Einstein Equations for Tetrad Fields

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Every metric tensor can be expressed by the inner product of tetrad fields. We prove that Einstein equations for these fields have the same form as the stress-energy tensor of electromagnetism.

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It is agreed that gravitation can be best described by general relativity and that it cannot be explained by using fields as in electromagnetism or as in the case of any other interaction. Furthermore, it has been assumed that the metric tensor is the best mathematical argument to use to study on gravitation. Such opinions lead physicists to concentrate more on only the metric tensor and, hence, to change it according to circumstances. As a result, this method provides some important results about gravitation. However, it is also obvious that these results are not enough to understand gravitation as well as, perhaps, other interactions.

In the present paper, instead of concentrating on the metric tensor, we shall focus on tetrad fields. Our first objective will be to find some reasonable mathematical results with these fields. The complete interpretation of the results will be out of the scope of this paper.

Gravitation curves the space-time and this effect is related to the line element or invariant interval as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

where $g_{\mu\nu}$ is the metric tensor and its elements are some functions of the space-time.

The metric tensor with tetrad fields is given by [1], [2]

$$g_{\mu\nu} = \mathbf{e}_\mu \bullet \mathbf{e}_\nu \tag{1}$$

where $\mathbf{e}_\mu$ are basis vectors or tetrad fields, and these are some functions of the space-time also ($\mu, \nu = 0,1,2,3$).

Similar to (1), the inverse metric tensor can be written as

$$g^{\mu\nu} = \mathbf{e}^\mu \bullet \mathbf{e}^\nu$$

where $\mathbf{e}^\mu$ are basis vectors of the dual space or cotetrad fields. However, we will refer to these fields as inverse fields throughout this work.

There are some useful features of and equations for the tetrad fields and inverse fields. First

$$g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu{}_\nu$$

$$\mathbf{e}^\mu \bullet \mathbf{e}^\alpha g_{\alpha\nu} = \delta^\mu{}_\nu$$

$$\mathbf{e}^\mu \bullet \mathbf{e}_\nu = \delta^\mu{}_\nu \tag{2}$$

Other equations and all detailed calculations are given in the appendix section.

If the metric tensor is determined, it is well-known that it is demanding work to find the Einstein equations. The Christoffel symbols for the metric tensor (1) are

$$\Gamma^{\alpha}{}_{\mu\nu} = \frac{1}{2}\mathbf{f}^{\alpha}{}_{\nu} \bullet \mathbf{e}_{\mu} = \frac{1}{2}\mathbf{f}^{\alpha}{}_{\mu} \bullet \mathbf{e}_{\nu}$$

where $\mathbf{f}^{\alpha}{}_{\nu} = \partial^{\alpha}\mathbf{e}_{\nu} - \partial_{\nu}\mathbf{e}^{\alpha}$.

The Riemann tensor for the above Christoffel symbols is

$$R^{\alpha}{}_{\mu\beta\nu} = \frac{1}{2}\partial^{\alpha}\mathbf{f}_{\beta\nu} \bullet \mathbf{e}_{\mu} + \frac{1}{4}\mathbf{f}^{\alpha}{}_{\nu} \bullet \mathbf{f}_{\beta\mu} + \frac{1}{4}\mathbf{f}^{\alpha}{}_{\beta} \bullet \mathbf{f}_{\mu\nu}.$$

the Ricci tensor is

$$R_{\mu\nu} = \mathbf{j}_{\nu} \bullet \mathbf{e}_{\mu} + \frac{1}{4}\mathbf{f}^{\alpha}{}_{\nu} \bullet \mathbf{f}_{\alpha\mu},$$

and the Ricci scalar is

$$R = \mathbf{j}_{\beta} \bullet \mathbf{e}^{\beta} + \frac{1}{8}\mathbf{f}^{\alpha\beta} \bullet \mathbf{f}_{\alpha\beta}$$

where $\mathbf{j}_{\nu} = \frac{1}{2}\partial^{\alpha}\mathbf{f}_{\alpha\nu} = \partial^{\alpha}\partial_{\alpha}\mathbf{e}_{\nu}$.

Finally the Einstein Tensor can be expressed as

$$G_{\mu\nu} = \frac{1}{4}\left[\mathbf{f}^{\alpha}{}_{\nu} \bullet \mathbf{f}_{\alpha\mu} - g_{\mu\nu}\left(\frac{1}{4}\mathbf{f}^{\alpha\beta} \bullet \mathbf{f}_{\alpha\beta} + \mathbf{j}_{\alpha} \bullet \mathbf{e}^{\alpha}\right)\right]. \tag{3}$$

The expression in square brackets is the same as the stress-energy tensor of electromagnetism except for the inner products. Despite this difference, the equations of motion of the tetrad fields have the same form as the Maxwell equations; that is $\partial^{\alpha}\partial_{\alpha}\mathbf{e}_{\nu} = \mathbf{j}_{\nu}$.

Several results can be obtained from (3). However, the most significant of these is that the Einstein equations for the tetrad fields certainly give the electromagnetic stress-energy tensor. More precisely, the general relativity reveals that there are some inherent constraints for tetrad fields. This means there are also definite limits for the metric tensor. Since every metric tensor can be written in terms of tetrad fields, metric tensors cannot be chosen or adjusted arbitrarily. Instead, metric tensors must be found as inner products of tetrad fields after these fields are determined to be consistent with $\partial^{\alpha}\partial_{\alpha}\mathbf{e}_{\nu} = \mathbf{j}_{\nu}$.

Appendix

In this section detailed calculations and some useful equations are given for convenience, although some of these can be found from several sources and in different forms.

Since $\mathbf{e}_{\alpha}\bullet\mathbf{e}^{\lambda} = \delta_{\alpha}{}^{\lambda}$, the partial derivatives of $\mathbf{e}_{\mu}$ can be written as $\partial_{\nu}\mathbf{e}_{\rho} = \omega_{\rho\nu\lambda}\mathbf{e}^{\lambda}$, where $\omega_{\nu\rho\lambda}$ are some coefficients. Then

$$\partial_{\nu}\mathbf{e}_{\rho}\bullet\mathbf{e}_{\alpha} = \omega_{\rho\nu\lambda}\mathbf{e}^{\lambda}\bullet\mathbf{e}_{\alpha} = \omega_{\rho\nu\alpha} .$$

So

$$\partial_{\nu}\mathbf{e}_{\rho}\bullet\mathbf{e}_{\alpha} = \omega_{\rho\nu\lambda}\mathbf{e}^{\lambda}\bullet\mathbf{e}_{\alpha} = \omega_{\rho\nu\alpha} ,$$

$$\left(\partial_{\nu}\mathbf{e}_{\rho}\bullet\mathbf{e}_{\alpha}\right)\mathbf{e}^{\alpha} = \partial_{\nu}\mathbf{e}_{\rho} . \tag{A.1}$$

Similarly it can be shown that

$$\left(\partial_{\nu}\mathbf{e}_{\rho}\bullet\mathbf{e}^{\alpha}\right)\mathbf{e}_{\alpha} = \partial_{\nu}\mathbf{e}_{\rho} . \tag{A.2}$$

Another important equation can be derived by starting from $\partial_{\nu}\mathbf{e}_{\rho}\bullet\mathbf{e}^{\nu} = \partial_{\lambda}\mathbf{e}_{\rho}\bullet\mathbf{e}^{\lambda}$. Using (A.1)

$$\partial_{\nu}\mathbf{e}_{\rho}\bullet\mathbf{e}^{\nu} = \left(\partial_{\lambda}\mathbf{e}_{\rho}\bullet\mathbf{e}_{\nu}\right)\mathbf{e}^{\nu}\bullet\mathbf{e}^{\lambda}$$

$$\partial_\nu \mathbf{e}_\rho \bullet \mathbf{e}^\nu = \left[\left(\partial_\lambda \mathbf{e}_\rho \bullet \mathbf{e}_\nu \right) \mathbf{e}^\lambda \right] \bullet \mathbf{e}^\nu$$

$$\partial_\nu \mathbf{e}_\rho = \left(\partial_\lambda \mathbf{e}_\rho \bullet \mathbf{e}_\nu \right) \mathbf{e}^\lambda$$

The dot product of the last equation with $\mathbf{e}_\alpha$ is

$$\partial_\nu \mathbf{e}_\rho \bullet \mathbf{e}_\alpha = \left(\partial_\lambda \mathbf{e}_\rho \bullet \mathbf{e}_\nu \right) \mathbf{e}^\lambda \bullet \mathbf{e}_\alpha .$$

Since $\mathbf{e}_\alpha \bullet \mathbf{e}^\lambda = \delta_\alpha{}^\lambda$

$$\partial_\nu \mathbf{e}_\rho \bullet \mathbf{e}_\alpha = \partial_\alpha \mathbf{e}_\rho \bullet \mathbf{e}_\nu . \tag{A.3}$$

Similarly it can be found that

$$\partial_\nu \mathbf{e}_\rho \bullet \mathbf{e}^\alpha = \partial^\alpha \mathbf{e}_\rho \bullet \mathbf{e}_\nu . \tag{A.4}$$

Another equation can be derived if the derivative of (2) is rewritten as

$$\partial_\alpha \mathbf{e}^\mu \bullet \mathbf{e}_\nu = -\mathbf{e}^\mu \bullet \partial_\alpha \mathbf{e}_\nu . \tag{A.5}$$

Now we can start to calculate the Einstein equations. The Christoffel symbols are

$$\Gamma^\alpha{}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left[\partial_\mu \mathbf{e}_\nu \bullet \mathbf{e}_\beta + \partial_\mu \mathbf{e}_\beta \bullet \mathbf{e}_\nu + \partial_\nu \mathbf{e}_\mu \bullet \mathbf{e}_\beta \right]$$
$$+ \frac{1}{2} g^{\alpha\beta} \left[\partial_\nu \mathbf{e}_\beta \bullet \mathbf{e}_\mu - \partial_\beta \mathbf{e}_\mu \bullet \mathbf{e}_\nu - \partial_\beta \mathbf{e}_\nu \bullet \mathbf{e}_\mu \right].$$

Using (A.3), we get

$$\Gamma^\alpha{}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left(\partial_\mu \mathbf{e}_\beta \bullet \mathbf{e}_\nu + \partial_\nu \mathbf{e}_\beta \bullet \mathbf{e}_\mu \right). \tag{A.6}$$

Symmetries and characteristics of the tetrad fields enable to derive some helpful identities. First using (A.1)

$$\Gamma^{\alpha}{}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left(\left(\partial_{\mu} \mathbf{e}_{\beta} \bullet \mathbf{e}^{\rho} \right) \mathbf{e}_{\rho} \bullet \mathbf{e}_{\nu} + \left(\partial_{\nu} \mathbf{e}_{\beta} \bullet \mathbf{e}^{\rho} \right) \mathbf{e}_{\rho} \bullet \mathbf{e}_{\mu} \right).$$

Similarly, (A.5) enables us to write

$$\Gamma^{\alpha}{}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left(-\left(\partial_{\mu} \mathbf{e}^{\rho} \bullet \mathbf{e}_{\beta} \right) \mathbf{e}_{\rho} \bullet \mathbf{e}_{\nu} - \left(\partial_{\nu} \mathbf{e}^{\rho} \bullet \mathbf{e}_{\beta} \right) \mathbf{e}_{\rho} \bullet \mathbf{e}_{\mu} \right),$$

$$\Gamma^{\alpha}{}_{\mu\nu} = \frac{1}{2} \left(-\left(\partial_{\mu} \mathbf{e}^{\rho} \bullet \mathbf{e}^{\alpha} \right) \mathbf{e}_{\rho} \bullet \mathbf{e}_{\nu} - \left(\partial_{\nu} \mathbf{e}^{\rho} \bullet \mathbf{e}^{\alpha} \right) \mathbf{e}_{\rho} \bullet \mathbf{e}_{\mu} \right).$$

Using (A.3) and (A.4), we can obtain

$$\Gamma^{\alpha}{}_{\mu\nu} = \frac{1}{2} \left(-\left(\partial^{\alpha} \mathbf{e}^{\rho} \bullet \mathbf{e}_{\mu} \right) \mathbf{e}_{\rho} \bullet \mathbf{e}_{\nu} - \left(\partial^{\alpha} \mathbf{e}^{\rho} \bullet \mathbf{e}_{\nu} \right) \mathbf{e}_{\rho} \bullet \mathbf{e}_{\mu} \right),$$

$$\Gamma^{\alpha}{}_{\mu\nu} = \frac{1}{2} \left(\left(\partial^{\alpha} \mathbf{e}_{\mu} \bullet \mathbf{e}^{\rho} \right) \mathbf{e}_{\rho} \bullet \mathbf{e}_{\nu} + \left(\partial^{\alpha} \mathbf{e}_{\nu} \bullet \mathbf{e}^{\rho} \right) \mathbf{e}_{\rho} \bullet \mathbf{e}_{\mu} \right).$$

Also, by using (A.3) and (A.4)

$$\Gamma^{\alpha}{}_{\mu\nu} = \frac{1}{2} \left(\partial^{\alpha} \mathbf{e}_{\mu} \bullet \mathbf{e}_{\nu} + \partial_{\mu} \mathbf{e}_{\nu} \bullet \mathbf{e}^{\alpha} \right) \tag{A.7}$$

and using (A.5) again

$$\Gamma^{\alpha}{}_{\mu\nu} = \frac{1}{2} \left(\partial^{\alpha} \mathbf{e}_{\mu} \bullet \mathbf{e}_{\nu} - \partial_{\mu} \mathbf{e}^{\alpha} \bullet \mathbf{e}_{\nu} \right) = \frac{1}{2} \left(\partial^{\alpha} \mathbf{e}_{\mu} - \partial_{\mu} \mathbf{e}^{\alpha} \right) \bullet \mathbf{e}_{\nu}$$

can be found.

Although it can be proved easily that $\partial^{\alpha} \mathbf{e}_{\nu} = -\partial_{\nu} \mathbf{e}^{\alpha}$, for convenience let

$$\mathbf{f}^{\alpha}{}_{\nu} = \partial^{\alpha} \mathbf{e}_{\nu} - \partial_{\nu} \mathbf{e}^{\alpha}.$$

Then the Christoffel symbols are

$$\Gamma^{\alpha}{}_{\mu\nu} = \frac{1}{2} \mathbf{f}^{\alpha}{}_{\mu} \bullet \mathbf{e}_{\nu}. \tag{A.8}$$

With similar calculations, $\Gamma^{\alpha}{}_{\mu\nu}$ can be found in the following form

$$\Gamma^{\alpha}{}_{\mu\nu} = \frac{1}{2}\mathbf{f}^{\alpha}{}_{\nu} \bullet \mathbf{e}_{\mu}.$$

Or using (A.4), (A.7) can be rewritten as

$$\Gamma^{\alpha}{}_{\mu\nu} = \frac{1}{2}\left(\partial_{\nu}\mathbf{e}_{\mu} + \partial_{\mu}\mathbf{e}_{\nu}\right) \bullet \mathbf{e}^{\alpha}.$$

When the Einstein Equations are calculated, one of these $\Gamma^{\alpha}{}_{\mu\nu}$ can be used.

The Riemann tensor defined by

$$R^{\alpha}{}_{\mu\beta\nu} = \partial_{\beta}\Gamma^{\alpha}{}_{\nu\mu} - \partial_{\nu}\Gamma^{\alpha}{}_{\beta\mu} + \Gamma^{\alpha}{}_{\beta\lambda}\Gamma^{\lambda}{}_{\nu\mu} - \Gamma^{\alpha}{}_{\nu\lambda}\Gamma^{\lambda}{}_{\beta\mu}.$$

using the above Christoffel symbols

$$R^{\alpha}{}_{\mu\beta\nu} = \frac{1}{2}\partial_{\beta}\mathbf{f}^{\alpha}{}_{\nu} \bullet \mathbf{e}_{\mu} + \frac{1}{2}\mathbf{f}^{\alpha}{}_{\nu} \bullet \partial_{\beta}\mathbf{e}_{\mu} - \frac{1}{2}\partial_{\nu}\mathbf{f}^{\alpha}{}_{\beta} \bullet \mathbf{e}_{\mu} - \frac{1}{2}\mathbf{f}^{\alpha}{}_{\beta} \bullet \partial_{\nu}\mathbf{e}_{\mu}$$
$$+ \frac{1}{4}\mathbf{f}^{\alpha}{}_{\beta} \bullet \left(\partial_{\nu}\mathbf{e}_{\mu} + \partial_{\mu}\mathbf{e}_{\nu}\right) - \frac{1}{4}\mathbf{f}^{\alpha}{}_{\nu} \bullet \left(\partial_{\beta}\mathbf{e}_{\mu} + \partial_{\mu}\mathbf{e}_{\nu}\right),$$

$$R^{\alpha}{}_{\mu\beta\nu} = \frac{1}{2}\partial_{\beta}\left(\mathbf{f}^{\alpha}{}_{\nu} \bullet \mathbf{e}_{\mu}\right) - \frac{1}{2}\partial_{\nu}\left(\mathbf{f}^{\alpha}{}_{\beta} \bullet \mathbf{e}_{\mu}\right)$$
$$+ \frac{1}{4}\left(\mathbf{f}^{\alpha}{}_{\beta} \bullet \mathbf{e}_{\lambda}\right)\left(\left(\partial_{\nu}\mathbf{e}_{\mu} + \partial_{\mu}\mathbf{e}_{\nu}\right) \bullet \mathbf{e}^{\lambda}\right) - \frac{1}{4}\left(\mathbf{f}^{\alpha}{}_{\nu} \bullet \mathbf{e}_{\lambda}\right)\left(\left(\partial_{\beta}\mathbf{e}_{\mu} + \partial_{\mu}\mathbf{e}_{\nu}\right) \bullet \mathbf{e}^{\lambda}\right),$$

$$R^{\alpha}{}_{\mu\beta\nu} = \frac{1}{2}\left(\partial_{\beta}\mathbf{f}^{\alpha}{}_{\nu} - \partial_{\nu}\mathrm{f}^{\alpha}{}_{\beta}\right) \bullet \mathbf{e}_{\mu} + \frac{1}{2}\mathbf{f}^{\alpha}{}_{\nu} \bullet \left(\partial_{\beta}\mathbf{e}_{\mu} - \frac{1}{2}\left(\partial_{\beta}\mathbf{e}_{\mu} + \partial_{\mu}\mathbf{e}_{\beta}\right)\right)$$
$$+ \frac{1}{2}\mathbf{f}^{\alpha}{}_{\beta} \bullet \left(\frac{1}{2}\left(\partial_{\nu}\mathbf{e}_{\mu} + \partial_{\mu}\mathbf{e}_{\nu}\right) - \partial_{\nu}\mathbf{e}_{\mu}\right),$$

$$R^{\alpha}{}_{\mu\beta\nu} = \frac{1}{2}\left(\partial_{\beta}\mathbf{f}^{\alpha}{}_{\nu} + \partial_{\nu}\mathrm{f}_{\beta}{}^{\alpha}\right)\bullet\mathbf{e}_{\mu} + \frac{1}{4}\mathbf{f}^{\alpha}{}_{\nu}\bullet\mathbf{f}_{\beta\mu} + \frac{1}{4}\mathbf{f}^{\alpha}{}_{\beta}\bullet\mathbf{f}_{\mu\nu}.$$

As a result of $\partial_{\beta}\mathbf{f}^{\alpha}{}_{\nu} + \partial_{\nu}\mathbf{f}_{\beta}{}^{\alpha} + \partial^{\alpha}\mathbf{f}_{\nu\beta} = 0$, the last Riemann tensor can be simplified. For this, write $\partial_{\beta}\mathbf{f}^{\alpha}{}_{\nu} + \partial_{\nu}\mathbf{f}_{\beta}{}^{\alpha} = -\partial^{\alpha}\mathbf{f}_{\nu\beta} = \partial^{\alpha}\mathbf{f}_{\beta\nu}$. Thus,

$$R^{\alpha}{}_{\mu\beta\nu} = \frac{1}{2}\partial^{\alpha}\mathbf{f}_{\beta\nu}\bullet\mathbf{e}_{\mu} + \frac{1}{4}\mathbf{f}^{\alpha}{}_{\nu}\bullet\mathbf{f}_{\beta\mu} + \frac{1}{4}\mathbf{f}^{\alpha}{}_{\beta}\bullet\mathbf{f}_{\mu\nu}.$$

The Ricci tensor is

$$R^{\alpha}{}_{\mu\beta\nu} = \frac{1}{2}\partial^{\alpha}\mathbf{f}_{\beta\nu}\bullet\mathbf{e}_{\mu} + \frac{1}{4}\mathbf{f}^{\alpha}{}_{\nu}\bullet\mathbf{f}_{\beta\mu} + \frac{1}{4}\mathbf{f}^{\alpha}{}_{\beta}\bullet\mathbf{f}_{\mu\nu}.$$

Let $\mathbf{j}_{\nu} = \frac{1}{2}\partial^{\alpha}\mathbf{f}_{\alpha\nu} = \partial^{\alpha}\partial_{\alpha}\mathbf{e}_{\nu}$. Thus the Ricci tensor becomes

$$R_{\mu\nu} = \mathbf{j}_{\nu}\bullet\mathbf{e}_{\mu} + \frac{1}{4}\mathbf{f}^{\alpha}{}_{\nu}\bullet\mathbf{f}_{\alpha\mu}.$$

and the Ricci scalar is

$$R = R_{\alpha\beta}g^{\alpha\beta} = \mathbf{j}_{\beta}\bullet\mathbf{e}^{\beta} + \frac{1}{8}\mathbf{f}^{\alpha\beta}\bullet\mathbf{f}_{\alpha\beta}.$$

Here $\mathbf{f}^{\alpha\beta}\bullet\mathbf{f}_{\alpha\beta}$ is multiplied by $\frac{1}{2}$ because of twofold summation.

The Einstein tensor is defined as $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$. Then,

$$G_{\mu\nu} = \mathbf{j}_{\nu}\bullet\mathbf{e}_{\mu} + \frac{1}{4}\mathbf{f}^{\alpha}{}_{\nu}\bullet\mathbf{f}_{\alpha\mu} - \frac{1}{2}g_{\mu\nu}\left[\mathbf{j}_{\alpha}\bullet\mathbf{e}^{\alpha} + \frac{1}{8}\mathbf{f}^{\alpha\beta}\bullet\mathbf{f}_{\alpha\beta}\right].$$

The last expression can be simplified if we start with

$$\mathbf{j}_{\mu}\bullet\mathbf{e}^{\mu} = \mathbf{j}_{\beta}\bullet\mathbf{e}^{\beta}.$$

As a result of (2), $\mathbf{e}_\mu \bullet \mathbf{e}^\mu = 4$. So $\dfrac{\mathbf{e}_\mu \bullet \mathbf{e}^\mu}{4} = 1$ and we can write

$$\mathbf{j}_\mu \bullet \mathbf{e}^\mu = \left(\mathbf{j}_\beta \bullet \mathbf{e}^\beta\right)\frac{\left(\mathbf{e}_\mu \bullet \mathbf{e}^\mu\right)}{4},$$

$$\mathbf{j}_\mu \bullet \mathbf{e}^\mu = \frac{1}{4}\left(\left(\mathbf{j}_\beta \bullet \mathbf{e}^\beta\right)\mathbf{e}_\mu\right)\bullet \mathbf{e}^\mu,$$

$$\mathbf{j}_\mu = \frac{1}{4}\left(\mathbf{j}_\beta \bullet \mathbf{e}^\beta\right)\mathbf{e}_\mu .$$

The dot product of the last equation with $\mathbf{e}_\nu$ yields

$$\mathbf{j}_\mu \bullet \mathbf{e}_\nu = \frac{1}{4}\left(\mathbf{j}_\beta \bullet \mathbf{e}^\beta\right)\mathbf{e}_\mu \bullet \mathbf{e}_\nu = \frac{1}{4}\left(\mathbf{j}_\beta \bullet \mathbf{e}^\beta\right) g_{\mu\nu} .$$

Finally, the Einstein tensor becomes

$$G_{\mu\nu} = \frac{1}{4}\left[\mathbf{f}^{\alpha}{}_{\nu} \bullet \mathbf{f}_{\alpha\mu} - g_{\mu\nu}\left(\frac{1}{4}\mathbf{f}^{\alpha\beta} \bullet \mathbf{f}_{\alpha\beta} + \mathbf{j}_\alpha \bullet \mathbf{e}^\alpha\right)\right].$$

References

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