Length Contraction on Rotating Disc: an Argument for the Lorentzian Approach to Relativity

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A thought experiment with rotating disc (Ehrenfest paradox) is considered. With minor exceptions, all resolutions of this paradox, from 1909 onward, have been formulated with the tacit assumption that special relativity theory (SRT) is irrefutable. This may cause a false impression that Ehrenfest paradox is apparent, and that its solution within SRT is possible or even already accomplished. We put forward arguments in favor of a different opinion, namely that the rotating disc cannot be contracted in the way predicted by SRT. We then conclude that Ehrenfest paradox can be effectively solved by the FitzGerald-Lorentz hypothesis of length contraction. Besides, we show that relativistic effects on rotating disc do not consist a basis for derivation of equivalence principle.

The problem considered in this paper is tightly connected with a concept of 'Born rigidity', a definition of rigid body in special relativity proposed by M. Born in 1909, and with a corresponding concept of 'rigid motion' formulated by Pauli [1]. In this formulation 'rigidity' means that distances between respective points of a body in question remain constant in the co-moving frame. This would refer also to non-relativistic mechanics; yet, as applied to SRT, it means that length contraction must satisfy a condition of Lorentz invariance. A paradoxical conclusion deduced from 'Born rigidity' is that it puts special restrictions upon acceleration. This consequence is widely known as 'Bell's spaceship paradox' [2]. Instead, the 'Ehrenfest paradox' [3] states that a disc (or, originally, an ideal cylinder) cannot rotate without violation of Lorentz invariance. However, P. Ehrenfest concluded that apparently Born's definition of rigidity does not comply with SRT. In turn, very likely in face of the arisen difficulties, M. Planck [4] has postulated to separate the problem of geometry on rotating disc from that what happens to the disc in the spin-up phase. In his opinion, the latter requires employing a relevant theory of elasticity. In the same year (1910) T. Kaluza suggested that geometry on rotating disc is non-Euclidean. In his 1916 paper on general relativity [5] A. Einstein considered a thought experiment with rotating disc in aim to introduce the gravitation theory conceived in terms of the space-time curvature. Later on, in the book "The Meaning of Relativity" [6] he described again this experiment as leading to GRT. The relevant citation brings us directly in the essence of the problem:

"(…) let K' be a system of co-ordinates whose z'-axis coincides with the z-axis of K, and which rotates about the latter axis with constant angular velocity. Are the configurations of rigid bodies, at rest relatively to K', in accordance with the laws of Euclidean geometry? Since K' is not an inertial system, we do not know directly the laws of configuration of rigid bodies with respect to K', nor the laws of nature, in general. But we do know these laws with respect to the inertial system K and we can therefore infer their form with respect to K'. Imagine a circle drawn about the origin in the $z'y'$ plane of K', and a diameter of this circle. Imagine, further, that we have given a large number of rigid rods, all equal to each other. We suppose these laid in series along the periphery and the diameter of the circle, at rest relatively to K'. If U is the number of these rods along the periphery, D the number along the diameter, then, if K' does not rotate relatively to K, we shall have

$$\frac{U}{D} = \pi .$$

But if K' rotates we get a different result. Suppose that at a definite time t, of K we determine the ends of all the rods. With respect to K all the rods upon the periphery experience the Lorentz contraction, but the rods upon the diameter do not experience this contraction (along their lengths!). It therefore follows that

$$\frac{U}{D} > \pi .$$

It therefore follows that the laws of configuration of rigid bodies with respect to K' do not agree with the laws of configuration of rigid bodies that are in accordance with Euclidean geometry. If, further, we place two similar clocks (rotating with K'), one upon the periphery, and the other at the centre of the circle, then, judged from K, the clock on the periphery will go slower than the clock at the centre. The same thing must take place, judged from K', if we do not define time with respect to K' in a wholly unnatural way, (that is, in such a way that the laws with respect to K' depend explicitly upon the time). Space and time, therefore, cannot be defined with respect to K' as they were in the special theory of relativity with respect to inertial systems. But, according to the principle of equivalence, K' may also be considered as a system at rest, with respect to which there is a gravitational field (field of centrifugal force, and force of Coriolis). We therefore arrive at the result: the gravitational field influences and even determines the metrical laws of the space-time continuum. If the laws of configuration of ideal rigid bodies are to be expressed geometrically, then in the presence of a gravitational field the geometry is not Euclidean."

According to the above description, Einstein's interpretation looks as follows (Fig. 1a, b).

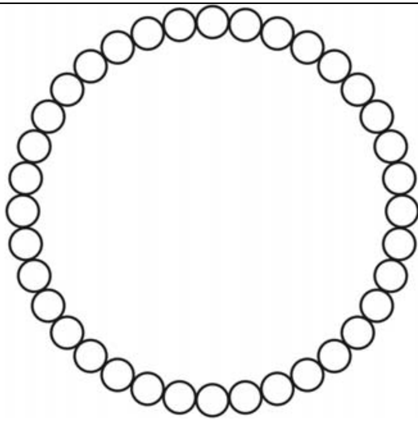

Fig. 1a
The disc at rest ($U/D = \pi$).

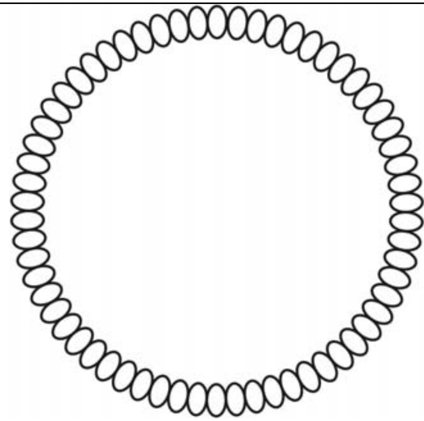

Fig. 1b
The disc in rotation according to Einstein. The particles upon the periphery are contracted due to their linear velocity. The number of particles laid along the periphery is higher than the respective number for the disc at rest ($U/D > \pi$). For the observer in K' the particles remain unaffected, therefore the geometry appears non-Euclidean.

The expression "number of rods" is somewhat ambiguous yet, certainly, by postulating $U/D > \pi$, Einstein didn't mean any miraculous multiplication of rods in result of rotation. It is quite clear that, in his considerations, Einstein didn't care about the spin-up phase. Instead, he concentrated solely on the geometry (and therefore likely preferred to speak of a circle than of a disc), separately deduced for the case of a disc at rest and in rotary motion.

However, if one considers the case as a physical experiment, it seems reasonable to assume that U and D do not refer to the possible number of rods "laid in series along the periphery and the diameter" but to the real number of rods, particles or anything else composing the respective parts of a disc. Say thus that a disc's periphery is composed by a certain number of particles (denoted by U) laid in series close to each other. If then a disc, formerly at rest, begins to rotate then following conditions, as judged from the system K, should be fulfilled:

1) The length of periphery remains unchanged during rotation (this is a consequence of the fact that diameter does not undergo contraction). This can be written as $R_0 = R$ where R_0 and R denote the radius of a disc at rest and in rotation respectively.

2) Each particle laid along periphery is contracted in line with the (instantaneous) linear velocity vector, due to the value of Lorentz factor.

3) The number of particles composing the periphery of a disc at rest remains unchanged during the spin-up phase.

Provided that all the above conditions are satisfied, the length contraction on the periphery of rotating disc may be figured as follows (Fig. 2a, b).

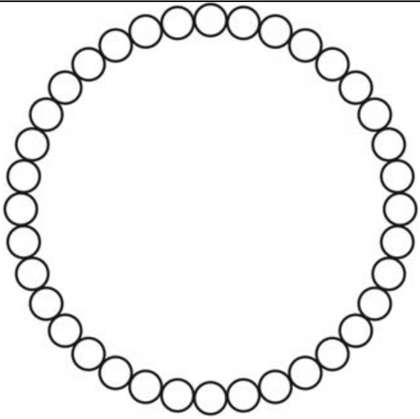

Fig. 2a
The disc at rest ($U/D = \pi$).

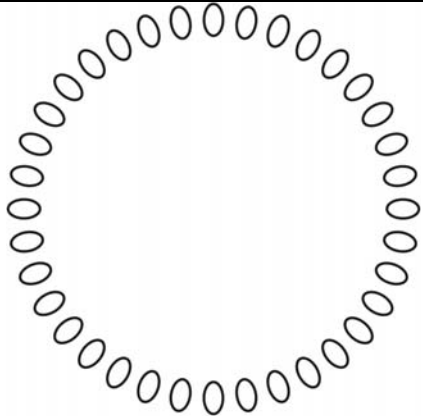

Fig. 2b
The same disc in rotation. The particles upon the periphery are contracted due to their linear velocity. The number of particles is conserved since they are treated as actual entities. In consequence, the distance between neighboring particles increases in comparison with the respective distance at rest. Thus, $U/D = \pi$ but this does not determine the geometry.

As far as the second and third points seem quite obvious (considering the quoted assumption), a reservation may be raised to the first point. Namely, one may suggest that a realistic description of the spin-up phase of a material disc demands consideration of the centrifugal force and radial expansion. According to the calculations carried out by E. L. Hill, if the speed of sound within a rigid disc equals to the speed of light, then contraction would cancel the radial expansion; in other (realistic) cases it would only lessen it. Here the concept of 'rigidity' is used in its classical meaning defined by the stress/strain ratio (elastic modulus).

However, this solution, though seems to overcome the difficulties connected with the coherent explanation of the spin-up phase within SRT, in fact confuses the problem instead of enlightening it. It makes up a conditional prediction as to what would occur in case when a rigid disc expands. But, in itself, it does not prove the necessity of radial expansion since centrifugal force is not related to rigidity but to mass, radius and angular velocity, in accord with $F = m\omega^2 R$. This makes possible to assume the limiting case with a combination of these three quantities such that we get any value of Lorentz contraction (on periphery) together with the radial expansion tending to zero. Regarding that, it's much better to abide by the original Einstein's assumption that D remains unchanged during rotation, *i.e.* that $R = R_0$.

Instead, the geometry on rotating disc in the phase of relaxation is currently defined as Riemannian manifold with Langevin-Landau-Lifschitz metric. The essential feature of this metric is that simultaneity, comprehended in accord to SRT rules, cannot be defined on the whole circumference, *e.g.* by joining up local infinitesimal planes of simultaneity along the periphery, or in any other way. The reason is that for the Langevin observer on a disc, the plane of simultaneity makes up a helix in spacetime. This, however, entails paradoxical consequences. The world line of Langevin observer (which also makes up a helix in spacetime but a stretched one) crosses the successive events set on the helix of simultaneity, which implies that number of events from the past and from the future are simultaneous to the observer at one point. In other words, we get sequence of events, simultaneous to each other, laid on the observer's world line. This may inspire sf writers but is, for sure, impossible.

Let us return now to the length contraction. The questions are: Which way of contraction from those represented on Fig. 1b and Fig. 2b is compatible with SRT? And which from them is possible?

As long as we consider inertial systems in uniform motion, the essence of relativistic effects predicted by Lorentz transformation is that they are relative, *i.e.*, they depend solely on the choice of reference system. In such cases 'Born rigidity' is satisfied on the strength of relativity principle. The difficulties arise when we introduce acceleration. Then 'Born rigidity' is no longer satisfied unless special conditions are fulfilled. Either the body must be accelerated *gently* (but then we get only approximation of 'Born rigidity'), or each point of the body must be accelerated with the use of an appropriate force. Such a way of assurance of 'rigid motion' cannot be, however, regarded as a natural way of accelerating, and thus considered to be a satisfactory solution to the 'Bell's spaceship paradox'. If we limit our consideration to the question of contraction, 'Ehrenfest paradox' does not essentially differ from 'Bell's spaceship paradox'. The only difference is that 'Born rigidity' cannot be satisfied on rotating disc, even with the help of an artificial procedure.

Hence, the answer to the above questions is: The way of contraction predicted by Einstein (Fig. 1b) is admittedly consistent with SRT (at least if we consider a small sector of the disc's periphery) but, at the same time, it is also impossible. Any reasonable hypothesis cannot explain (or assume) the increase of the number of particles!

The situation looks as deadlock, at least until we stay on the ground of the standard opinion. Meanwhile, an appropriate tool to solve this problem is well known as it has been found before this opinion originated. In a short note published in 1889 [7] G.F. FitzGerald introduced a hypothesis of length contraction in aim to explain the null result of Michelson-Morley experiment. A few years later, H.A. Lorentz independently proposed the same idea [8] and then developed it in the framework of his electron theory. The essence of this hypothesis consists in assumption that length contraction is a real physical process of dynamic origin connected with the inner properties of matter, and caused by the absolute motion. As employed to ‘Ehrenfest paradox’ (in particular to the spin-up phase) the FitzGerald-Lorentz hypothesis does not predict ‘Born rigidity’. It predicts instead that all relatively self-contained parts of the matter composing the disc (such as rigid rods, particles or atoms) undergo contraction in line of their motion, due to the respective values of Lorentz factor. In result of this, distances between them increase in comparison with their sizes, which makes the whole process absolute, *i.e.* detectable both in K and K'.

Let us refer now to the main conclusion of the quoted citation, that is to equivalence principle, as deduced from the measurements performed on rotating disc in the phase of relaxation. To some extent, it is (in accord with Planck’s suggestion) a separate problem. Following the Einstein’s way of reasoning, we may assume that in the reference system K' one gets $U/D > \pi$, which means that the circumference of a disc is longer than $2\pi R_0$. The observer in K' may ascribe this effect to the presence of gravity, identified with the centrifugal force. It is doubtful, however, if the measurements performed on a disc validate the equivalence principle. The reason for this doubt is the following. The length contraction and time dilation, as measured in K, depend solely on linear velocity (v) of the respective parts of a disc (say, of the periphery), due to the value of Lorentz factor. Meanwhile, linear velocity is not directly related to the centrifugal force. Since $v = \sqrt{FR/m}$ then, considering a given constant mass (*e. g.* the mass of an observer on the disc), the linear velocity can be coupled with different values of centrifugal force, dependently of the radius length. Therefore the Einstein's claim that "gravitational field influences and even determines the metrical laws of the space-time continuum" cannot be effectively deduced from the case of rotating disc.

Conclusions

The condition of 'Born rigidity' is inherently connected with special relativity. The 'Ehrenfest paradox' reveals that 'Born rigidity' cannot be satisfied within SRT. This implies the need of introducing a different theory. The FitzGerald-Lorentz hypothesis of length contraction shows the right way.

Though the geometry on rotating disc seems to be non-Euclidean, it does not give the sufficient grounds to derive the equivalence principle. The reason is that linear velocity, responsible for relativistic effects, is not directly coupled with centrifugal force.

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