The stability of the QED vacuum in the temporal gauge

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The stability of the vacuum for QED in the temporal gauge will be examined. It is generally assumed that the vacuum state is the quantum state with the lowest energy. However, it is easy to show that this is not true in general but depends on the nature of the Hamiltonian that describes the system. It will be shown that this assumption does not hold for a system consisting of a fermion field coupled to a quantized electromagnetic field in the temporal gauge.

I. Introduction.

In this article we will examine the problem of the stability of the vacuum in quantum field theory. If $|\Omega\rangle$ is a normalized state vector and $\hat{H}$ is the Hamiltonian then the energy is given by,

$$E\left(|\Omega\rangle\right)=\langle\Omega|\hat{H}|\Omega\rangle \tag{1.1}$$

The question we want to examine is whether or not there exists a lower bound to the energy of a quantum state. That is, does there exist a normalized state vector $\left|\Omega_{vac}\right\rangle$, usually considered to the be the vacuum state, where,

$$E\left(\left|\Omega\right\rangle\right) - E\left(\left|\Omega_{vac}\right\rangle\right) \geq 0 \text{ for all normalized state vectors } \left|\Omega\right\rangle \quad (1.2)$$

The answer to this question obviously depends on the nature of the Hamiltonian. For example consider the following Hamiltonian for a self-interacting scalar field,

$$\hat{H} = \hat{H}_0 + \hat{H}_I \quad (1.3)$$

where,

$$\hat{H}_0 = \frac{1}{2}\int\left(\hat{\pi}^2 + \left|\nabla\hat{\varphi}\right|^2 + m^2\hat{\varphi}^2\right)d\mathbf{x} \text{ and } \hat{H}_I = \omega\int\hat{\varphi}^3 d\mathbf{x} \quad (1.4)$$

In the above expression $\hat{\pi}(\mathbf{x})$ and $\hat{\varphi}(\mathbf{x})$ are the usual field operators, m is the mass, $\hat{H}_0$ is the interaction free Hamiltonian, $\hat{H}_I$ is the interaction, and ω is a positive coupling constant. Note that throughout this discussion we will use $\hbar = c = 1$. Also vectors are indicated by bold text. In addition, for sections I and II we will suppress the time dependence because are analyzing the quantum systems at a given instant of time. Later, in Section III, where we consider the time evolution of a state, we will work in the Schrödinger picture and assign time dependence to the state vector.

Now is there a lower bound to the energy for this system? Does there exist a state $\left|\Omega_{vac}\right\rangle$ for which the relationship (1.2) is valid? For the moment let us first consider the above question for the case where $\hat{\pi}$ and $\hat{\varphi}$ are not operators but classical quantities, i.e., real numbers. In this case it is evident that $\hat{H}_0 \geq 0$ for any combination of $\hat{\pi}$ and $\hat{\varphi}$. However the interaction term $\hat{H}_I$ will be negative if $\hat{\varphi}$ is negative. As $\hat{\varphi}$ increases in magnitude $\hat{H}_I$ will dominate the expressions and it is evident that there is no lower bound to the energy. So much for the classical case.

Now let us consider the quantized system. The system is quantized by having the field operators obey the commutation relationships,

$$\left[\hat{\varphi}(\mathbf{x}),\hat{\pi}(\mathbf{y})\right]=i\delta^3(\mathbf{y}-\mathbf{x});\ \left[\hat{\varphi}(\mathbf{x}),\hat{\varphi}(\mathbf{y})\right]=0;\ \left[\hat{\pi}(\mathbf{x}),\hat{\pi}(\mathbf{y})\right]=0 \quad (1.5)$$

Based on the above discussion we would strongly suspect that there is no lower bound to the energy of the quantized system. We will now prove that this is case as follows. If $|\Omega\rangle$ is a normalized state vector then it is always possible to produce another state vector by operating on $|\Omega\rangle$ with the field operators [1]. Let $|\Omega'\rangle$ be defined by,

$$|\Omega'\rangle = e^{-i\hat{F}}|\Omega\rangle \quad (1.6)$$

where,

$$\hat{F} = \int \hat{\pi}\chi d\mathbf{x} \quad (1.7)$$

and where $\chi(\vec{x})$ is a real valued function. Due to the fact that $\hat{\pi}$ is real (so that $\hat{\pi} = \hat{\pi}^\dagger$) we have $\hat{F} = \hat{F}^\dagger$ so that the state dual to $|\Omega'\rangle$ is,

$$\langle\Omega'| = \langle\Omega|e^{i\hat{F}} \quad (1.8)$$

From the above relationships we obtain $\langle\Omega'|\Omega'\rangle = \langle\Omega|\Omega\rangle$. So that $|\Omega'\rangle$ is normalized since $|\Omega\rangle$ is normalized. The energy of $|\Omega'\rangle$ is then,

$$\langle\Omega'|\hat{H}|\Omega'\rangle = \langle\Omega|e^{+i\hat{F}}\hat{H}e^{-i\hat{F}}|\Omega\rangle \quad (1.9)$$

To evaluate the above expression we will use the following relationships. If $\hat{O}_1$ and $\hat{O}_2$ are operators then,

$$e^{+i\hat{F}}\hat{O}_1\hat{O}_2e^{-i\hat{F}} = \left(e^{+i\hat{F}}\hat{O}_1e^{-i\hat{F}}\right)\left(e^{+i\hat{F}}\hat{O}_2e^{-i\hat{F}}\right) \tag{1.10}$$

From (1.5) we obtain,

$$\left[\hat{\varphi}(\mathbf{x}),\hat{F}\right] = i\chi(\mathbf{x});\quad \left[\hat{\pi}(\vec{x}),\hat{F}\right] = 0 \tag{1.11}$$

Also we will use the Baker-Campell-Hausdorff relationships [2] which state that,

$$e^{+\hat{O}_1}\hat{O}_2e^{-\hat{O}_1} = \hat{O}_2 + \left[\hat{O}_1,\hat{O}_2\right] + \frac{1}{2}\left[\hat{O}_1,\left[\hat{O}_1,\hat{O}_2\right]\right] + \ldots \tag{1.12}$$

Using these relationships we obtain,

$$e^{+i\hat{F}}\hat{\varphi}(\mathbf{x})e^{-i\hat{F}} = \hat{\varphi}(\mathbf{x}) + \chi(\mathbf{x});\quad e^{+i\hat{F}}\pi(\mathbf{x})e^{-i\hat{F}} = \pi(\mathbf{x}) \tag{1.13}$$

Use these results to obtain,

$$\begin{aligned} e^{+i\hat{F}}\hat{H}e^{-i\hat{F}} &= \frac{1}{2}\int\left(\hat{\pi}^2 + \left|\nabla(\hat{\varphi}+\chi)\right|^2 + m^2(\hat{\varphi}+\chi)^2\right)d\mathbf{x} \\ &\quad + \omega\int(\hat{\varphi}+\chi)^3\,d\mathbf{x} \end{aligned} \tag{1.14}$$

Apply this to equation (1.9) to obtain,

$$\begin{aligned} \langle\Omega'|\hat{H}|\Omega'\rangle &= \langle\Omega|\hat{H}|\Omega\rangle + \frac{1}{2}\int\langle\Omega|\begin{pmatrix} 2\nabla\hat{\varphi}\cdot\nabla\chi + |\nabla\chi|^2 \\ +m^2\left(2\hat{\varphi}\chi + \chi^2\right)\end{pmatrix}|\Omega\rangle d\mathbf{x} \\ &\quad + \omega\int\langle\Omega|\left(3\hat{\varphi}^2\chi + 3\hat{\varphi}\chi^2 + \chi^3\right)|\Omega\rangle\,d\mathbf{x} \end{aligned} \tag{1.15}$$

It is evident that as $|\chi(\mathbf{x})| \to \infty$ the above expression will be dominated by the χ^3 term. Therefore as χ approaches negative infinity the energy of the state $\langle \Omega' | \hat{H} | \Omega' \rangle$ will also approach negative infinity. Therefore there is no lower bound to the energy and equation (1.2) does not hold for the Hamiltonian given by (1.3) and (1.4).

The above example was somewhat trivial and was introduced to illustrate the fact that we cannot simply assume that a lower bound exists to the energy for a quantum system. This must be checked for the Hamiltonian in question.

II. The QED Hamiltonian

Now let us apply the results of the above section to the QED Hamiltonian which describes the interaction between a quantized fermion field and quantized electromagnetic field. It will be convenient to work in the temporal gauge. In the temporal gauge the gauge condition is given by the relationship $A_0 = 0$ [3,4,5,6] where A_0 is the scalar component of the electric potential. The advantage of the temporal gauge is due to the simplicity of the commutation relationship between the electromagnetic field quantities which are given below. In the coulomb gauge, for instance, these are more complicated. Due to this fact the temporal gauge is particularly useful in the treatments of QED which use the functional Schrödinger equation [5,6]. The Hamiltonian $\hat{H}$ is given by [5],

$$\hat{H} = \hat{H}_{0,D} + \hat{H}_{0,M} - \int \hat{\mathbf{J}}(\mathbf{x}) \cdot \hat{\mathbf{A}}(\mathbf{x}) d\mathbf{x} \tag{2.1}$$

The quantities in the above expression are defined by,

$$\hat{H}_{0,D} = \frac{1}{2} \int \left[\hat{\psi}^\dagger(\mathbf{x}), H_{0,D} \hat{\psi}(\mathbf{x}) \right] d\mathbf{x}\,; \quad H_{0,D} = -i\boldsymbol{\alpha} \cdot \nabla + \beta m \tag{2.2}$$

$$\hat{H}_{0,M} = \frac{1}{2}\int\left(\hat{\mathbf{E}}^2 + \hat{\mathbf{B}}^2\right)d\mathbf{x}\,;\;\; \hat{\mathbf{B}}(\mathbf{x}) = \nabla\times\hat{\mathbf{A}}(\mathbf{x}) \tag{2.3}$$

$$\hat{\mathbf{J}}(\mathbf{x}) = \frac{q}{2}\left[\hat{\psi}^\dagger(\mathbf{x}), \boldsymbol{\alpha}\hat{\psi}(\mathbf{x})\right] \tag{2.4}$$

In the above expressions m is the fermion mass, $\boldsymbol{\alpha}$ and β are the usual 4x4 matrices, q is the electric charge, $\hat{H}_{0,D}$ is the Dirac Hamiltonian, $\hat{H}_{0,M}$ is the Hamiltonian for the electromagnetic field, and $\hat{\mathbf{J}}(\mathbf{x})$ is the current operator. The fermion field operators are $\hat{\psi}(\mathbf{x})$ and $\hat{\psi}^\dagger(\mathbf{x})$ and the field operators for the electromagnetic field are $\hat{\mathbf{A}}(\mathbf{x})$ and $\hat{\mathbf{E}}(\mathbf{x})$. The electromagnetic field operators are real so that $\hat{\mathbf{A}}^\dagger(\mathbf{x}) = \hat{\mathbf{A}}(\mathbf{x})$ and $\hat{\mathbf{E}}^\dagger(\mathbf{x}) = \hat{\mathbf{E}}(\mathbf{x})$.

The field operators obey the following relationships [4,5],

$$\begin{gathered}\left[\hat{A}^i(\mathbf{x}), \hat{E}^j(\mathbf{y})\right] = -i\delta_{ij}\delta^3(\mathbf{x}-\mathbf{y}) \\ \left[\hat{A}^i(\mathbf{x}), \hat{A}^j(\mathbf{y})\right] = \left[\hat{E}^i(\mathbf{x}), \hat{E}^j(\mathbf{y})\right] = 0\end{gathered} \tag{2.5}$$

and

$$\begin{gathered}\left\{\hat{\psi}_a^\dagger(\mathbf{x}), \hat{\psi}_b(\mathbf{y})\right\} = \delta_{ab}\delta(\mathbf{x}-\mathbf{y}) \\ \left\{\hat{\psi}_a^\dagger(\mathbf{x}), \hat{\psi}_b^\dagger(\mathbf{y})\right\} = \left\{\hat{\psi}_a(\mathbf{x}), \hat{\psi}_b(\mathbf{y})\right\} = 0\end{gathered} \tag{2.6}$$

where “a” and “b” are spinor indices. In addition, all commutators between the electromagnetic field operators and fermion field operators are zero, i.e.,

$$\left[\hat{\mathbf{A}}(\mathbf{x}), \hat{\psi}(\mathbf{y})\right] = \left[\hat{\mathbf{E}}(\mathbf{x}), \hat{\psi}(\mathbf{y})\right] = \left[\hat{\mathbf{A}}(\mathbf{x}), \hat{\psi}^\dagger(\mathbf{y})\right] = \left[\hat{\mathbf{E}}(\mathbf{x}), \hat{\psi}^\dagger(\mathbf{y})\right] = 0 \tag{2.7}$$

Next define,

$$\hat{G}(\mathbf{x}) = \nabla \cdot \hat{\mathbf{E}}(\mathbf{x}) - \hat{\rho}(\mathbf{x}) \tag{2.8}$$

where the current operator $\hat{\rho}(\mathbf{x})$ is defined by,

$$\hat{\rho}(\mathbf{x}) = \frac{q}{2}\left[\hat{\psi}^{\dagger}(\mathbf{x}), \hat{\psi}(\mathbf{x})\right] \tag{2.9}$$

All physically acceptable state vectors $|\Omega\rangle$ must satisfy the gauss's law constraint [5],

$$\hat{G}(\mathbf{x})|\Omega\rangle = 0 \tag{2.10}$$

Now we want to determine if there is a lower bound to the energy for the QED Hamiltonian. Is the relationship (1.2) true for this case? Proceeding along the lines of the discussion in the previous section let us assume for the moment that the quantities in the expression for the Hamiltonian are not operators but classical quantities, i.e., complex numbers in the case of the fermion field and real numbers in the case of the electromagnetic field. This is, of course, not a mathematically correct way to analyze the problem but is simply used to guide our intuition and to motivate further study. The interaction term consists of a fermion current multiplying an electric potential. It is evident that this term can make an arbitrarily large negative contribution to the energy. Therefore it is possible that there may not be a lower bound to the energy at this, initial, level of analysis. This suggests that it would be of value to examine the situation in more detail. We will do this using the techniques of the last section.

Start by assuming that there exists a normalized state $|\Omega_1\rangle$ which satisfies Gauss's law and for which the divergence of the current expectation value is non-zero, that is,

$$\nabla \cdot \langle\Omega_1|\mathbf{J}(\mathbf{x})|\Omega_1\rangle \neq 0 \text{ in some region of space.} \tag{2.11}$$

Before proceeding we must ask the question "how do we know that a state $\left|\Omega_1\right\rangle$ can be found where the above condition holds?". The answer is that if quantum mechanics is a correct model of the real world then there must exist many states where the above condition holds because in the real world there are many examples where the divergence of the current is non-zero over some region of space. For example in classical physics one can envision a point charge moving at some velocity. For this case the divergence of the classical current is obviously non-zero. The quantum mechanics approximation to this is a wave packet confined to some small region of space and moving with some velocity. In this case the divergence of the current expectation value will be non-zero. Next define some new state as follows,

$$\left|\Omega_2\right\rangle = e^{-i\hat{C}}\left|\Omega_1\right\rangle \tag{2.12}$$

where the operator $\hat{C}$ is defined by,

$$\hat{C} = \int \hat{\mathbf{E}}(\mathbf{x}) \cdot \nabla\chi(\mathbf{x})\, d\mathbf{x} \tag{2.13}$$

and where $\chi(\mathbf{x})$ is an arbitrary real valued function. Note that the dual state is,

$$\left\langle\Omega_2\right| = \left\langle\Omega_1\right| e^{+i\hat{C}^\dagger} = \left\langle\Omega_1\right| e^{+i\hat{C}} \tag{2.14}$$

where we have used $\hat{C}^\dagger = \hat{C}$ since $\hat{\mathbf{E}}(\mathbf{x})$ and $\chi(\mathbf{x})$ are both real. From this we have that $\left\langle\Omega_2 | \Omega_2\right\rangle = \left\langle\Omega_1 | \Omega_1\right\rangle = 1$ where we use the relationship,

$$e^{+i\hat{C}} e^{-i\hat{C}} = 1 \tag{2.15}$$

Now is $|\Omega_2\rangle$ a valid state, i.e., does it satisfy (2.10)? Based on the commutator relationships (2.5) and (2.7) we see that the operator $\hat{C}$ commutes with both $\hat{\mathbf{E}}(\mathbf{x})$ and $\hat{\rho}(\mathbf{x})$. Therefore $\hat{G}(\mathbf{x})|\Omega_2\rangle = e^{-i\hat{C}}\hat{G}(\mathbf{x})|\Omega_1\rangle = 0$ so that $|\Omega_2\rangle$ satisfies (2.10) since $|\Omega_1\rangle$ has been assumed to satisfy $\hat{G}(\mathbf{x})|\Omega_1\rangle = 0$.

Next we want to evaluate the energy of the state $|\Omega_2\rangle$. To do this use (2.1) and (1.1) to obtain,

$$E(|\Omega_2\rangle) = \langle\Omega_2|\hat{H}_{0,D}|\Omega_2\rangle + \langle\Omega_2|\hat{H}_{0,M}|\Omega_2\rangle - \langle\Omega_2|\int \hat{\mathbf{J}}(\mathbf{x})\cdot\hat{\mathbf{A}}(\mathbf{x})d\mathbf{x}|\Omega_2\rangle \tag{2.16}$$

Consider first the term $\langle\Omega_2|\hat{H}_{0,D}|\Omega_2\rangle$. To evaluate this use the fact that $\hat{\mathbf{E}}(\mathbf{x})$, and thereby $\hat{C}$, commutes with the fermion field operators $\hat{\psi}(\mathbf{x})$ and $\hat{\psi}^{\dagger}(\mathbf{x})$. Use this fact along with (2.15) to obtain,

$$\langle\Omega_2|\hat{H}_{0,D}|\Omega_2\rangle = \langle\Omega_1|\hat{H}_{0,D}|\Omega_1\rangle \tag{2.17}$$

Next consider the term $\langle\Omega_2|\hat{H}_{0,M}|\Omega_2\rangle$. From (2.5) we obtain,

$$\left[\hat{\mathbf{A}}(\mathbf{x}),\hat{C}\right] = -i\nabla\chi(\mathbf{x}) \tag{2.18}$$

Use this result to obtain,

$$\left[\hat{\mathbf{B}}(\mathbf{x}),\hat{C}\right] = \nabla\times\left[\hat{\mathbf{A}}(\mathbf{x}),\hat{C}\right] = -i\nabla\times\nabla\chi(\mathbf{x}) = 0 \tag{2.19}$$

Therefore $\hat{C}$ commutes with $\hat{H}_{0,M}$ so that,

$$\langle\Omega_2|\hat{H}_{0,M}|\Omega_2\rangle = \langle\Omega_1|\hat{H}_{0,M}|\Omega_1\rangle \tag{2.20}$$

Now for last term in (2.16) use the fact that $\hat{C}$ commutes with $\hat{\mathbf{J}}(\mathbf{x})$ to obtain,

$$\left\langle\Omega_2\left|\int\hat{\mathbf{J}}(\mathbf{x})\cdot\hat{\mathbf{A}}(\mathbf{x})d\mathbf{x}\right|\Omega_2\right\rangle=\left\langle\Omega_1\left|\int\hat{\mathbf{J}}(\mathbf{x})\cdot\left(e^{+i\hat{C}}\hat{\mathbf{A}}(\mathbf{x})e^{-i\hat{C}}\right)d\mathbf{x}\right|\Omega_1\right\rangle \tag{2.21}$$

To evaluate the above expression further use the Baker-Campell-Hausdorff relationships (1.12) along with (2.5) and (2.18) to obtain,

$$e^{+i\hat{C}}\hat{\mathbf{A}}(\mathbf{x})e^{-i\hat{C}}=\hat{\mathbf{A}}(\mathbf{x})-\nabla\chi(\mathbf{x}) \tag{2.22}$$

Use this result in (2.21) to obtain,

$$\left\langle\Omega_2\left|\int\hat{\mathbf{J}}(\mathbf{x})\cdot\hat{\mathbf{A}}(\mathbf{x})d\mathbf{x}\right|\Omega_2\right\rangle=\left\langle\Omega_1\left|\int\hat{\mathbf{J}}(\mathbf{x})\cdot\left(\hat{\mathbf{A}}(\mathbf{x})-\nabla\chi(\mathbf{x})\right)d\mathbf{x}\right|\Omega_1\right\rangle \tag{2.23}$$

Use the above results in (2.16) to yield,

$$\begin{aligned}E\left(\left|\Omega_2\right\rangle\right)=&\left\langle\Omega_1\left|\left(\hat{H}_{0,D}+\hat{H}_{0,M}-\int\hat{\mathbf{J}}(\mathbf{x})\cdot\hat{\mathbf{A}}(\mathbf{x})d\mathbf{x}\right)\right|\Omega_1\right\rangle\\&+\int\left\langle\Omega_1\left|\hat{\mathbf{J}}(\mathbf{x})\right|\Omega_1\right\rangle\cdot\nabla\chi(\mathbf{x})d\mathbf{x}\end{aligned} \tag{2.24}$$

Next use (2.1) and (1.1) in the above and integrate the last term by parts, assuming reasonable boundary conditions (i.e. let $\chi(\mathbf{x})\to 0$ as $|\mathbf{x}|\to\infty$), to obtain,

$$E\left(\left|\Omega_2\right\rangle\right)=E\left(\left|\Omega_1\right\rangle\right)-\int\chi(\mathbf{x})\nabla\cdot\left\langle\Omega_1\left|\hat{\mathbf{J}}(\mathbf{x})\right|\Omega_1\right\rangle d\mathbf{x} \tag{2.25}$$

Next subtract the energy of the vacuum state, $E\left(\left|\Omega_{vac}\right\rangle\right)$, from both sides to obtain,

$$\begin{aligned}E\left(\left|\Omega_2\right\rangle\right)-E\left(\left|\Omega_{vac}\right\rangle\right)=&\left(E\left(\left|\Omega_1\right\rangle\right)-E\left(\left|\Omega_{vac}\right\rangle\right)\right)\\&-\int\chi(\mathbf{x})\nabla\cdot\left\langle\Omega_1\left|\hat{\mathbf{J}}(\mathbf{x})\right|\Omega_1\right\rangle d\mathbf{x}\end{aligned} \tag{2.26}$$

Now in the above expression the quantities $\left(E\left(\left|\Omega_1\right\rangle\right)-E\left(\left|\Omega_{vac}\right\rangle\right)\right)$ and $\left\langle\Omega_1\left|\hat{\mathbf{J}}(\mathbf{x})\right|\Omega_1\right\rangle$ are independent of $\chi(\mathbf{x})$. Recall that we have picked the quantum state $\left|\Omega_1\right\rangle$ so that $\nabla\cdot\left\langle\Omega_1\left|\hat{\mathbf{J}}(\mathbf{x})\right|\Omega_1\right\rangle$ is nonzero. Based on this we can always find a $\chi(\mathbf{x})$ so that $\left(E\left(\left|\Omega_2\right\rangle\right)-E\left(\left|\Omega_{vac}\right\rangle\right)\right)$ is a negative number. For example, let $\chi(\mathbf{x})=\lambda\nabla\cdot\left\langle\Omega_1\left|\hat{\mathbf{J}}(\mathbf{x})\right|\Omega_1\right\rangle$ where λ is a constant. Then (2.26) becomes,

$$E\left(\left|\Omega_2\right\rangle\right)-E\left(\left|\Omega_{vac}\right\rangle\right)=\left(E\left(\left|\Omega_1\right\rangle\right)-E\left(\left|\Omega_{vac}\right\rangle\right)\right) -\lambda\int\left(\nabla\cdot\left\langle\Omega_1\left|\hat{\mathbf{J}}(\mathbf{x})\right|\Omega_1\right\rangle\right)^2 d\mathbf{x} \tag{2.27}$$

Now, since $\nabla\cdot\left\langle\Omega_1\left|\hat{\mathbf{J}}(\mathbf{x})\right|\Omega_1\right\rangle$ is nonzero, the integral must be positive so that as $\lambda\to\infty$ the quantity $\left(E\left(\left|\Omega_2\right\rangle\right)-E\left(\left|\Omega_{vac}\right\rangle\right)\right)\to-\infty$. Therefore the energy of the state $\left|\Omega_2\right\rangle$ is less than that of the vacuum state $\left|\Omega_{vac}\right\rangle$ by an arbitrarily large amount. Therefore there is no lower bound to the energy of a QED quantum state in the temporal gauge.

III. Interaction with Classical fields

In the previous section we have shown that if there exists a state $\left|\Omega_1\right\rangle$ that satisfies (2.11) then there exists a state $\left|\Omega_2\right\rangle$ whose energy is less that that of $\left|\Omega_1\right\rangle$ by an arbitrarily large amount. This suggests the possibility that it would be possible to extract an arbitrarily large amount of energy from a quantum state through the interaction with an external field. It will be shown in this section that this is theoretically possible. We will work this problem in the Schrödinger picture in which the field operators do not depend on time and the time dependence of the state vector $\left|\Omega(t)\right\rangle$ is given by,

$$i\frac{\partial\left|\Omega(t)\right\rangle}{\partial t}=\hat{H}\left|\Omega(t)\right\rangle \tag{3.1}$$

In the absence of external interactions the energy of a quantum state remains constant. In order to change the energy we must allow the field operators to interact with external sources or fields. This is done by adding an interaction term to the Hamiltonian. Let this term be,

$$\hat{H}_{\text{int}}=-\int\mathbf{S}(\mathbf{x},t)\cdot\hat{\mathbf{A}}(\mathbf{x})d\mathbf{x}-\int\hat{\mathbf{J}}(\mathbf{x})\cdot\mathbf{R}(\mathbf{x},t)d\mathbf{x} \tag{3.2}$$

In the above expression $\mathbf{S}(\mathbf{x},t)$ is a classical field that interacts with the quantized electromagnetic field and $\mathbf{R}(\mathbf{x},t)$ is a separate classical field that interacts with the fermion current operator. It should not be assumed that the classical fields $\mathbf{S}(\mathbf{x},t)$ and $\mathbf{R}(\mathbf{x},t)$ correspond to physical fields that actually exist. For the purposes of this discussion these fields are fictitious. They have been introduced for the purposes of perturbing the Hamiltonian in order to change the energy of some initial state. It will be shown that for properly applied fields $\mathbf{S}(\mathbf{x},t)$ and $\mathbf{R}(\mathbf{x},t)$ an arbitrarily large amount of energy can be extracted from some initial state. Therefore even though these fields do not correspond to actual physical objects we believe that the following results are mathematically interesting. The reason we pick these fields is because for particular values of the interaction we obtain an exact solution to the Schrödinger equation. This is demonstrated in the following discussion.

When the interaction is included the Schrödinger equation becomes,

$$i\frac{\partial\left|\Omega(t)\right\rangle}{\partial t}=\hat{H}_T\left|\Omega(t)\right\rangle \tag{3.3}$$

where,

$$\hat{H}_T=\hat{H}+\hat{H}_{\text{int}}=\hat{H}-\int\mathbf{S}(\mathbf{x},t)\cdot\hat{\mathbf{A}}(\mathbf{x})d\mathbf{x}-\int\hat{\mathbf{J}}(\mathbf{x})\cdot\mathbf{R}(\mathbf{x},t)d\mathbf{x} \tag{3.4}$$

Now we will solve (3.3) for the following interaction,

$$\mathbf{R}(\mathbf{x},t)=0 \text{ for } t<t_1;\ \mathbf{R}(\mathbf{x},t)=-g(t)\nabla\chi(\mathbf{x}) \text{ for } t_1\le t\le t_2;$$
$$\mathbf{R}(\mathbf{x},t)=0 \text{ for t} > \text{t}_2$$

and,

$$\mathbf{S}(\mathbf{x},t)=0 \text{ for t} < \text{t}_1;\ \mathbf{S}(\mathbf{x},t)=\ddot{g}(t)\nabla\chi(\mathbf{x}) \text{ for } t_1\le t\le t_2;$$
$$\mathbf{S}(\mathbf{x},t)=0 \text{ for t} > \text{t}_2$$

where the double dots represent the second derivative with respect to time. In addition to the above $g(t)$ satisfies the following relationship at time t_2,

$$\dot{g}(t_2)=0 \text{ and } g(t_2)=-1 \tag{3.5}$$

According to the above expressions the interaction is turned on at time t_1 and turned off at time $t>t_2$. During this time energy is exchanged between the quantized fermion-electromagnetic field and the classical fields $\mathbf{S}(\mathbf{x},t)$ and $\mathbf{R}(\mathbf{x},t)$. At some initial time $t_i<t_1$ the state vector is given by $\left|\Omega(t_i)\right\rangle$. We are interested in determining

the state vector $\left|\Omega\left(t_f\right)\right\rangle$ at some final time $t_f > t_2$. Based on the above remarks the state vector $\left|\Omega(t)\right\rangle$ satisfies,

$$i\frac{\partial\left|\Omega(t)\right\rangle}{\partial t} = \hat{H}\left|\Omega(t)\right\rangle \text{ for t < t}_1 \tag{3.6}$$

$$i\frac{\partial\left|\Omega(t)\right\rangle}{\partial t} = \begin{pmatrix} \hat{H} - \ddot{g}(t)\int\nabla\chi(\mathbf{x})\cdot\hat{\mathbf{A}}(\mathbf{x})d\mathbf{x} \\ +g(t)\int\hat{\mathbf{J}}(\mathbf{x})\cdot\nabla\chi(\mathbf{x})d\mathbf{x} \end{pmatrix}\left|\Omega(t)\right\rangle \text{ for } t_1 \le t \le t_2 \tag{3.7}$$

$$i\frac{\partial\left|\Omega(t)\right\rangle}{\partial t} = \hat{H}\left|\Omega(t)\right\rangle \text{ for t > t}_2 \tag{3.8}$$

Since these equations are first order differential equations the boundary conditions at t_1 and t_2 are,

$$\left|\Omega\left(t_1+\varepsilon\right)\right\rangle \underset{\varepsilon\to 0}{=} \left|\Omega\left(t_1-\varepsilon\right)\right\rangle \text{ and } \left|\Omega\left(t_2+\varepsilon\right)\right\rangle \underset{\varepsilon\to 0}{=} \left|\Omega\left(t_2-\varepsilon\right)\right\rangle \tag{3.9}$$

The solution to (3.6) is,

$$\left|\Omega(t)\right\rangle = e^{-i\hat{H}(t-t_i)}\left|\Omega\left(t_i\right)\right\rangle \text{ for } t < t_1 \tag{3.10}$$

It is shown in Appendix A that the solution to (3.7) is,

$$\left|\Omega(t)\right\rangle = e^{ig(t)\hat{C}}e^{i\dot{g}(t)\hat{D}}e^{iw(t)}e^{-i\hat{H}(t-t_1)}\left|\Omega\left(t_1\right)\right\rangle \text{ for } t_2 \ge t \ge t_1 \tag{3.11}$$

where the operator $\hat{D}$ is defined by,

$$\hat{D} = \int\hat{\mathbf{A}}(\mathbf{x})\cdot\nabla\chi(\mathbf{x})d\mathbf{x} \tag{3.12}$$

and

$$w(t)=\int_{t_1}^{t}\left(\frac{\dot{g}(t')^2}{2}\int\left|\nabla\chi\right|^2 d\mathbf{x}+\ddot{g}(t')g(t')\int\left|\nabla\chi\right|^2 d\mathbf{x}\right)dt' \quad (3.13)$$

The solution to (3.8) is,

$$\left|\Omega(t_f)\right\rangle = e^{-i\hat{H}(t_f-t_2)}\left|\Omega(t_2)\right\rangle \text{ where } t_f>t_2 \quad (3.14)$$

Use the boundary conditions (3.9) in the above to obtain,

$$\left|\Omega(t_f)\right\rangle = e^{-i\hat{H}(t_f-t_2)}e^{ig(t_2)\hat{C}}e^{i\dot{g}(t_2)\hat{D}}e^{iw(t_2)}e^{-i\hat{H}(t_2-t_i)}\left|\Omega(t_i)\right\rangle \quad (3.15)$$

Use (3.5) in the above to obtain,

$$\left|\Omega(t_f)\right\rangle = e^{-i\hat{H}(t_f-t_2)}e^{-i\hat{C}}e^{iw(t_2)}\left|\Omega_0(t_2)\right\rangle \quad (3.16)$$

where $\left|\Omega_0(t_2)\right\rangle$ is defined by,

$$\left|\Omega_0(t_2)\right\rangle = e^{-i\hat{H}(t_2-t_i)}\left|\Omega(t_i)\right\rangle \quad (3.17)$$

$\left|\Omega_0(t_2)\right\rangle$ is the state vector that the initial state $\left|\Omega(t_i)\right\rangle$ would evolve into, by the time t_2, in the absence of the interactions. Use (3.16) in (1.1) to show that the energy of the state $\left|\Omega(t_f)\right\rangle$ is,

$$E\left(\left|\Omega(t_f)\right\rangle\right) = \left\langle\Omega_0(t_2)\right|e^{i\hat{C}}\hat{H}e^{-i\hat{C}}\left|\Omega_0(t_2)\right\rangle \quad (3.18)$$

From the discussion leading up to equation (2.25) we obtain,

$$E\left(\left|\Omega(t_f)\right\rangle\right) = E\left(\Omega_0(t_2)\right)-\int\chi(\mathbf{x})\nabla\cdot\left\langle\Omega_0(t_2)\right|\hat{\mathbf{J}}(\mathbf{x})\left|\Omega_0(t_2)\right\rangle d\mathbf{x} \quad (3.19)$$

Now, as before, assume that we select an initial state $\left|\Omega(t_i)\right\rangle$ so that $\nabla\cdot\left\langle\Omega_0(t_2)\right|\hat{\mathbf{J}}(\mathbf{x})\left|\Omega_0(t_2)\right\rangle$ is non-zero. Recall that $\left|\Omega_0(t_2)\right\rangle$ is the state that $\left|\Omega\left(t_i\right)\right\rangle$ evolves into in the absence of interactions. Therefore $E\left(\Omega_0\left(t_2\right)\right)=E\left(\Omega\left(t_i\right)\right)$ and $\left|\Omega_0\left(t_2\right)\right\rangle$ is independent of $\chi(\mathbf{x})$. The function $\chi(\mathbf{x})$ can take on any value without affecting $\nabla\cdot\left\langle\Omega_0\left(t_2\right)\left|\hat{\mathbf{J}}(\mathbf{x})\right|\Omega_0\left(t_2\right)\right\rangle$. Let $\chi(\mathbf{x})=\lambda\nabla\cdot\left\langle\Omega_0\left(t_2\right)\left|\hat{\mathbf{J}}(\mathbf{x})\right|\Omega_0\left(t_2\right)\right\rangle$ so that (3.19) becomes,

$$E\left(\left|\Omega\left(t_f\right)\right\rangle\right)=E\left(\Omega\left(t_i\right)\right)-\lambda\int\left(\nabla\cdot\left\langle\Omega_0\left(t_2\right)\left|\hat{\mathbf{J}}(\mathbf{x})\right|\Omega_0\left(t_2\right)\right\rangle\right)^2 d\mathbf{x} \quad (3.20)$$

Define ΔE_{ext} as the amount of energy extracted from the quantum state due to its interaction with the classical fields. From the above equation,

$$\Delta E_{ext}=\lambda\int\left(\nabla\cdot\left\langle\Omega_0\left(t_2\right)\left|\hat{\mathbf{J}}(\mathbf{x})\right|\Omega_0\left(t_2\right)\right\rangle\right)^2 d\mathbf{x} \quad (3.21)$$

Obviously as $\lambda\to\infty$ then $\Delta E_{ext}\to\infty$.

In conclusion, it has been shown that we cannot assume that there exists a lower bound to the energy of a quantum state. The existence of a lower bound depends on the Hamiltonian in question and cannot be assumed but must be verified by mathematical techniques. When we examine the QED Hamiltonian consisting of a fermion field interacting with an electromagnetic field we find that is no lower bound to the energy. If an initial state interacts with properly applied classical fields then it is possible to extract an arbitrarily large amount of energy from the initial state. The classical fields that were applied in this article are mathematical objects that are not assumed to correspond to real physical objects. A possible next step in this research would be to determine if these same results can be obtained through the interaction of real existing fields and to extend this analysis to other Hamiltonians.

Appendix A

It will be shown that (3.11) is the solution to (3.7). Take the time derivative of (3.7) and multiply by "i" to obtain,

$$i\frac{\partial}{\partial t}\left|\Omega(t)\right\rangle = -\left(\dot{g}\hat{C}+\dot{w}\right)\left|\Omega(t)\right\rangle - \left|\Omega_a\right\rangle + \left|\Omega_b\right\rangle \tag{A.1}$$

where,

$$\left|\Omega_a\right\rangle = e^{ig\hat{C}}\ddot{g}\hat{D}e^{i\dot{g}\hat{D}}e^{iw(t)}e^{-i\hat{H}(t-t_1)}\left|\Omega(t_1)\right\rangle \tag{A.2}$$

and

$$\left|\Omega_b\right\rangle = e^{ig\hat{C}}e^{i\dot{g}\hat{D}}e^{iw(t)}\hat{H}e^{-i\hat{H}(t-t_1)}\left|\Omega(t_1)\right\rangle \tag{A.3}$$

To evaluate (A.2) we will use the following relationships.

$$e^{ig\hat{C}}\hat{D} = \left(e^{ig\hat{C}}\hat{D}e^{-ig\hat{C}}\right)e^{ig\hat{C}} \tag{A.4}$$

Use (2.5) and (1.12) to obtain,

$$e^{ig\hat{C}}\hat{D}e^{-ig\hat{C}} = \hat{D} + ig\left[\hat{C},\hat{D}\right] = \hat{D} - g\int\left|\nabla\chi\right|^2 d\mathbf{x} \tag{A.5}$$

Use these results in (A.2) to yield,

$$\left|\Omega_a\right\rangle = \ddot{g}\left(\hat{D} - g\int\left|\nabla\chi\right|^2 d\mathbf{x}\right)\left|\Omega(t)\right\rangle \tag{A.6}$$

Next evaluate (A.3). Use (1.12) and the commutation relationships to obtain,

$$e^{i\dot{g}\hat{D}}\hat{H}e^{-i\dot{g}\hat{D}} = \hat{H} + i\dot{g}\left[\hat{D},\hat{H}\right] - \frac{\dot{g}^2}{2}\left[\hat{D},\left[\hat{D},\hat{H}\right]\right] \tag{A.7}$$

where,

$$\left[\hat{D},\hat{H}\right] = \int\left[\hat{\mathbf{A}},\hat{H}\right]\cdot\nabla\chi d\mathbf{x} = \int\left[\hat{\mathbf{A}},\hat{H}_{0,M}\right]\cdot\nabla\chi d\mathbf{x} = -i\hat{C} \tag{A.8}$$

and,

$$\left[\hat{D},\left[\hat{D},\hat{H}\right]\right]=-i\left[\hat{D},\hat{C}\right]=-i\left[\int\hat{\mathbf{A}}\cdot\nabla\chi d\mathbf{x},\int\hat{\mathbf{E}}\cdot\nabla\chi d\mathbf{x}\right]=-\int\left|\nabla\chi\right|^{2}d\mathbf{x} \quad (A.9)$$

Therefore,

$$e^{i\dot{g}\hat{D}}\hat{H}e^{-i\dot{g}\hat{D}}=\hat{H}+\dot{g}\hat{C}+\frac{\dot{g}^{2}}{2}\int\left|\nabla\chi\right|^{2}d\mathbf{x} \quad (A.10)$$

Use this in (A.3) to obtain,

$$\left|\Omega_{b}\right\rangle=e^{ig\hat{C}}\left(\hat{H}+\dot{g}\hat{C}+\frac{\dot{g}^{2}}{2}\int\left|\nabla\chi\right|^{2}d\mathbf{x}\right)e^{i\dot{g}\hat{D}}e^{iw(t)}e^{-i\hat{H}(t-t_{1})}\left|\Omega\left(t_{1}\right)\right\rangle \quad (A.11)$$

To evaluate this further use,

$$e^{ig\hat{C}}\hat{H}e^{-ig\hat{C}}=\hat{H}+ig\left[\hat{C},\hat{H}\right]=\hat{H}+g\left(\int\hat{\mathbf{J}}\cdot\nabla\chi d\mathbf{x}\right) \quad (A.12)$$

where we have used,

$$\left[\hat{C},\hat{H}\right]=-i\int\hat{\mathbf{J}}\cdot\nabla\chi d\mathbf{x} \quad (A.13)$$

and the fact that $\left[\hat{C},\left[\hat{C},\hat{H}\right]\right]=0$. Therefore,

$$\left|\Omega_{b}\right\rangle=\left(\hat{H}+g\left(\int\hat{\mathbf{J}}\cdot\nabla\chi d\mathbf{x}\right)+\dot{g}\hat{C}+\frac{\dot{g}^{2}}{2}\int\left|\nabla\chi\right|^{2}d\mathbf{x}\right)\left|\Omega\left(t\right)\right\rangle \quad (A.14)$$

Use this along with (A.11) and (A.6) in (A.1) to obtain,

$$i\frac{\partial}{\partial t}\left|\Omega\left(t\right)\right\rangle=\left\{\begin{array}{l}\left(\hat{H}+g\left(\int\hat{\mathbf{J}}\cdot\nabla\chi d\mathbf{x}\right)+\dot{g}\hat{C}+\frac{\dot{g}^{2}}{2}\int\left|\nabla\chi\right|^{2}d\mathbf{x}\right)\\-\left(\dot{g}\hat{C}+\dot{w}\right)-\ddot{g}\left(\hat{D}-g\int\left|\nabla\chi\right|^{2}d\mathbf{x}\right)\end{array}\right\}\left|\Omega\left(t\right)\right\rangle \quad (A.15)$$

Rearrange terms and do some simple algebra to obtain,

$$i\frac{\partial}{\partial t}\left|\Omega(t)\right\rangle = \left\{ \begin{array}{l} \left(\hat{H} + g\int \hat{\mathbf{J}}\cdot\nabla\chi d\mathbf{x} - \ddot{g}\int \hat{\mathbf{A}}\cdot\nabla\chi d\mathbf{x}\right) \\ -\left(\dot{w} - \frac{\dot{g}^2}{2}\int\left|\nabla\chi\right|^2 d\mathbf{x} - \ddot{g}g\int\left|\nabla\chi\right|^2 d\mathbf{x}\right) \end{array} \right\}\left|\Omega(t)\right\rangle \quad \text{(A.16)}$$

Now let

$$\dot{w} = \frac{\dot{g}^2}{2}\int\left|\nabla\chi\right|^2 d\mathbf{x} + \ddot{g}g\int\left|\nabla\chi\right|^2 d\mathbf{x} \quad \text{(A.17)}$$

to obtain

$$i\frac{\partial}{\partial t}\left|\Omega(t)\right\rangle = \left(\hat{H} + g\int \hat{\mathbf{J}}\cdot\nabla\chi d\mathbf{x} - \ddot{g}\int \hat{\mathbf{A}}\cdot\nabla\chi d\mathbf{x}\right)\left|\Omega(t)\right\rangle \quad \text{(A.18)}$$

which is (3.7) in the text. This completes the proof.

References

[1] S. Weinberg. *The Quantum Theory of Fields. Vol. I. Cambridge University Press*, Cambridge, 1995.

[2] W. Greiner and J. Reinhardt. *Field Quantization*, Springer-Verlag, Berlin (1996).

[3] G. Leibbrandt. Introduction to noncovariant gauges. *Rev of Modern Phys*, **59** (Oct. 1987)1067-1119 .

[4] M. Creutz, Quantum Electrodynamics in the Temporal Gauge. *Ann. Phys.*, **117** (Feb. 1979) 471-483.

[5] B. Hatfield. *Quantum Field Theory of Point Particles and Strings*. Addison-Wesley, Reading, Massachusetts (1992).

[6] C. Kiefer and A. Wiepf , Functional Schrodinger equation for Fermions in External Gauge Fields. *Annals Phys.*, **236** (Dec. 1994)241-245 .