Lanczos invariant as an important element in Riemannian 4-spaces

J. López-Bonilla, E. Ramírez-García
SEPI-ESIME-Zacatenco.
Instituto Politécnico Nacional.
Edif. Z-4, 3er. Piso, Col. Lindavista, CP 07738 México DF.
E-mail: jlopezb@ipn.mx

J. Yalja Montiel.
Escuela Superior de Cómputo,
Av. Bátiz S/N, Col. Nueva Industrial Vallejo, CP 07738
México DF, E-mail: yalja@ipn.mx

We show the importance that the Lanczos invariant has in the study of R_4 embedded into E_5, in the analysis of non-null constant vectors, and in the existence of the Lanczos potential for the Weyl tensor.
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Introduction.

The Lanczos scalar is defined by [1]:

$$K_2 = {}^{*}R^{*\,ijrc} R_{ijrc}, \tag{1}$$

where R_{ijrc} is the curvature tensor [2] and its double dual is given by:

$$ {}^{*}R^{*ij}{}_{ac} = \frac{1}{4}\eta^{ijrt} R_{rt}{}^{pq} \eta_{pqac}, \tag{2} $$

with η_{ijrc} denoting the Levi-Civita tensor. The Bianchi identities [2] for the Riemann tensor adopt the following compact form [3]:

$$ {}^{*}R^{*ijac}{}_{;c} = 0, \tag{3} $$

where *;c* denotes covariant derivative.

Here we show the usefulness of (1) in several topics of general relativity. In fact, the embedding of Riemannian 4-spaces into E_5 [2, 4] has an algebraic character if $K_2 \neq 0$ and, furthermore, in this case it is possible to construct Gauss-Codazzi equations [5, 6] for the inverse matrix of the corresponding second fundamental form. On the other hand, K_2 is an ordinary divergence [7-9] which implies the existence of the Lanczos potential [3, 10-12], whose physical meaning is an open problem. Finally, when $K_2 \neq 0$ the spacetime does not accept non-null constant vectors, that is, the presence of a non-null constant vector leads [13] to $K_2 = 0$, which is a result of interest in various studies of Riemannian geometry.

R_4 embedded into E_5

A 4-space can be embedded into E_5 (that is, R_4 has class one) if and only if there exists the second fundamental form $b_{ac} = b_{ca}$ satisfying the Gauss-Codazzi equations [2, 4-6, 11, 12]:

$$ R_{acij} = \varepsilon\left(b_{ai}\, b_{cj} - b_{aj}\, b_{ci}\right), \varepsilon = \underline{+}1, \tag{4} $$

$$b_{ij;c} = b_{ic;j}, \tag{5}$$

It is well known [14] that whenever $\det\left(b^{i}{}_{j}\right) \neq 0$ then (4) implies (5); in other words, if a non-singular matrix $\underset{\sim}{b}$ satisfies the Gauss equation then the Codazzi equation is verified automatically. However, in the general case the computation of $\underset{\sim}{b}$ for a given spacetime should involve the analysis of both (4) and (5) together.

From (4) it is not difficult to obtain the relation [5, 6, 15]:

$$\det\left(b^{i}{}_{j}\right) = -\frac{K_2}{24}, \tag{6}$$

thus $K_2 \neq 0$ means that the embedding process has algebraic nature because it is only necessary to satisfy (4), and, in addition the inverse matrix $b^{-1}{}_{ij}$ exists. Now we may deduce an interesting relationship between ${}^{*}R^{*}{}_{ijac}$ and $b^{-1}{}_{rt}$ which is similar to (4). In fact, Yakupov [16] showed that, for any R_4 of class one, it is valid the expression:

$${}^{*}R^{*\,ijrt} R_{acrt} = \frac{K_2}{12}\left(\delta_a^i\ \delta_c^j - \delta_c^i\ \delta_a^j\right), \tag{7}$$

then substituting (4) into (7), and after multiplying by $b^{-1}{}_{m}{}^{a}\ b^{-1}{}_{n}{}^{c}$ we get [5, 6]:

$$\frac{24}{K_2}\ {}^{*}R^{*}{}_{ijmn} = \varepsilon\left(b^{-1}{}_{im}\ b^{-1}{}_{jn}\ -\ b^{-1}{}_{in}\ b^{-1}{}_{jm}\right), \tag{8}$$

which represents the Gauss equation for $\underset{\sim}{b}^{-1}$. The relation (8) has the same structure as (4) hence illustrating the analogous role that the Riemann tensor and its double dual play. Thus, when $K_2 \neq 0$ the embedding problem is reduced to analyzing (4) or (8).

From (8) it is easy to obtain $\underset{\sim}{b}^{-1}$ explicitly [17]:

$$K_2 \; b^{-1}{}_{i\,m} = 8\varepsilon\, {}^*R^*{}_{i\,j\,m\,n} b^{j\,n}\,, \tag{9}$$

this means that $\underset{\sim}{b}^{-1}$ is essentially the projection of $\underset{\sim}{b}$ over the double dual of the curvature tensor. Equations (5), (9) and the Bianchi identities (3) imply the differential condition [17]:

$$\left(K_2 \; b^{-1\,i}{}_m\right)_{;\,i} = 0\,. \tag{10}$$

The application of (3) and (10) to (8) leads to one more differential restriction on $\underset{\sim}{b}^{-1}$:

$$b^{-1}{}_{i\,j\,;\,r} \; b^{-1\,r}{}_c = b^{-1}{}_{i\,c\,;\,r} \; b^{-1\,r}{}_j\,, \tag{11}$$

which also is obtained if we apply *;c* to the relation $g_{i\,j} = b^{-1}{}_i{}^r \; b_{r\,j}$ and we employ (5), remembering that $g_{i\,j\,;\,c} = 0$. Then we say that (10) and (11) are the Codazzi equations for $\underset{\sim}{b}^{-1}$.

The Leverrier-Faddeev-Takeno method [17,18-26] permits to construct the characteristic polynomial of $\underset{\sim}{b}^{-1}$, and the Cayley-Hamilton theorem [27] affirms that it is satisfied by this inverse matrix, thus we deduce for $\underset{\sim}{b}^{-1}$ an expression alternative to (9):

$$\frac{K_2}{24} b^{-1}{}_{i\,j} = \varepsilon b_{i\,r} G^r{}_j - p\, g_{i\,j}\,, \tag{12}$$

where $G_{i\,j} = {}^*R^{*\,c}{}_{i\,j\,c}$ is the Einstein tensor, and [4]:

$$p = \frac{\varepsilon}{3} b_{a\,c} G^{a\,c}\,, \tag{13}$$

with the property [5, 6, 11, 12, 15, 28, 29] ($R = -G^a{}_a$ is the scalar curvature):

$$p^2 = -\frac{\varepsilon}{6}\left(\frac{R}{24}K_2 + R_{imnj}G^{ij}G^{mn}\right) \geq 0\,, \tag{14}$$

that is, the intrinsic geometry of R_4 determines p and ε, then (12) gives us $\underset{\sim}{b}^{-1}$ and its trace [17]:

$$b^{-1\,r}{}_{r} = -\frac{24p}{K_2}\,, \tag{15}$$

The analysis exhibited in (4)-(15) shows the usefulness of the Lanczos invariant $K_2 \neq 0$ in the study of a spacetime embedded into E_5.

Lanczos potential

If we make use of the Lagrangian:

$$L = \sqrt{-g}\;K_2\,,\; g = \det(g_{ij})\,, \tag{16}$$

in a Hilbert type variational principle (*Htvp*) [30, 31], $\delta\int L\,d^4x = 0$, we obtain [1] the identity *0 = 0*, from which one suspects that the density L is an exact divergence for any R_4:

$$L = \left(\sqrt{-g}\;B^r\right)_{,r} \tag{17}$$

where $, r = \frac{\partial}{\partial x^r}$. This suspicion turned out to be correct because Goenner-Kohler [7] and Buchdal [32, 33] got non-tensorial expressions for B^r; while, Horndeski [13] found an expression for B^r strictly tensorial.

Lanczos [3], with an appropriate variational use of (16) (that is, with a non-*Htvp*), proved the existence of a potential K_{ijr} [34-40] for the Weyl tensor:

$$C_{pqjb} = K_{pqj;b} - K_{pqb;j} + K_{jbp;q} - K_{jbq;p} + g_{pb} K_{jq} - g_{pj} K_{qb} + g$$
$$- g_{qb} K_{pj} \tag{18}$$

such that:

$$K_{rab} = -K_{arb}, \; K_{r\;\;a}^{\;a} = 0,$$

$$K_{abc} + K_{bca} + K_{cab} = 0, K_{rj\;\;;a}^{\;\;\;a} = 0,$$

$$K_{ab} \equiv K_{a\;\;b;r}^{\;r} = K_{ba}, \tag{19}$$

For empty spacetimes $\left(G_{ab} = 0,\; C^{rjpq}{}_{;r} = 0\right)$ it is possible [8, 9] to employ (18) and (19) to deduce (17) with:

$$B^r = 2\, C^{rjpq} K_{pqj}\;, \tag{20}$$

which has a tensorial nature because it is the projection of the Lanczos potential over the conformal tensor. For example, (20) is valid in the Kerr geometry [41] whose K_{pqj} was studied in [34, 35, 37, 38, 40].

We have just commented that the Lagrangian L does not lead to field equations under *Htvp*, but it is interesting to note that L contributes [42] to the gravitational energy-momentum distribution, this B^r deserves a more careful analysis, which could help to elucidate the elusive physical meaning of Lanczos potential.

Non-null constant vectors

Here we shall consider the Horndeski's expression [13] for B^r verifying (17):

$$B^r = \frac{8}{A}\left({}^*R^{*\,r\,t}{}_{i\,j} + \frac{1}{3A}\delta^{m\,t\,r\,n}_{p\,j\,a\,i} A^p{}_{;m}\, A^a{}_{;n} \right) A^i A^j{}_{;t}\,, \qquad (21)$$

where $\delta^{m\,t\,r\,n}_{p\,j\,a\,i}$ is the generalized Kronecker delta [30] and A^b is an arbitrary non-null vector:

$$A \equiv A^b A_b = \text{constant} \neq 0\,. \qquad (22)$$

The relation (21) is correct for any spacetime, and it is therefore more general than (20) which is valid only for vacuum 4-spaces, however, (20) does not contain an arbitrary element A^r as in the case of (21).

With (17) and (21) it is easy to show the result:

"If R_4 accepts a non-null constant vector A^r,

that is, $A^r{}_{;c} = 0$, then $K_2 = 0$". (23)

For example, the Gödel metric [2, 11, 12, 36, 37, 43]:

$$ds^2 = -\left(dx^1\right)^2 - 2\, e^{x^4} dx^1 dx^2 - \frac{1}{2} e^{2x^4}\left(dx^2\right)^2 + \left(dx^3\right)^2 + \left(dx^4\right)^2, \qquad (24)$$

has a spacelike constant vector:

$$\left(A^r\right) = (0,\ 0,\ 1,\ 0)\,,\ A = 1,\ A^r{}_{;t} = 0\,, \qquad (25)$$

then (23) implies $K_2 = 0$ for this cosmological model, without the necessity of long computations as in the definition (1).

From (23) it is evident that:

"$K_2 \neq 0$ implies the non-existence of non-null constant vectors".(26)

The spacetimes of Schwarzschild, Taub, C, Kerr, . . ., have [2] $K_2 \neq 0$, then by [26] we conclude that these 4-spaces do not admit non-null constant vectors.

We know [2] that the presence of non-null constant vectors has impact in the embedding class: If R_4 has one of these vectors, then it can be embedded into E_7, which occurs with (24). However, it is an open question by now [44, 45] whether the Gödel metric accepts an embedding into E_6.

References

[1] C. Lanczos, Ann. Math. 39 (1938) 842

[2] D. Kramer, H. Stephani, M. MacCallum and E. Herlt, Exact solutions of Einstein's field equations, Cambridge Univ. Press (1980)

[3] C. Lanczos, Rev. Mod. Phys. 34 (1962) 379

[4] J. López-Bonilla, J. Sosa-Pedroza and S. Vidal-Beltrán, Apeiron 12, N. 4 (2005) 363

[5] J. López-Bonilla and H. N. Núñez-Yépez, Pramana J. Phys. 46 (1996) 219

[6] J. H. Caltenco, J. López-Bonilla and G. Ovando, J. Bangladesh Acad. Sci. 25 (2001) 95

[7] H. Goenner and M. Kohler, Nuovo Cim. B25 (1975) 308

[8] V. Gaftoi, J. López-Bonilla and G. Ovando, Nuovo Cim. B113 (1998) 159

[9] J. López-Bonilla, G. Ovando and O. Puente, Indian J. Theor Phys. 50 (2002) 177

[10] J. H. Caltenco, J. López-Bonilla, G. Ovando, and J Rivera, Apeiron 9, N. 1. (2002) 38

[11] J. H. Caltenco, R. Linares y M. and J. López-Bonilla, Proc. Pakistan Acad. Sci. 42, N. 2 (2005) 153

[12] M. Acevedo, J. López-Bonilla and S. Vidal-Beltrán, Proc. Pakistan Acad. Sci. 42, N. 4 (2005) in print

[13] G. W. Horndeski, Proc. Cambridge Phil. Soc. 72 (1972) 77

[14] T.Y. Thomas, Acta Math. 67 (1936) 169

[15] R. Fuentes, J. López-Bonilla, T. Matos and G. Ovando, Gen. Rel. Grav. 21 (1989) 777
[16] M. Sh. Yakupov, Sov. Phys. Dokl. (Engl. Transl.) 13 (1968) 585
[17] J. López-Bonilla, J. Morales, G. Ovando and E. Ramírez-García, Proc. Pakistan Acad. Sci 43 (2006) in print
[18] U. J. J. Leverrier, J. de Math. Pures Appl. Série 1, 5 (1840) 220
[19] A. N. Krylov, Izv. Akad. Nauk. SSSR. Ser. Fiz.-Mat. 4 (1931) 491
[20] H. Takeno, Tensor New Series, 3 (1954) 119
[21] V. N. Faddeeva, Computational methods of linear algebra, Dover NY (1959)
[22] D. K. Faddeev and V. N. Faddeeva, Methods in linear algebra, W. H. Freeman, San Fco., USA (1963)
[23] E. B. Wilson, J. C. Decius and P. C. Cross, Molecular vibrations, Dover NY (1980)
[24] J. C. Gower, Linear Algebra Appls. 31 (1980) 61
[25] J. H. Caltenco, J. López-Bonilla and R. Peña-Rivero, Aligarh Bull. Math. 19 (2000) 55
[26] J. C. Gower, Computational Statistics & Data Analysis, 50 (2006) 89
[27] C. Lanczos, Applied analysis, Dover NY (1988)
[28] G. González, J. López-Bonilla and M. Rosales, Pramana J. Phys. 42 (1994) 85
[29] J. López-Bonilla, J. Morales and G. Ovando, Indian J. Math. 42 (2000) 309
[30] D. Lovelock and H. Rund, Tensors, differential forms and variational principles, John Wiley and Sons (1975)
[31] J. López-Bonilla, J. Morales and G. Ovando, “Relativity, Gravitation, Cosmology: New Development”, Nova Science Pub. NY (2005) in print
[32] H. A. Buchdal, J. Math. Phys. 1 (1960) 537
[33] H. A. Buchdal, J. Austral. Math. Soc. 6 (1966) 402 and 424
[34] V. Gaftoi, J. López-Bonilla and G. Ovando, Nuovo Cim. B113 (1998) 1493
[35] J. López-Bonilla, J. Morales and G. Ovando, Gen. Rel. Grav. 31 (1999) 413
[36] J. López-Bonilla and G. Ovando, Gen. Rel. Grav. 31 (1999) 1071
[37] J. López-Bonilla, J. Morales, G. Ovando and J. Rivera, Indian J. Theor. Phys. 48 (2000) 289
[38] J. H. Caltenco, J. López-Bonilla, J. Morales and G. Ovando, Chinese J. Phys. 39 (2001) 397

[39] V. Gaftoi, J. López-Bonilla and G. Ovando, Czech. J. Phys. 52 (2002) 811

[40] M. Acevedo, J. López-Bonilla and S. Vidal-Beltrán, Grav. & Cosm. 10 (2004) 328

[41] R. P. Kerr, Phys. Rev. Lett. 11 (1963) 237

[42] H. Goenner and M. Kohler, Nuovo Cim. B22 (1974) 79

[43] K. Gödel, Rev. Mod. Phys. 21 (1949) 447

[44] C. D. Collinson, J. Math. Phys. 9 (1968) 403

[45] J. López-Bonilla, G. Ovando and J. Rivera, Aligarh Bull. Math. 17 (1997-1998) 63