Alternative method of developing Maxwell's fields

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A method to derive Maxwell's fields form Gauss' law and the Lorentz force law is presented. Additionally we assume retardation of electric interactions and that the electric field is independent of the time derivatives of the field-source acceleration.

Keywords: Maxwell's fields, Gauss' law, force transformation

Introduction

It is well known that the classical theory of electromagnetic fields rests on four Maxwell's equations and the Lorentz force law. On the other hand, the Lorentz transformation establishes some relations between Maxwell's equations (for example Ampere-Maxwell law follows from Gauss' law) and between the electric and magnetic fields. It suggests that it is possible to reduce the electromagnetic theory to fewer number of equations and in this way elucidate the basis of the theory.

This paper is an attempt to give a support to this idea. We argue that the whole theory of electromagnetism may be based essentially on two laws only, the Lorenz force law and Gauss' law. However, besides these laws some additional reasonable restrictions on the mathematical form of their solutions must be introduced. In this work we show that the general Maxwell's fields can be developed from the following postulates:

Assumption 1. *In any inertial reference frame the Lorentz force law is valid:*

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}. \tag{1}$$

Assumption 2. *For all inertial reference frames the electric field is determined by Gauss' law:*

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} \sum_i Q_i \delta \left(\vec{r} - \vec{r}_{Q_i} \right), \tag{2}$$

Assumption 3. *The contribution of the source charge to the electric field at an observation point at time t depends on the position and motion of the charge at the retarded time only.* The retarded time is $\tau = t - R_{ret}/c$, where R_{ret} is the distance between the retarded position of the charge Q, $\vec{r}_Q(\tau)$, and the observation point $\vec{r}(t)$ at the moment t. This is equivalent to assuming that any electric effect is transmitted through space with the velocity c.

Assumption 4. *The electric field does not depend on time derivatives of the source position of order higher than the acceleration.*

Note that from Assumption 2 follows that the electric field will somehow depend on the position of the test particle (i.e.

the point of observation of the field) but may *not* depend on the test particle velocity or acceleration. The reason is that Gauss' law does not contain any information about the test particle dynamics so that the velocity or acceleration of the test particle may never enter the solutions of Gauss' law.

Assumption 3 can be regarded as a consequence of a postulate that the speed of interactions between charges must be a Lorentz invariant. In such a case the instantaneous interactions must be discarded because of the relativity of synchronism. In turn, the only relativistically invariant speed of interaction is the velocity of light, which leads just to the Assumption 3.

Assumption 4 has basically an operational meaning. It allows us to find the simpler possible solutions of Gauss' law. As the solutions appear to be precisely Maxwell's fields, this assumption obtains the status of a general postulate.

One can find similar postulates in the work by Frisch and Wilets [1] who use them to derive Maxwell's fields as well. However the crucial solution (see their Eq. 21) is rather guessed instead of being systematically developed. The authors emphasize also the test-particle-velocity independence of the electric field as a separate assumption, which is unnecessary if we postulate Gauss' law. Other authors use alternative assumptions to find Maxwell's fields. Rosser [2] gets the electromagnetic potentials merely for uniformly moving sources and then postulates they are correct for accelerated ones. Tessman [3] method requires Newton's third law for steady state charge distributions. Several authors [4-6] simply generalize Gauss' law to a Lorentz covariant four-vector form, which is a very strong and unjustified assumption. Among other conceptions to derive Maxwell's equations let as mention the work by Galeriu [7] in which the author obtains the homogenous equations from Stokes theorem and the approach of Gersten [8] based on the quantum equation for the wave function.

Comparing to the methods mentioned above our approach represents a systematic mathematical reasoning without extraordinary assumptions. The method may seem to be anachronistic because we use three-vectors instead of tensors. But we want to emphasize that having Gauss' law referring to the electric field *3-vector* as the only premise we are compelled to work within 3-dimentional formalism. Especially, because we have only one of Maxwell's equations, no four-vector potential $(\phi, \vec{A})$ may be introduced to represent electromagnetic fields (note that one of important relations for the vector potential $\vec{A}$, $\vec{B} = \vec{\nabla} \times \vec{A}$, is based on the law $\vec{\nabla} \cdot \vec{B} = 0$ we are deprived of in our approach). Postulating that the theory must have a four-dimensional structure with a four-vector $(\phi, \vec{A})$ involved would be an extra assumption we want to avoid.

Mathematical simplicity of our method is achieved because of performing the main considerations in the frame $S_{V_{ret}=0}$ in which the retarded velocity of the accelerating source is *zero*. What is more, in the frame $S_{V_{ret}=0}$ we use the spherical coordinates. Thanks to symmetry of the kinematical situation some of the spherical components of the fields appear to vanish. In this way Gauss' law written in the spherical coordinates becomes quite simple and the total electric field may be easily determined. In turn, to obtain the magnetic field we notice that the validity of Gauss' law in any frame appears to be equivalent to the Ampere-Maxwell law: $\vec{\nabla} \times \vec{B} - \partial\vec{E}/c^2\partial t = \vec{j}/c^2\epsilon_0$. The Ampere-Maxwell law allows us to find the magnetic field in the frame $S_{V_{ret}=0}$. Next, the electromagnetic fields for an arbitrarily moving source are obtained by performing the Lorentz transformation of fields established in $S_{V_{ret}=0}$ to an appropriate system of reference.

Uniformly moving source

If in a frame S' the source is at rest (with acceleration equal to zero), the postulated Gauss' law is equivalent to Coulomb's law. Then the electric field is:

$$\vec{E'} = \frac{Q}{4\pi\epsilon_0}\frac{\vec{R'}}{R'^3}, \tag{3}$$

where $\vec{R'} = \vec{r'} - \vec{r'}_Q$.

To obtain the force $\vec{F}$ exerted by the source charge Q when it moves with a constant velocity $\vec{V}$ it is enough to perform the Lorentz transformation of the force $\vec{F'} = q\vec{E'}$ from the rest frame S' of the charge Q to a system of reference S moving with the velocity $-\vec{V}$ with respect to S'. As the result we obtain [9]:

$$\vec{F} = \vec{F}'_{\parallel} + \gamma\vec{F}'_{\perp} + \gamma\vec{v} \times \left(\frac{\vec{V}}{c^2} \times \vec{F'}\right), \tag{4}$$

where $\gamma = (1 - V^2/c^2)^{-1/2}$, $\vec{v}$ is a velocity of the charge q measured in the frame S and the indices $\parallel$ and $\perp$ refer to the directions parallel and perpendicular to the velocity $\vec{V}$. The last equation may be rewritten as follows:

$$\vec{F} = q\vec{E} + q\vec{v} \times \left(\frac{\vec{V}}{c^2} \times \vec{E}\right), \tag{5}$$

where

$$\vec{E} \equiv \vec{E}'_{\parallel} + \gamma\vec{E}'_{\perp} \tag{6}$$

defines the electric field in the frame S. In Eq. (5) there has appeared also the magnetic field:

$$\vec{B} \equiv \frac{\vec{V}}{c^2} \times \vec{E} \tag{7}$$

Using Eqs. (3) and (6) we can find the field $\vec{E}$ in an explicit form. To have the field $\vec{E}$ expressed by means of the quantities from the frame S we have to transform $\vec{R}'$ to the frame S remembering about Assumption 3. It means that the position $\vec{r}_Q$ of the charge Q must be taken at the retarded moment τ:

$$\vec{r}\,'_{\parallel} = \gamma\left(\vec{r}(t)_{\parallel} - \vec{V}t\right), \quad \vec{r}\,'_{\perp} = \vec{r}(t)_{\perp}, \tag{8}$$

$$\vec{r}\,'_{Q\parallel} = \gamma\left(\vec{r}_Q(\tau)_{\parallel} - \vec{V}\tau\right), \quad \vec{r}\,'_{Q\perp} = \vec{r}_Q(\tau)_{\perp}, \tag{9}$$

$$\vec{R}\,'_{\parallel} = \gamma\left(\vec{r}(t)_{\parallel} - \vec{r}_Q(\tau)_{\parallel} - \vec{V}(t-\tau)\right) = \gamma\left(\vec{R}_{ret\parallel} - \vec{V}R_{ret}/c\right),$$
$$\vec{R}\,'_{\perp} = \vec{r}(t)_{\perp} - \vec{r}_Q(\tau)_{\perp} = \vec{R}_{ret\perp}, \tag{10}$$

where, as we see, there appeared the retarded relative position $\vec{R}_{ret} = \vec{r}(t) - \vec{r}_Q(\tau)$. It is easy to show also that:

$$R\,' = \gamma(R_{ret} - \vec{R}_{ret} \cdot \vec{\beta}) \tag{11}$$

Inserting the above relations to Coulomb's field (3) and using the definition (6) we get:

$$\vec{E}_V = \frac{Q}{4\pi\epsilon_0}\left[\frac{\vec{n} - \vec{\beta}}{\gamma^2(1 - \vec{\beta}\cdot\vec{n})^3 R^2}\right]_{ret}, \tag{12}$$

where: $\vec{\beta}_{ret} = \vec{V}/c$ is the source charge (actual and retarded) velocity and $\vec{n}_{ret} = \vec{R}_{ret}/R_{ret}$. Now we find the magnetic field from the definition (7). Note that for the field $\vec{E}_V$ it is equivalent to the relation:

$$\vec{B} = \frac{\vec{n}_{ret}}{c} \times \vec{E}. \tag{13}$$

Accelerating source

Let in a reference frame $S_{V_{ret}=0}$ the source charge Q be at rest at the *retarded* moment but have at this instant a retarded acceleration $\vec{a}_{ret}$. Let us introduce the spherical system of coordinates (r, θ, ϕ) with the origin placed at the retarded position of the source Q and the polar angle θ measured with respect to the direction of $\vec{a}_{ret}$ (see Fig. 1).

It means that in this frame the radius vector $\vec{r} = \vec{R}_{ret}$. Note also that in $S_{V_{ret}=0}$ all the derivatives over the *spatial* coordinates at the observation point are equal to the derivatives over the components of the retarded relative position of the observation point $\vec{R}_{ret}$:

$$\partial/\partial r = \partial/\partial R_{ret}, \quad \partial/\partial\theta = \partial/\partial\theta_{ret}, \quad \partial/\partial\phi = \partial/\partial\phi_{ret}. \tag{14}$$

To prove this let us take for example $\partial/\partial r = (\partial/\partial R_{ret})(\partial R_{ret}/\partial r)$. Since $R_{ret} = \sqrt{(\vec{r} - \vec{r}_Q(\tau))^2}$ we get:

$$\partial R_{ret}/\partial r = \frac{\vec{r} - \vec{r}_Q(\tau)}{\sqrt{(\vec{r} - \vec{r}_Q(\tau))^2}} \cdot \left(\frac{\partial \vec{r}}{\partial r} - \frac{\partial \vec{r}_Q(\tau)}{\partial r}\right) = \vec{n}_{ret} \cdot \left(\vec{n}_{ret} - \frac{\partial \vec{r}_Q}{\partial \tau}\frac{\partial \tau}{\partial r}\right) \tag{15}$$

because $\partial\vec{r}_Q/\partial\tau = \vec{V}_{ret} = 0$. Similar calculations apply for the other components, which ends our proof.

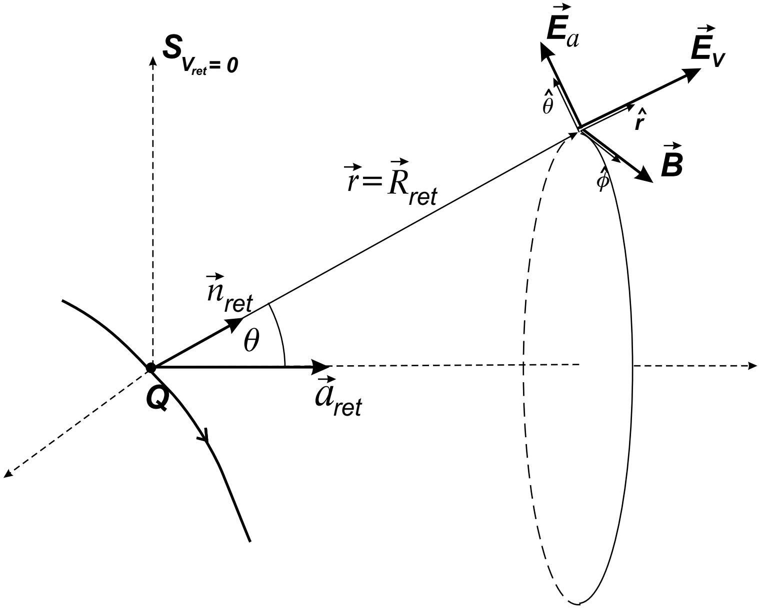

Figure 1. Spherical coordinates in the frame $S_{V_{ret}=0}$. At the retarded moment the source charge Q is at the origin of the frame and its retarded velocity is zero. As shown below, at the observation point the electric field has two components, $\vec{E}_V$ along the radius vector and $\vec{E}_a$ in the direction of $\hat{\theta}$. In turn, the magnetic field $\vec{B}$ has only the ϕ-component.

So, while working in the frame $S_{V_{ret}=0}$, the values of coordinates (r, θ, ϕ) are equivalent to the values $(R_{ret}, \theta_{ret}, \phi_{ret})$ of the retarded relative vector radius $\vec{R}_{ret}$; and similarly for the respective spatial derivatives. To shorten notation we will omit during the calculations the index *"ret"* and will write simply R, θ and ϕ remembering that we deal with the retarded quantities. The same refers to the notation of the retarded acceleration.

Electric field

According to Gauss' law the divergence of the field $\vec{E}$ must vanish everywhere, except at the source. In the spherical system of coordinates we have then for the observation points outside the source:

$$\vec{\nabla}\cdot\vec{E} = \frac{2}{R}E_R + \frac{\partial E_R}{\partial R} + \frac{\cos\theta}{R\sin\theta}E_\theta + \frac{1}{R}\frac{\partial E_\theta}{\partial\theta} = 0, \tag{16}$$

Note that because the electric field does not depend on the test particle velocity and due to symmetry of this kinematical situation in the frame $S_{V_{ret}=0}$ the electric force $q\vec{E}$ may act only in the plane of $\vec{a}$ and $\vec{R}$. It means that the electric field cannot have the third component E_ϕ.

Now suppose the component E_R depends on the retarded acceleration of source $\vec{a}$. In such a case there would occur in Eq. (16) a term $\partial E_R(\vec{a})/\partial R \sim (\partial\vec{a}(\tau)/\partial\tau)(\partial\tau/\partial R) = (\partial\vec{a}(\tau)/\partial\tau)(-1/c)$ i.e. a term proportional to the time derivative of acceleration. To have the divergence of $\vec{E}$ equal to zero this term must be canceled out by some other term proportional to the time derivative of acceleration. This additional term cannot come from the derivative $\partial E_\theta(\vec{a})/\partial\theta \sim (\partial\vec{a}(\tau)/\partial\tau)(\partial\tau/\partial\theta)$ because $\partial\tau/\partial\theta = 0$. To ensure vanishing of the divergence of the electric field there should be in Eq. (16) a term proportional to the time derivative of acceleration coming from the field $\vec{E}$ itself, i.e. form the terms containing the components E_R and E_θ. But it is excluded on the basis of Assumption 4. It follows then that our initial assumption cannot be satisfied, that is the component E_R cannot depend on the acceleration of source. (Important is, however, that there is no obstacle for the component E_θ to depend on acceleration because, as we have shown, the term proportional to

$\partial\vec{a}(\tau)/\partial\tau$ coming form $\partial\vec{E}_\theta(\vec{a})/\partial\theta$ is multiplied by $\partial\tau/\partial\theta = 0$, so it does not occur in the divergence.)

Looking for the electric field produced by the accelerating source charge we assume its form is:

$$\vec{E} = \vec{E}_V + \vec{E}_a \tag{17}$$

where $\vec{E}_V$ is given in Eq. (12) and $\vec{E}_a$ is a component of the field depending on the retarded acceleration. From the previous discussion follows that $\vec{E}_a$ has only the θ-component in the frame $S_{V_{ret}=0}$. In turn, from Eq. (12) we get that in this frame $\vec{E}_V$ has only the R-component equal to $Q/4\pi\epsilon_0 R^2$. To have the partial derivatives of $\vec{E}_V$ one have to differentiate the expression given in Eq. (12) and not forget that $\vec{V}_{ret} \equiv \vec{V}(\tau)$. Because the retarded moment τ depends on the retarded position of the source, we have to differentiate also the retarded velocity. For example $\partial\vec{V}(\tau)/\partial R = (\partial\vec{V}(\tau)/\partial\tau)(\partial\tau/\partial R) = \vec{a}\,(-1/c)$. *After* differentiation we put $\beta_{ret} = 0$. As the result we get:

$$\frac{\partial E_{V_R}}{\partial R} = -\frac{Q}{2\pi\epsilon_0}\left(\frac{1}{R^3} + \frac{a\cos\theta}{c^2R^2}\right), \tag{18}$$

$$\frac{\partial E_{V_R}}{\partial\theta} = 0. \tag{19}$$

Using the last equalities and remembering that in the frame $S_{V_{ret}=0}$ $E_R = E_{V_R}$ and $E_\theta = E_{a_\theta}$, Gauss' law (16) obtains the form:

$$\frac{Q}{2\pi\epsilon_0}\frac{a\cos\theta}{c^2R^2} = \frac{\cos\theta}{R\sin\theta}E_{a_\theta} + \frac{1}{R}\frac{\partial E_{a_\theta}}{\partial\theta}. \tag{20}$$

The last equation is extremely simple and its solution is:

$$E_{a_\theta} = \frac{Q}{4\pi\epsilon_0}\frac{a\sin\theta}{c^2R}. \tag{21}$$

This is the only component of $\vec{E}_a$ so we can rewrite the last equation in vector form as (see Fig. 1):

$$\vec{E}_a = \frac{Q}{4\pi\epsilon_0 c^2}\left[\frac{\vec{n}\times(\vec{n}\times\vec{a})}{R}\right]_{ret}. \qquad (22)$$

In effect the total electric field measured in the frame $S_{V_{ret}=0}$ is:

$$\vec{E} = \frac{Q}{4\pi\epsilon_0}\left[\frac{\vec{n}}{R^2}\right]_{ret} + \frac{Q}{4\pi\epsilon_0 c^2}\left[\frac{\vec{n}\times(\vec{n}\times\vec{a})}{R}\right]_{ret} \qquad (23)$$

Precisely, the first term on the right side should be written as:

$$\vec{E}_V = \frac{Q}{4\pi\epsilon_0}\left[\frac{\vec{n}-\vec{\beta}}{\gamma^2(1-\vec{\beta}\cdot\vec{n})^3 R^2}\right]_{ret} \quad \text{with} \;\; \beta_{ret} = 0, \qquad (24)$$

to show that any differentiation of the field $\vec{E}_V$ must refer also to the retarded velocity.

Magnetic field

To find the magnetic field we use the Ampere-Maxwell law $\vec{\nabla}\times\vec{B}-\partial\vec{E}/c^2\partial t = \vec{j}/c^2\epsilon_0$ in the area outside the point source charge where the current density $\vec{j}$ is zero. As proved in Appendix, this law is a consequence of Gauss' law postulated in Assumption 2.

Let us first imagine a test particle that moves with a velocity $\vec{v}$ in the plain defined by the retarded radius $\vec{R}$ and the retarded acceleration $\vec{a}$. Then the only directions that are meaningful are restricted to the plain of $\vec{R}$ and $\vec{a}$ (still we are in the frame $S_{V_{ret}=0}$, thus the retarded velocity of the source is zero); so the magnetic force $q\vec{v}\times\vec{B}$, for *any* direction of velocity $\vec{v}$ lying in the plain of

$\vec{R}$ and $\vec{a}$, may act only in that plane. It means that the magnetic field $\vec{B}$ may have only the ϕ-component. But $\vec{B}$ does not depend on the test particle velocity $\vec{v}$ (it follows from the fact that $\vec{B}$ may be expressed, via the Ampere-Maxwell law, by means of the test-particle-velocity-independent field $\vec{E}$). Thus our conclusion that $\vec{B}$ has in $S_{V_{ret}=0}$ only the ϕ-component we have deduced for the specially chosen velocity $\vec{v}$ must be generally valid for any possible direction of $\vec{v}$.

To determine the field $\vec{B}$ it is enough to consider the R-component of the Ampere-Maxwell law written in the spherical coordinates. The R-component of the Ampere-Maxwell law (with $\vec{j}=0$) is:

$$\frac{1}{c^2}\frac{\partial E_R}{\partial t} = \frac{1}{R}\frac{\partial B_\phi}{\partial \theta} + \frac{\cos\theta}{R\sin\theta}B_\phi. \tag{25}$$

Let us recall that only $\vec{E}_V$ contributes the R-component to the total field $\vec{E}$. From Eq. (24) we know precisely what the field $\vec{E}_V$ is, so we can find the left side of the above equation:

$$\frac{1}{c^2}\frac{\partial E_R}{\partial t} = \frac{1}{c^2}\frac{\partial E_{V_R}}{\partial t} = \frac{Q}{2\pi\epsilon_0}\frac{a\cos\theta}{c^3R^2}. \tag{26}$$

Thus now Eq. (25) becomes an explicit equation for B_ϕ:

$$\frac{Q}{2\pi\epsilon_0}\frac{a\cos\theta}{c^3R^2} = \frac{1}{R}\frac{\partial B_\phi}{\partial \theta} + \frac{\cos\theta}{R\sin\theta}B_\phi. \tag{27}$$

The solution of the above equation can be easily found and we get:

$$B_\phi = \frac{Q}{4\pi\epsilon_0}\frac{a\sin\theta}{c^3R}. \tag{28}$$

This is the only component of the field $\vec{B}$ in the frame $S_{V_{ret}=0}$. Comparing to Eq. (21) we se that the value of $\vec{B}$ differs from the value of $\vec{E}_a$ only by the factor $1/c$. Taking into account the directions of these fields we can express $\vec{B}$ by $\vec{E}_a$ as follows (see Fig. 1): $\vec{B} = \vec{n}_{ret}/c \times \vec{E}_a$. But because the component $\vec{E}_V$ of the total field (17) has the direction of $\vec{n}_{ret}$, thus one can substitute in the expression for $\vec{B}$ the total field $\vec{E}$:

$$\vec{B} = \frac{\vec{n}_{ret}}{c} \times \vec{E}, \tag{29}$$

which is identical to Eq. (13) we have found for the uniformly moving source.

Electromagnetic fields in general

To get the most general solution for the fields produced by accelerating source with *non-zero* retarded velocity $\vec{V}_{ret}$ it is enough to transform the force given by Eq. (1) from the rest frame $S_{V_{ret}=0}$ to some frame S'' in which the source has the required retarded velocity. Passing to the frame S'' we transform the force (1) according to the relation (4), where $\vec{V} = \vec{V}_{ret}$. In effect we get again the Lorentz force:

$$\vec{F}'' = q\vec{E}'' + q\vec{v}'' \times \vec{B}'', \tag{30}$$

where $\vec{E}''$ and $\vec{B}''$ are related to the fields $\vec{E}$ and $\vec{B}$ from the frame $S_{V_{ret}=0}$ as follows [1]:

$$\vec{E}'' = \vec{E}_{\parallel} + \gamma\vec{E}_{\perp} - \gamma\vec{V}_{ret} \times \vec{B}, \tag{31}$$

$$\vec{B}'' = \vec{B}_{\parallel} + \gamma\vec{B}_{\perp} + \frac{\gamma}{c^2}\vec{V}_{ret} \times \vec{E}, \tag{32}$$

where $\gamma = (1 - V_{ret}^2/c^2)^{-1/2}$ and the indices $\parallel$ and $\perp$ refer to the directions parallel and perpendicular to the velocity $\vec{V}_{ret}$. We encourage the reader to verify himself by direct calculations that:

$$\vec{B}'' = \frac{\vec{n}''}{c} \times \vec{E}'', \tag{33}$$

which shows that the law (29) is Lorentz covariant.

Inserting the fields $\vec{E}$ and $\vec{B}$ given by Eqs. (23) and (29) into Eq. (31), transforming also coordinates and acceleration to the frame S'', we can obtain the electric field in the frame S''. The calculations are elementary but very tedious and there is no need to present them here in more detail. The final result is (dropping primes):

$$\vec{E} = \frac{Q}{4\pi\epsilon_0}\left[\frac{\vec{n} - \vec{\beta}}{\gamma^2(1 - \vec{\beta}\cdot\vec{n})^3 R^2}\right]_{ret} + \frac{Q}{4\pi\epsilon_0 c^2}\left[\frac{\vec{n} \times (\vec{n} - \vec{\beta}) \times \vec{a}}{(1 - \vec{\beta}\cdot\vec{n})^3 R}\right]_{ret} \tag{34}$$

and the field $\vec{B}$ is determined by Eq. (29). We have then arrived at the general solutions of Maxwell's equations.

Summary

We have shown that Maxwell's fields may be obtained from Gauss' law and the Lorentz force law with help of additional conditions imposing mathematical restrictions on the possible solutions of Gauss' law. Although the classical methods to solve Maxwell's equations based on the introduction of the electromagnetic potentials are very concise and elegant, and we do not attempt to replace them by our own method, the conceptual virtue of alternative attitudes consists in showing a deeper basis of Maxwell's theory. One of the examples is the idea presented in this work to reduce Maxwell's equations in essence to Gauss' law only, which suggests the major role of this law among the other Maxwell's laws of electromagnetism.

Appendix

Let us prove that the validity of Gauss' law in any inertial reference frame entails that in any frame the Ampere-Maxwell law must also be true.

Assume a reference frame S' is boosted along the x-axis of a frame S with a velocity $\vec{V}$. Then:

$$x = \gamma(x' + Vt'), \quad t = \gamma(t' + Vx'/c^2), \quad y = y', \quad z = z'. \tag{35}$$

The respective derivatives transform as follows:

$$\frac{\partial}{\partial x'} = \gamma\left(\frac{\partial}{\partial x} + \frac{V}{c^2}\frac{\partial}{\partial t}\right), \quad \frac{\partial}{\partial t'} = \gamma\left(\frac{\partial}{\partial t} + V\frac{\partial}{\partial x}\right). \tag{36}$$

According to Assumption 2 Gauss' law is satisfied both in a frame S, that is:

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0} \tag{37}$$

and in the frame S':

$$\vec{\nabla}'\cdot\vec{E}' = \frac{\rho'}{\epsilon_0}. \tag{38}$$

Due to charge conservation law the continuity equation $\vec{\nabla}\cdot\vec{j}+\partial\rho/\partial t = 0$ holds in any frame. It can be written in a covariant form $\partial_\mu J^\mu = 0$, which shows that $J^\mu = (c\rho, \vec{j})$ is a four vector. If so, its components transform as follows:

$$\rho' = \gamma(\rho - Vj_x/c^2), \quad j'_x = \gamma(j_x - V\rho), \quad j'_y = j_y, \quad j'_z = j_z. \tag{39}$$

In turn, from Eq. (31) we find the transformation of the electric field:

$$E'_x = E_x, \quad E'_y = \gamma(E_y - VB_z), \quad E'_z = \gamma(E_z + VB_y). \tag{40}$$

Now we can express the primed quantities in Gauss' law (38) by the unprimed ones. Using Eqs. (36, 39, 40) we obtain:

$$\gamma\left(\frac{\partial}{\partial x} + \frac{V}{c^2}\frac{\partial}{\partial t}\right)E_x + \frac{\partial}{\partial y}\gamma(E_y - VB_z) + \frac{\partial}{\partial z}\gamma(E_z + VB_y) = \frac{1}{\epsilon_0}\gamma(\rho - Vj_x \tag{41}$$

Three terms on the left side combine to $\gamma\vec{\nabla}\cdot\vec{E}$ while on the right side there occurred a term $\gamma\rho/\epsilon_0$. On the basis of Eq. (37) these terms cancel out and from Eq. (41) we have:

$$\frac{1}{c^2}\frac{\partial E_x}{\partial t} + \frac{\partial B_y}{\partial z} - \frac{\partial B_z}{\partial y} = -\frac{1}{c^2\epsilon_0}j_x, \tag{42}$$

or

$$\left(\vec{\nabla}\times\vec{B}\right)_x - \frac{1}{c^2}\frac{\partial E_x}{\partial t} = \frac{j_x}{c^2\epsilon_0}. \tag{43}$$

If the frame S' was boosted in an arbitrary direction, we would have obtained the last equation in the general vector form:

$$\vec{\nabla}\times\vec{B} - \frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} = \frac{\vec{j}}{c^2\epsilon_0}, \tag{44}$$

which is the desired Ampere-Maxwell law.

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