The Yukawa Lagrangian Density is Inconsistent with the Hamiltonian

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It is proved that no Hamiltonian exists for the real Klein-Gordon field used in the Yukawa interaction. It is also shown that a real Klein-Gordon particle can be neither in a free isolated state nor in a bound state having an angular momentum $l > 0$. The experimental data support these conclusions. This outcome is in a complete agreement with Dirac's negative opinion on the Klein-Gordon equation.

Keywords: Yukawa field, Hamiltonian.

1. Introduction

About 70 years ago, the Yukawa interaction was proposed as a quantum mechanical interpretation of the nuclear force (see [1], p. 78). This interaction is derived from the Lagrangian density of a system of a Dirac field and a Klein-Gordon (KG) field (see [2], p. 79)

$$\mathcal{L}_Y = \mathcal{L}_D + \mathcal{L}_{KG} - g\phi\bar{\psi}\psi. \tag{1}$$

Here the first term on the right hand side represents the Lagrangian density of a free Dirac field (see [2], p. 43)

$$\mathcal{L}_D = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi \tag{2}$$

and the second term represents the Lagrangian density of a free KG field (see [2], p. 16)

$$\mathcal{L}_{KG} = \frac{1}{2}(g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - m^2\phi^2). \tag{3}$$

The last term of (1) represents the interaction. Since the Hamiltonian is a Hermitian operator, the KG function ϕ used here is real.

In this work, Greek indices run from 0 to 3 and Latin indices run from 1 to 3. The Lorentz metric is diagonal and its entries are (1,-1,-1,-1). Units where $\hbar = c = 1$ are used. The symbol $_{,\mu}$ denotes the partial differentiation with respect to x^μ.

Difficulties concerning the KG Lagrangian density of a complex KG function have been pointed out recently. Thus, it is proved that a KG particle cannot interact with electromagnetic fields: an application of the linear interaction $j^\mu A_\mu$, where the KG 4-current j^μ is independent of the external 4-potential A_μ, fails [3]; if the quadratic expression $(p^\mu - eA^\mu)(p_\mu - eA_\mu)$ is used then the inner product of the Hilbert space of the KG wave function ϕ is destroyed. In addition to that, there is no covariant differential operator representing the Hamiltonian of a complex KG particle [4].

Another difficulty is the inconsistency of the 4-force derived from the Yukawa potential

$$u(r) = -g^2 e^{-mr}/r \tag{4}$$

with the relativistic requirement where the 4-acceleration must be orthogonal to the 4-velocity

$$a^\mu v_\mu = 0. \tag{5}$$

This requirement is satisfied by the electromagnetic interaction, where the Lorentz force is

$$ma^\mu = eF^{\mu\nu} v_\nu. \tag{6}$$

Here the electromagnetic field tensor is antisymmetric $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$ and this property satisfies (5)

$$a^\mu v_\mu = \frac{e}{m} F^{\mu\nu} v_\nu v_\mu = 0. \tag{7}$$

On the other hand, the scalar function ϕ cannot yield an antisymmetric tensor. Therefore, the force found in the classical limit of the Yukawa interaction is inconsistent with special relativity.

2. Theoretical Problems with the Yukawa Field

The purpose of the present work is to prove that the Lagrangian density (1) of the real KG field ϕ is inconsistent with the fundamental quantum mechanical equation

$$i\frac{\partial \phi}{\partial t} = H\phi. \tag{8}$$

This task extends the validity range of the proof of [4] where the complex KG field is discussed.

The Euler-Lagrange equations of a given Lagrangian density are obtained from the following general expression (see [2], p. 16)

$$\frac{\partial}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial \frac{\partial \phi}{\partial x^\mu}} - \frac{\partial \mathcal{L}}{\partial \phi} = 0. \tag{9}$$

Applying (9) to the KG function ϕ of (1), one obtains an *inhomogeneous* KG equation

$$(\Box + m^2)\phi = g\bar{\psi}\psi. \tag{10}$$

The following argument proves that the Euler-Lagrange equation (10) obtained from the Yukawa Lagrangian density (1) is inconsistent with the existence of a Hamiltonian. Indeed, the term $\partial^2/\partial t^2$ of (10) and the independence of its right hand side on the KG wave function ϕ, prove that it is a *second order inhomogeneous* partial differential equation. On the other hand, the Hamiltonian equation (8) is a *first order homogeneous* equation. Now, assume that at a certain instant t_0, a solution ϕ_0 of (8) solves (10) too. Using the fact that (10) is a second order differential equation, one finds that its first derivative with respect to time is a free parameter. This degree of freedom proves that an infinite number of different solutions of (10) agree with the *single* solution ϕ_0 of (8) at t_0. Thus, for $t > t_0$, just one solution of (10) agrees with the solution of the Hamiltonian (8) and all other solutions differ from it.

Moreover, if ϕ_0 solves the *homogeneous* equation (8), then $c\phi_0$, where c is a constant, solves it too. Therefore, at t_0, an infinite number of physically equivalent solutions that solve the

Hamiltonian equation (8) correspond to every solution of the *inhomogeneous* Euler-Lagrange equation (10) obtained from the Yukawa Lagrangian density. (This free factor is used in a construction of an orthonormal basis for the Hilbert space of solutions of the Hamiltonian.)

Either of these results proves that the Yukawa Lagrangian density (1) is inconsistent with the existence of a Hamiltonian. It is interesting to note that the Dirac Hamiltonian agrees perfectly with the Euler-Lagrange equation obtained from the Dirac Lagrangian density (2) (see [4], p. 32).

Another aspect of the lack of a Hamiltonian is the fact that the real KG wave function has no expression for a conserved density (see [5], pp. 42, 43). (As a matter of fact, also the complex KG function has no expression for a positive definite density. The corresponding quantity used for the complex KG function is a positive or negative charge density (see [6], Section 2)). Hence, without having a self-consistent expression for density, one cannot normalize the real KG wave function ϕ. Thus, no basis for a Hilbert space can be constructed and a matrix representation for the Hamiltonian cannot exist. It means that one cannot use a Hamiltonian density for a construction of a Hamiltonian.

Another problem of the Yukawa Lagrangian density (1) is that its wave function ϕ is real. Hence, the real Yukawa function ϕ cannot be an energy-momentum eigenfunction (namely, an eigenfunction of the operators $(i\partial/\partial t, -i\nabla)$), because an energy-momentum eigenfunction has a complex factor $e^{i(kx-\omega t)}$. Therefore, the Yukawa particle cannot be in an isolated free state.

An analogous argument proves that a Yukawa particle cannot exist in a bound state where the angular quantum number $l >$

0. Indeed, in this case one finds legitimate states where the quantum number $m \neq 0$. However, the angular part of each of these states takes the form (see [7], p. 510).

$$Y_{lm}(\theta, \phi) = \frac{1}{\sqrt{2\pi}} e^{im\phi}\Theta(\theta) \tag{11}$$

Evidently, for every $m \neq 0$ this function is complex. Hence, a Yukawa particle cannot be in a bound state where $l > 0$.

A related aspect of the real Yukawa wave-function pertains to the fundamental quantum mechanical equation (8). Thus, real wave-functions have real time derivatives and pure imaginary Hamiltonian eigenvalues. This is unacceptable because Hamiltonian eigenvalues represent energy and must be real for a stable particle and must have a nonvanishing real part for a decaying particle.

At this point it is clear that the real Yukawa field cannot be a part of the current structure of quantum mechanics. Thus, one may ask whether or not an alternative theory can describe wave properties of a *massive* particle characterized by a *real* wave function. The following argument proves that such a theory cannot find an expression for the particle's energy.

(The need for a self-consistent energy expression is mandatory for the real Yukawa field described by the Lagrangian density (1). Thus, the Dirac part of (1) as well as its interaction term represent energy. Hence, in order to maintain energy balance, one needs an energy expression for the real Yukawa field.)

As argued above, one cannot construct an expression for density of a particle described by a real wave function χ (see [5], pp. 42, 43). Hence, one must use a differential operator $\hat{O}$. Now, energy is a 0-component of a 4-vector and its dimension is $[L^{-1}]$.

It follows that a differential operator for energy must be $\partial/\partial t$ multiplied by a dimensionless factor K

$$\hat{O}\chi = K\frac{\partial\chi}{\partial t} = E\chi. \tag{12}$$

Now, since energy is a real quantity and so is the wave function χ, one concludes that K is a real number.

Let us examine the simplest case of a free massive particle which is motionless in the laboratory frame. Here the energy takes the value of the particle's mass $E = m > 0$. It follows that the time-dependence of the real function χ is

$$\chi(t) = \chi(t = 0)e^{Et/K}. \tag{13}$$

Thus, an unreasonable result is derived where the real wave function, describing a static state increases or decreases exponentially with time. This conclusion casts doubts on the possibility of constructing an alternative wave theory for a *massive* particle described by a *real* wave function.

3. Experimental Problems with the Yukawa Theory

Turning to the experimental side, it is not surprising to find that Nature does not provide an experimental support for the Yukawa theory. Thus, the KG field function $\phi(x^\mu)$ depends on a *single* set of space-time coordinates. Hence, like the Dirac field $\psi(x^\mu)$, it describes a structureless pointlike particle. Now, unlike Dirac particles (electrons, muons, quarks etc.), the existence of pointlike KG particles has not been established. In particular, it is now recognized that π mesons, which are regarded as the primary example of a KG particle, are made of a quark and an

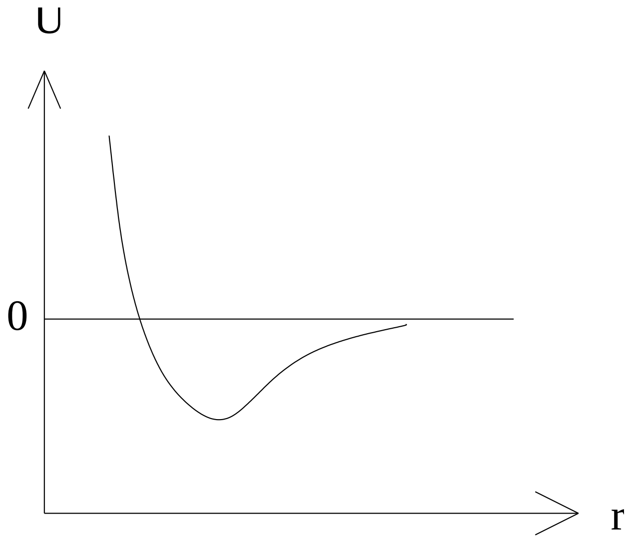

Figure 1: A qualitative description of the nucleon-nucleon phenomenological potential as a function of the distance between the nucleons' centers.

antiquark. Hence, π mesons are not pointlike particles. Experimental data confirms this conclusion (see [8], p. 499).

The Yukawa particle is a particular case of a KG particle and π^0 is the primary candidate of such particles. However, the foregoing discussion provides a proof showing that π^0 is not a Yukawa particle. Indeed, as stated above, it is not a pointlike particle. Moreover, the lifetime of π^0 is about 10^{-16} seconds (see [8], p. 500). Thus, having a relativistic velocity, the length of its path is more than 10^7 fermi. This length is much larger than the nucleon's radius which is about 1.2 fermi. Hence, π^0 is a free particle for the most of its lifetime, contrary to the above mentioned restriction on a Yukawa particle, stating that it cannot be in a free state.

The actual nuclear potential is inconsistent with the Yukawa formula (4). Indeed, the nuclear potential is characterized by a hard (repulsive) core and at its outer side there is a rapidly decreasing attractive force. Its general form is described in fig. 1 (see [1], p. 97).

Thus, the figure proves that the actual nuclear potential and its derivative with respect to r change sign. This is certainly inconsistent with the Yukawa formula (4). Indeed, neither the Yukawa potential nor its derivative change sign.

4. Conclusions

The discussion carried out above proves that the experimental side and the theoretical analysis carried out above, do not support the validity of the Yukawa theory. This conclusion is in a complete agreement with Dirac's negative opinion on the KG equation [9].

As stated in the first sentence of this work, the Yukawa theory has been proposed a very long time ago. This theory utilizes the real KG field. However, as of today, no textbook presents a Hamiltonian for this field. Note that the existence of a Hamiltonian density (see eg. [10], p.26; [11] pp. 177, 178) does not yield a Hamiltonian because, as stated above, it is proved that density cannot be defined for real fields (see [5], pp. 42, 43).) Hence, a

Hamiltonian operator cannot be extracted from the Hamiltonian density. Similarly, a Hamiltonian matrix cannot be constracted because one cannot define a Hilbert space without a self consistent expression for density. It is further explained above that the usage of any Hamiltonian that operates on a real wave function is inconsistent with the standard form of quantum mechanics. Considering these facts, one applies commonsense and concludes that such a Hamiltonian does not exist. Hence, there is a need for a proof showing this point. The present work fills this gap.

It is clear that the theoretical difficulties of the Yukawa field are derived here for its original version which takes the form of a real one component Lorentz scalar wave function used in the Lagrangian density (1). Other kinds of wave functions which may be related to the Yukawa idea are beyond the scope of this work.

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