach, the variational principle, causality and homogeneity of space-time are regarded here as the basis for the discussion. The present work examines these approaches within the validity domains of classical electrodynamics and of the associated quantum physics. Thus, the electrodynamic theory that regards Maxwell equations and the Lorentz law of force as cornerstones of the theory is called here Maxwell-Lorentz electrodynamics (MLE). The theory that relies on the variational principle is called here variational electrodynamics (VE). MLE and VE are very closely related theories. Thus, Maxwell equations and the Lorentz law of force can be derived from the variational principle (see [1], pp. 49-51,70,71,78-80; [2], 572-578,595-597). On the other hand, MLE and VE rely on two different sets of axioms. Therefore, the validity of their equivalence is not a priori clear. The first part of the discussion carried out here analyzes the two approaches within the realm of classical electrodynamics and proves that MLE is *not equivalent* to VE and that VE imposes further restrictions on the theory's structure. Quantum mechanics is strongly linked to the variational approach (see [3], pp. 2-23). Thus, it is proved in this work that the same results are obtained for quantum mechanics.

The specific subject discussed here is the role of gauge transformations and of gauge invariance in MLE and in VE. The

following argument indicates the need for a further clarification of this subject. It is very well known that all terms of a physical expression must have the same dimensions (otherwise, a change in the unit system destroys numerical balance). Now, let $F(q)$ be an analytic function used in a description of a physical relation. If the power series of $F(q)$ contains more than one term (e.g. $aq^m + bq^n$, where a and b are nonzero pure numbers and $m \neq n$), then it is required that q be dimensionless. Thus, for example, the exponential factor used in the Maxwell-Boltzmann distribution takes the form $e^{-E/KT}$ and the product KT has the dimensions of energy. If $F(q)$ belongs to a relativistic expression then covariance arguments prove that q must also be a Lorentz scalar. The wave function's phase $e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$ satisfies the two requirements.

Now, let $\Phi(x^\mu)$ be a gauge function used in VE and its 4-derivative $\Phi(x^\mu)_{,\nu}$ is subtracted from a 4-potential in a gauge transformation. In quantum mechanics, the charged particle's sector contains the gauge dependent factor $e^{ie\Phi(x^\mu)}$ (see [4], p. 78). Note that the symbol e in the exponent denotes the particle's electric charge. Now, in the system of units used here (see later in this Section) the electric charge is a pure number $e^2 \simeq 1/137$. Thus, the analytic properties of the exponential function and the laws described in the previous paragraph prove that in quantum mechanics, the gauge function $\Phi(x^\mu)$ must be a dimensionless Lorentz scalar. As of today, this restriction is not implemented and the standard gauge transformation used in the literature regards $\Phi(x^\mu)$ as a free function of space-time coordinates (see [1], p. 52; [4], p. 78). This example provides a reason for the investigation of the role of gauge transformations which is carried out here.

It is interesting to note that other problems emerging from gauge transformation are already pointed out in the literature (see [2], pp. 222, 223). Thus, in a Coulomb gauge, a transverse electric current is found throughout the entire space, in spite of actual charge localization.

It is proved in this work that if one adheres to VE together with causality and space-time homogeneity then the 4-potentials of electrodynamics are defined uniquely. On the other hand, the 4-potentials play no explicit role in Maxwell equations and in the Lorentz law of force. Hence, one may apply any gauge transformation without affecting MLE. This is the underlying reason for the claim that MLE is *not equivalent* to VE.

In the present work, units where the speed of light $c = 1$ and $\hbar = 1$ are used. Thus, one kind of dimension exists and the length $[L]$ is used for this purpose. Greek indices run from 0 to 3. The metric is diagonal and its entries are (1,-1,-1,-1). The symbol $_{,\mu}$ denotes the partial differentiation with respect to x^μ . A_μ denotes the 4-potentials and $F^{\mu\nu}$ denotes the antisymmetric tensor of the electromagnetic fields

$$F^{\mu\nu} = g^{\mu\alpha} g^{\nu\beta} (A_{\beta,\alpha} - A_{\alpha,\beta}) = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}. \quad (1)$$

In the second Section, the main point of this work is proved for classical physics. The third Section describes a specific example that substantiates the proof included in Section 2. The fourth Section proves that the same results are obtain for quantum physics. Several implications that may be connected to the

analysis presented herein are discussed in the fifth Section. The last Section contains concluding remarks.

2. Gauge Transformations and Variational Electrodynamics

The standard form of the Lagrangian density used for a derivation of Maxwell equations is (see [1], pp. 78-80; [2], pp. 596-597)

$$\mathcal{L} = -\frac{1}{16\pi}F^{\mu\nu}F_{\mu\nu} - j^\mu A_\mu, \quad (2)$$

where the first term represents free fields and the second term represents the interaction of the fields with charged matter. The following analysis examines a closed system of charges and fields. For the simplicity of the discussion, let us examine the fields associated with one charged particle e whose motion is given. This approach can be justified because, due to the linearity of Maxwell equations, one finds that the fields of a closed system of charges is a superposition of the fields of each individual charge belonging to the system. Let us examine the electromagnetic fields at a given space-time point x^μ . Using Maxwell equation and the principle of causality, one can derive the retarded Lienard-Wiechert 4-potentials (see [1], pp. 173-174; [2], pp. 654-656)

$$A_\mu = e \frac{v_\mu}{R^\alpha v_\alpha}. \quad (3)$$

Here v_μ is the charge's 4-velocity at the retarded time and R^μ is the 4-vector from the retarded space-time point to the field point x^μ . These 4-potentials define the fields uniquely.

A gauge transformation of (3) is (see [1], pp. 52-53; [2], pp.

220-223)

$$A'_{\mu} = A_{\mu} - \Phi_{,\mu}. \quad (4)$$

In the following lines, the laws of VE are used in an investigation of the form of the gauge function $\Phi(x^{\mu})$.

Relying on the variational principle, one finds constraints on terms of the Lagrangian density. Thus, the action is a Lorentz scalar and in the unit system used here where $\hbar = 1$, it is dimensionless. This property means that every term of the Lagrangian density must have the dimension $[L^{-4}]$. Now, components of the 4-current j^{μ} represent charge and current densities and their dimension is $[L^{-3}]$. Therefore, the 4-potentials A_{μ} must be a 4-vector whose dimension is $[L^{-1}]$. These requirements are satisfied by the Lienard-Wiechert 4-potentials (3). Thus, also $\Phi_{,\mu}$ of (4) is a 4-vector whose dimension is $[L^{-1}]$ and Φ must be a dimensionless Lorentz scalar function of the space-time coordinates.

Now, the coordinates are entries of a 4-vector. Let us first find the general form of a physically acceptable Lorentz scalar function depending *only* on the space-time coordinates. The following expression is a scalar function of the coordinates

$$f_{a,b}(x^{\mu}) = (x^{\mu} - x_a^{\mu})(x_{\mu} - x_{b\mu}), \quad (5)$$

where x_a^{μ} and x_b^{μ} denote specific space-time points. The first objective is to find a definition of the form of one scalar term T . This term must be a tensorial expression which is completely contracted. Thus, it can be cast into a product of powers of functions like (5)

$$T = f_{a,b}^{\alpha}(x^{\mu}) f_{c,d}^{\beta}(x^{\mu}) \dots f_{u,v}^{\gamma}(x^{\mu}), \quad (6)$$

where Latin subscripts denote specific coordinate points and Greek letters denote the power of each function. It follows that any scalar function of the coordinates can be written as a sum of terms where each of which is a product of positive or negative powers of functions like (5).

Relying on causality and homogeneity of space-time, one finds that in the case discussed here there is just one specific point x_a^μ , which is the retarded position of the charge. Thus, (5) boils down into the following form

$$f_{a,b}(x^\mu) \rightarrow R^\mu R_\mu. \quad (7)$$

This outcome proves that the gauge function $\Phi(x^\mu)$, which is a dimensionless quantity, must be a constant. (As a matter of fact, the retardation conditions prove that (7) vanishes identically.)

At this point it is clear that the expression $(x^\mu - x_a^\mu)(x_{c\mu} - x_{b\mu})$ cannot be used in place of (5). Indeed, here $x_{c\mu} = x_{b\mu}$ and the second factor vanishes.

These arguments complete the proof showing that if one adheres to VE then the gauge function Φ is a constant and the gauge 4-vector $\Phi_{,\mu}$ vanishes identically. Hence, the Lienard-Wiechert 4-vector (3) is unique.

3. An Example

Let us examine a simple system which consists of one motionless particle whose mass and charge are m , e , respectively. The particle is located in a spatial region where the external fields vanish. Therefore, the Lorentz force exerted on the particle vanishes too and it remains motionless as long as these conditions do not change. Thus, the system's energy is a con-

stant of the motion. This property holds for MLE, where the particle's energy is a constant

$$E = m. \quad (8)$$

Now, let us examine this system from the point of view of VE. For this purpose, the external 4-potentials should be defined. Thus, the null external fields are derived from null 4-potentials

$$A_{(ext)\mu} = 0 \rightarrow F_{(ext)}^{\mu\nu} = 0. \quad (9)$$

In order to define the particle's energy one must construct the Hamiltonian. Here the general expression is (see [1], pp. 47-49; [2], pp. 575)

$$H = [m^2 + (\mathbf{P} - e\mathbf{A})^2]^{1/2} + e\phi, \quad (10)$$

where \mathbf{P} denotes the canonical momentum and the components of the 4-potentials are (ϕ, \mathbf{A}) . Substituting the null values of (9) into (10) and putting there $\mathbf{P} = 0$ for the motionless particle and the vanishing 4-potentials, one equates the energy to the Hamiltonian's value and obtains

$$E = m. \quad (11)$$

At this point, one finds that result (8) of MLE is identical to (11) of VE.

This system is used here as an example showing how far can one proceed if gauge transformation freedom is permissible. To this end, let us apply a specific gauge transformation to the null external 4-potentials (9). The gauge function and its 4-potentials are

$$\Phi = t^2 \rightarrow A'_{(ext)\mu} = -\Phi_{,\mu} = (-2t, 0, 0, 0). \quad (12)$$

In MLE nothing changes, because the equations of motion depend on electromagnetic fields and their null value does not change

$$F'^{\mu\nu} = F^{\mu\nu} = 0. \quad (13)$$

Hence, the energy value (8) continues to hold and the gauge transformation (12) is acceptable in MLE.

The following points show several arguments proving that this conclusion does not hold for the VE theory.

1. The gauge function of (12) has the dimensions $[L^2]$, whereas in VE it must be dimensionless.
2. The gauge function of (12) is the entry U^{00} of the second rank tensor $U^{\mu\nu} = x^\mu x^\nu$. On the other hand, in VE the gauge function must be a Lorentz scalar.
3. Substituting the gauge 4-vector $A'_{(ext)\mu}$ of (12) into the Hamiltonian (10), one finds

$$H' = m - 2et. \quad (14)$$

Hence, if gauge transformations are allowed in VE then the energy of a closed system is not a constant of the motion.

4. The previous argument can be observed from another point of view. Thus, the physical state of the single motionless particle is time independent. Hence, one expects that energy is a constant of the motion. This point holds within the framework of MLE which is unaffected by any gauge transformation. It is also satisfied in VE, provided one uses the null 4-potential (9). However, the gauge degree of freedom allows one to use the gauge transformation (12). This

transformation casts the trivial time-independent Hamiltonian into the time-dependent expression (14). As is well known, if the Hamiltonian is time-dependent then energy is not a constant of the motion (see [5], p. 132). Hence, an application of the gauge degree of freedom deprives the VE theory from having an acceptable expression for the energy of a physically time-independent state. Here one finds a specific example showing that MLE and VE are not equivalent theories.

These four conclusions prove that the gauge degree of freedom destroys VE.

4. Gauge Transformations and Quantum Physics

As stated in the Introduction, quantum physics is very closely related to VE. Moreover, the Ehrenfest theorem (see [6], pp. 25-27, 138) shows that the classical limit of quantum mechanics agrees with the laws of classical physics. For these reasons, one expects that the laws of VE are relevant to quantum physics. A direct examination of gauge transformations proves this matter.

The Lagrangian density of the Dirac field is (see [3], p. 84; [4], p. 78)

$$\mathcal{L} = \bar{\psi}[\gamma^\mu(i\partial_\mu - eA_\mu) - m]\psi, \quad (15)$$

This Lagrangian density yields the Dirac Hamiltonian (see[7], p. 48)

$$H = \boldsymbol{\alpha} \cdot (\mathbf{P} - e\mathbf{A}) + \beta m + e\phi. \quad (16)$$

Now, in quantum mechanics, the gauge transformation (4) is accompanied by an appropriate transformation of the particle's wave function. Thus, the quantum mechanical form of gauge

transformation is (see [4], p. 78)

$$A'_\mu = A_\mu - \Phi_{,\mu}; \quad \psi'(x^\mu) = e^{ie\Phi(x^\mu)}\psi(x^\mu) \quad (17)$$

(Note that the symbol e in the exponent denotes the particle's electric charge.) Substituting the gauge transformation (17) into the Lagrangian density (15), one realized that it is gauge invariant indeed (see e.g. [4], p. 78).

It is interesting to note that in quantum mechanics the gauge function Φ is used in the exponent of the particle's wave function. As explained in the introduction, general laws of physics restrict $\Phi(x^\mu)$ to be a dimensionless Lorentz scalar. This form substantiates the classical arguments presented in Section 2, where it is explained why VE requires that the gauge function should be a dimensionless Lorentz scalar.

Now let us examine the quantum mechanical version of the example discussed in Section 3. The Dirac wave function of the spin-up state of a motionless particle is (see [7], p. 10)

$$\psi(x^\mu) = e^{-imt}(1, 0, 0, 0). \quad (18)$$

Thus, one uses the fundamental quantum mechanical equation and obtains the particle's energy from an application of the Dirac Hamiltonian to the wave function (18)

$$E\psi = H\psi = i\frac{\partial\psi}{\partial t} = m\psi \rightarrow E = m. \quad (19)$$

Now, let us examine the gauge transformation (17) for the specific case (12). The wave function (18) transforms as follows

$$\psi'(x^\mu) = e^{iet^2}e^{-imt}(1, 0, 0, 0). \quad (20)$$

Using the gauge transformed wave function (20), one applies a straightforward calculation and obtains the expectation value of the Hamiltonian. Here the result differs from the original value

$$H'\psi' = i\frac{\partial\psi'}{\partial t} = (m - 2et)\psi' \rightarrow \langle H' \rangle = m - 2et. \quad (21)$$

This is precisely the same discrepancy which was found above for the gauge transformation of VE of classical physics (14). Indeed, a gauge transformation casts a time independent Hamiltonian into a time-dependent expression and energy calculation is destroyed. Thus, one concludes that gauge transformations are inconsistent with quantum physics. This specific example illustrates the general argument written in the paragraph that begins below (17).

5. Tentative Consequences

In this Section several consequences that may result from the foregoing analysis are presented. This list probably does not exhaust all possibilities.

1. All applications of gauge functions continue to hold within MLE. Indeed, The equations of motion of MLE - Maxwell equations and the Lorentz law of force - depend on electromagnetic fields. These fields are not affected by *any* gauge transformation. Hence, all results derived from these equations remain intact.

As an illustration, let us examine the dimensional and the covariance arguments discussed in points 1 and 2 of Section 3. Within the scope of MLE, a gauge transformation adds

a zero to the fields. Now a zero is consistent with *all* dimensions and with *all* tensorial quantities. Hence, within MLE, a gauge transformation is acceptable.

Another example is the solution of Maxwell equations obtained from an application of the the Green function of the d'Alembertian (see [1], p. 117; [2], pp. 220, 549). As is well known from the theory of differential equations, a solution of a linear homogeneous equation can be added to a specific solution of the corresponding inhomogeneous equation. Hence, gauge transformations remain an important tool for finding a solution to Maxwell equations.

2. One possibility that may hold for VE is that all operations and all restrictions associated with gauge transformations will continue to hold. In this case, the role of the present paper is to provide a stimulus for an analysis that will substantiate the freedom of gauge transformations in VE. *This assignment must settle all gauge related problems derived above.* Even in this case, MLE and VE are not exactly equivalent because VE needs the (yet unknown) theoretical structure mentioned in this item whereas MLE does not need it.
3. Another possibility is that all operations that use gauge transformations in VE will continue to hold but gauge invariance will stop to be a mandatory relation for the acceptability of electrodynamic expressions. This case may be regarded as the minimal theoretical change that emerges from the present work. (As a matter of fact, this possibility was the initial motivation that has led the Author to

carry out the present research.)

4. A more profound scenario that may result from this work is that some or all operations which are based on gauge transformations will be forbidden within VE. In this case electromagnetic relations which are derived today from gauge related procedures should be based on other kinds of proofs. In particular, electromagnetic relations that have been confirmed in experiments are expected to be proved successfully by other methods.

The alternative scenarios described above illustrate the nature of this work. It is not intended to present a comprehensive solution of a physical problem but to draw the attention of the physical community to a problem which deserves a further analysis.

6. Conclusions

The foregoing results indicate the difference between an electrodynamic theory where Maxwell equations and the Lorentz law of force are regarded as the theory's cornerstones and an electrodynamic theory based on the variational principle together with causality and space-time homogeneity. Indeed, if Maxwell equations and the Lorentz law of force are the theory's cornerstone then it is very well known that one is free to define the gauge function $\Phi(x^\mu)$ of (4) (see [1], pp. 52-53; [2], pp. 220-223). On the other hand, this work proves that gauge transformations are inconsistent with electrodynamics based on the variational principle. In particular, all terms of the Lagrangian density must be Lorentz scalars having the dimension $[L^{-4}]$. The

discussion presented in this work explains why the variational principle requires the usage of the Lienard-Wiechert 4-potentials as a unique expression. For this reason, one concludes that the two approaches are *not equivalent*. It is also proved that gauge transformations are forbidden in quantum physics.

The outcome of this work does not negate the well known gauge invariance of the Lagrangian density. Indeed, in the Dirac Lagrangian density (15), the two parts of the gauge transformation (17) cancel each other. (Hence, the action, the associated phase and the interference pattern are formally unaffected by a gauge transformation.) On the other hand, other problems emerge. Thus, the Dirac Hamiltonian (16) does not contain the time-derivative of the gauge transformed wave function (17). Therefore, one term has no counterpart and the Hamiltonian varies. This conclusion explains why the Lagrangian density (15) is invariant under the gauge transformation (17) whereas the corresponding Hamiltonian is *not* invariant under it. The specific example discussed above examines a free motionless charged particle. An application of a gauge transformation casts its Hamiltonian into a time-dependent expression. This is unacceptable because energy of a free particle should be a constant of the motion and its Hamiltonian should be time-independent. Another problem arises in quantum mechanics because the gauge function $\Phi(x^\mu)$ appears as an exponential factor of the particle's wavefunction. Hence, as explained above, it must be a constant and the associated gauge 4-vector vanishes identically.

This work aims to examine restrictions imposed on gauge transformations of electrodynamic systems. It introduces the examination of MLE and VE as two theories which may be different. The results justify this distinction because it is proved that

any gauge transformation is acceptable within MLE whereas VE requires a unique 4-potentials.

The outcome of this work is probably linked to Jackson's analysis of the Coulomb gauge, where he shows that an unphysical ghost-like transverse current arises in an empty space (see [2], pp. 222, 223).

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* Email: elic@tauphy.tau.ac.il

Internet site: <http://www-nuclear.tau.ac.il/~elic>

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