Hidden in Plain View: The Material Invariance of Maxwell-Hertz-Lorentz Electrodynamics

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Maxwell accounted for the apparent elastic behavior of the electromagnetic field through augmenting Ampere's law by the so-called displacement current much in the same way that he treated the viscoelasticity of gases. Original Maxwell constitutive relations for both electrodynamics and fluid dynamics were not material invariant, while combining Faraday's law and the Lorentz force makes the first of Maxwell's equation material invariant. Later on, Oldroyd showed how to make a viscoelastic constitutive law material invariant. The main assumption was that the proper description of a constitutive law must be material invariant. Assuming that the electromagnetic field is a material field, we show here that if the upper convected Oldroyd derivative (related to Lie derivative) is used, the displacement current becomes material invariant.

The new formulation ensures that the equation for conservation of charge is also material invariant which vindicates the choice of Oldroyd derivative over the standard convective derivative. A material invariant field model is by necessity Galilean invariant. We call the material field (the manifestation of which are the equations of electrodynamics the *metacontinuum*), in order to distinguish it form the standard material continua.

Keywords: Maxwell's Electrodynamics, Material Invariance, Oldroyd-Lie derivative

Introduction

> "... according to the general theory of relativity space is endowed with physical qualities; in this sense, therefore, there exists an ether. According to the general theory of relativity space without ether is unthinkable; ... ". A. Einstein [1, p.23]

The first attempt to explain the propagation of light as a field phenomenon was by Cauchy *circa* 1827 (see the account in [2]) who postulated the existence of an elastic continuum, through which light propagates as shear waves. Unfortunately, Cauchy's model of an elastic aether contradicted the natural perception of a particle moving *through* the field. As a result, it did not receive much attention and development because the notion of an elastic liquid was not available at that time. Subsequently came the contributions of Faraday and Ampere which led to the formulation of the electromagnetic model. The crucial advancement was achieved, however, when Maxwell [3] added the term $\frac{\partial \boldsymbol{E}}{\partial t}$ in Ampere's law and named it the "displacement current". It was very similar to the nonlocal term in the constitutive relation for gases [4] (see, also [5] for insightful review on viscoelastic models). We observe that the electric field vector is a clear analog of the stress vector in continuum mechanics. One can say that Maxwell enshrined an elastic constitutive relation by adding the displacement current to Ampere's law (more on this analogy can be found in [6, 7]). Indeed, the new term transformed the system of equations established in electrostatics into a hyperbolic system with characteristic speeds of wave propagation similar to the speed of sound in gases. Maxwell identified the characteristic speed with the speed of light and paved the way to understanding electromagnetic wave phenomena.

The advantage of Maxwell's system over the proposal of Cauchy was its intimate relation to the empirically observed laws, such as Faraday's, Ampere's and Biot-Savart's while the approach of Cauchy seemed unrelated to those fundamental observable laws. The most valuable achievement of the new formulation was deemed to be the fact that it allows one to derive the continuity equation for the charge as a corollary (see, e.g., [8], [9, Ch.7]).

The most puzzling aspect of Maxwell's model was its apparent lack of Galilean invariance. This was an indication that the linear form of Maxwell's electrodynamics was somehow divorced from the basic principles of mechanics, and continuum mechanics, in particular. Consequently, the continuity equation for the charge density was not Galilean invariant either, and it is usually augmented by the term $\rho \boldsymbol{v}$ (see [8, 9]).

Instead of trying to apply the principles of material invariance, scientific thought took another approach which assumed that Maxwell's equations were untouchable and in order to justify the lack of Galilean invariance of these equations, the electromagnetic field was decreed to be a field that is not a material continuum but something else. As a result, electrodynamics was exempted from the requirement to comply with the Galileo's principle of relativity.

The difficulties in establishing the full Galilean invariance lie in the fact that the constitutive relations proposed by Maxwell are not material invariant neither in theory of gases, nor in electrodynamics. The invariance in fluids was remedied by Oldroyd [10] who enunciated the principle of material invariance of a constitutive law. At the same time, the similar issue in electrodynamics is yet to be resolved. Unfortunately, the solution adopted in the beginning of the Twentieth century was more like wishful thinking rather than scientific approach. Because the Maxwell equations did not look at first sight as material invariant, the electromagnetic field was just exempted from the requirement to behave as a material continuum. Thus, science ended up with a concept of some kind of ghost field which was allowed to be material when harmonic oscillations are considered ("luminiferous aether"), and was deemed non-material when the invariance in a moving frame is at issue. It seems important to derive a material invariant formulation of the second of Maxwell's equations which will make the electromagnetic field a *bona fide* material field with rheology paralleling that of elastic liquids. The present paper is devoted to achieving this purpose.

1. Invariance of Electrodynamics in Moving Frames

The easiest way out of the perceived non-invariance of the electrodynamics was to enshrine a new principle, namely, that electrodynamics is Lorentz invariant, rather than Galilean invariant. It was discovered that some vestiges from the missing convective terms can be restored in the coordinate transformation, provided that time is no longer considered as an absolute parameter. Instead it was stipulated that time in the moving frame (parametrized by $x' = (x - vt)/\gamma$) transforms like $t' = (t - vx)/\gamma$, where γ is the Lorentz contraction factor. Such a transformation leaves the *form* of the linear wave equations for the potentials (Lorenz gauge) unchanged in a moving frame. This brought into view the idea of invariance in space-time (see the account in [11]) which is a different concept than the material invariance in three dimensions.

The Lorentz invariance can be viewed as a "poor man's material invariance" in the sense that the assumption of relativity of time (with mandatory time dilation) is a palliative solution to the problems of Maxwell's system in moving frames: a heuristic approach that can restore some terms of the convective derivatives. It is important to always keep in mind, however, that even though it works as a temporary fix for rigid frames that move translatory, the Lorentz invariance has not been and is impossible to generalize to the case of generally moving deformable frames. In this sense, it can never serve as a reliable substitute to the general principle of material invariance.

Indeed, when trying to work out the notion of Lorentz invariance for the field vectors $\boldsymbol{E}, \boldsymbol{B}$, it turned out that additional terms (forces) need to be added in the equations [12, 13]. This is nowadays called "Lorentz Covariance", however, as mentioned in [14], the terminology involving "covariance" and/or "invariance" is quite often very loose. In Faraday's law it was the electromotive force that acts upon a moving electrical charge in a magnetic field. Currently, it is called Lorentz force, because it was Lorentz who added it to Faraday's law as an integral part of the latter [18].

The fact that incorporating the Lorentz force in Faraday's law makes the latter material invariant was spotted by many authors (see, e.g., the authoritative monographs [8, pg.212-213],[12, 13]). To see this, we replace the electric field by the sum $\boldsymbol{E}' = \boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}$, and render Faraday's law as

$$-\nabla \times \boldsymbol{E}' = \frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) = \frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{B}, \qquad (1)$$

because $\nabla \cdot \boldsymbol{B} = 0$. Consequently, the prime is omitted without fear of confusion. Then, it is clear that in Eq. (1) we have exactly the convective derivative of $\boldsymbol{B}$. A similar situation one faces in Maxwell-Cattaneo model of waves in heat conduction. The importance of material derivatives for that problem is discussed in [19].

Now, if Maxwell-Ampere equation is to be valid in a moving frame, then $\boldsymbol{B}' = \boldsymbol{B} - c^{-2}\boldsymbol{v} \times \boldsymbol{E}$ has to replace the magnetic field [12, 13]. However, as pointed out in [20], one cannot have both additions simultaneously and the authors of [20] went on to derive different Galilean limits (see also the recent work [21]). We note here that in order to be consistent, however, the current has to be simultaneously augmented in the same equation by the convected part of the current, i.e. $\boldsymbol{j}' = \boldsymbol{j} + \rho\boldsymbol{v} = \boldsymbol{j}' + \epsilon_0(\nabla \cdot \boldsymbol{E})\boldsymbol{v}$.

Then

$$c^2\nabla\times\boldsymbol{B}' - \frac{\boldsymbol{j}'}{\epsilon_0} = \frac{\partial\boldsymbol{E}}{\partial t} + \nabla\times\boldsymbol{E} + \boldsymbol{v}(\nabla\cdot\boldsymbol{E}) = \frac{\partial\boldsymbol{E}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{E}. \quad (2)$$

The fact that terms added to the different Maxwell equations are different and, in fact mutually exclusive, speaks volumes about the insufficiency of the palliative solution called "Lorentz Covariance". As no surprise to any student of continuum mechanics, we found that the proper acknowledgment of all relevant terms in a moving frame, yields to the standard convective derivatives. Thus, after a full cycle through artificial devices as non-absolute time, the need of convective derivatives is affirmed.

Replacing the partial time derivatives in Maxwell's equations by convective time derivatives, was done at end of the Nineteenth century by Hertz, who regarded his formulation as an explanation of electromagnetic phenomena *inside* material bodies (see [15, Ch.14]). The fact that the Maxwell-Hertz equations are Galilean invariant (and in fact, material invariant) is usually overlooked in the literature. Apparently, this point was originally raised in [16], as reported in [17], where the case for Galilean invariance is forcefully argued.

The Hertz equations, Eqs. (1), (2), are still not the desired set of equations because one cannot derive from them an invariant equation for the conservation of charge. The way out of this situation is to exploit the above stated analogy between the second of Maxwell's equations and a constitutive law. The simplest constitutive laws, such as Hooke's law in elasticity, and the Navier–Stokes law for viscous liquids, establish pointwise connections between the stress tensor, and the tensor of strains, or rate of strains. Such constitutive laws are local and the material invariance is trivially established by the transformation rule. It is quite different a situation when a constitutive law involves time derivatives (relaxation of stresses or retardation of strains). It is beyond doubt that a partial time derivative is insufficient. It is interesting to note that employing a mere convective derivative works perfectly for momentum equations, but is not enough to make a constitutive law material invariant (see [10]).

2. Invariant Time Derivative of a Vector Density

Directional and other invariant derivatives of tensors are investigated in numerous mathematical and physical works but in order to make the paper self-contained and to clarify the physical meaning, we present here the pertinent derivations and highlight the concept of material invariance.

Consider a 3D space and a fixed system of coordinates, $\{x^i\}$, in it. The fixed coordinate system can be assumed to be Cartesian without loosing the generality. Together with the fixed coordinate system, consider a generally curvilinear moving coordinate system, $\{\bar{x}^i\}$, that is embedded in the material continuum occupying the geometrical space in the sense that coordinate lines of the moving system consist always of the same material particles. Then the transformation $x^j = f^j(\bar{x}^i; t)$ presents the law of motion of a material particle, parameterized by the coordinate $\bar{x}^i$. Assume that at time t, the two coordinate systems coincide. Then at time $t + \Delta t$, the law of motion gives the infinitesimal transformation $x^j = \bar{x}^j + v^j \Delta t$, which can be resolved for the material coordinates:

$$\bar{x}^i = x^i - v^i(x^j)\Delta t, \tag{3}$$

where v^i is the contravariant velocity vector.

$$\frac{\partial \bar{x}^i}{\partial x^j} = \delta^i_j - \Delta t \frac{\partial v^i}{\partial x^j} + o(\Delta t), \quad \frac{\partial x^j}{\partial \bar{x}^i} = \delta^j_i + \Delta t \frac{\partial v^j}{\partial x^i} + o(\Delta t),$$

Let $\boldsymbol{A}$ represent some mechanical quantity, e.g. stress vector, electric field, temperature flux, etc. For all these mechanical characteristics, the actual observable is the following integral (see, e.g. [25])

$$\int_D \boldsymbol{A} \mathrm{d}^3 \boldsymbol{x} = \int_{\bar{D}} \boldsymbol{A} \mathrm{d}\bar{\boldsymbol{x}}, \tag{4}$$

where D is the region occupied by the material parcel in the initial moment t, and $\bar{D}$ is the region occupied by the same material points in the moment $t + dt$ (the deformed parcel).

The principle of material invariance requires that this integral be invariant under coordinate transformation, which means that the vector $\boldsymbol{A}$ is a tensor density (or what is called "relative tensor"). In component form, the integral in the left-hand side can be rewritten as

$$\int_D \mathfrak{A}^k \mathrm{d}x^1 \mathrm{d}x^2 \mathrm{d}x^3 \equiv \int_{\bar{D}} \frac{\partial \bar{x}^k}{\partial x^j} J \mathfrak{A}^j \mathrm{d}\bar{x}^1 \mathrm{d}\bar{x}^2 \mathrm{d}\bar{x}^3,$$

where J is the Jacobian of the coordinate transformation,

$$J = \left| \frac{\partial x^i}{\partial \bar{x}^j} \right| = 1 + \Delta t \frac{\partial \bar{v}^i}{\partial x^i} + o\left(\Delta t\right), \tag{5}$$

and $\mathfrak{A}^k$ are the contravariant components of $\boldsymbol{A}$. Being reminded that D is an arbitrary region, one finds the transformation rule for a vector density in contravariant components

$$\bar{\mathfrak{A}}^k = J \frac{\partial \bar{x}^k}{\partial x^l} \mathfrak{A}^l, \tag{6}$$

where a summation is understood if an index appears once as a superscript and once as a subscript.

Material invariance (see [10]) requires that in constitutive laws, the total time variance of a tensor density,

$$\frac{\mathfrak{d}\mathfrak{A}^j}{\mathfrak{d}t} = \lim_{\Delta t\to 0}\frac{\bar{\mathfrak{A}}^j(\bar{x}^k;t+\Delta t)-\mathfrak{A}^j(x^k;t)}{\Delta t},$$

is used. Taylor series with Eq. (3) acknowledged, yields

$$\begin{aligned}\bar{\mathfrak{A}}^j(\bar{x}^k;t+\Delta t) &= \bar{\mathfrak{A}}^j(x^k;t)+\Delta t\left[\frac{\partial\bar{\mathfrak{A}}^j}{\partial t}+v^l\frac{\partial\bar{\mathfrak{A}}^j}{\partial\bar{x}^l}\right]+o(\Delta t)\\ &= \bar{\mathfrak{A}}^j(x^k;t)+\Delta t\left[\frac{\partial\mathfrak{A}^j}{\partial t}+v^l\frac{\partial\mathfrak{A}^j}{\partial x^l}\right]+o(\Delta t),\end{aligned}$$

where the fact is also acknowledged that at the moment t, vectors $\boldsymbol{A}$ and $\bar{\boldsymbol{A}}$ and their gradients coincide.

Now, the contravariant components $\mathfrak{A}^k$ transform according to the rule from Eq. (6), which gives

$$\bar{\mathfrak{A}}^j(x^k;t) = \left(1+\frac{\partial v^i}{\partial x^i}\Delta t\right)\left(\mathfrak{A}^j(x^k)-\frac{\partial v^k}{\partial x^m}\mathfrak{A}^m(x^k)\Delta t\right) \tag{7}$$

$$= \mathfrak{A}^j(x^k)+\frac{\partial v^i}{\partial x^i}\mathfrak{A}^j(x^k)\Delta t-\frac{\partial v^k}{\partial x^m}\mathfrak{A}^m(x^k)\Delta t+o(\Delta t).$$

After making use of Eq.(5) and neglecting the higher order terms in (Δt), Eq. (7) yields

$$\begin{aligned}\frac{\mathfrak{d}\mathfrak{A}^j}{\mathfrak{d}t} &\stackrel{\text{def}}{=} \lim_{\Delta t\to\infty}\frac{\bar{\mathfrak{A}}^j(x^k)-\mathfrak{A}^j(x^k)}{\Delta t} = \frac{\partial\mathfrak{A}^j}{\partial t}+\mathcal{L}_v\mathfrak{A}^j\\ &= \frac{\partial\mathfrak{A}^j}{\partial t}+v^k\frac{\partial\mathfrak{A}^j}{\partial x^k}-\frac{\partial v^j}{\partial x^m}\mathfrak{A}^m+\frac{\partial v^i}{\partial x^i}\mathfrak{A}^j\end{aligned} \tag{8}$$

where $\mathcal{L}_v$ is the Lie derivative along the vector field v^i (see [26] for a mixed tensor density of arbitrary rank). The first term in the invariant derivative (the partial time derivative) accounts for the changes of the components as functions of time, and the second term (the Lie derivative) represents the changes due to the fact that the coordinate system and the associated basis are also changing with time (being "convected" with the velocity field). In abstract vector notations valid in any coordinate system, the upper convected derivative of a vector density has the form

$$\frac{\mathfrak{d}\boldsymbol{A}}{\mathfrak{d}t} = \frac{\partial \boldsymbol{A}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{A} - \boldsymbol{A}\cdot\nabla\boldsymbol{v} + (\nabla\cdot\boldsymbol{v})\boldsymbol{A}. \tag{9}$$

For a pointed exposition of different issues connected with invariant derivatives, we refer the reader to the recent article [27] and the literature cited therein. What Oldroyd did, actually amounted to taking the directional derivative of a contravariant tensor density along the contravariant velocity vector of the material point at which the constitutive relation was written, which is a generalization of the advective part of the usual material derivative.

Following the established terminology [28], we can call Eq. (9) "the upper convected" material derivative of vector $\boldsymbol{A}$. Note that if $\boldsymbol{A}$ was not a tensor density, but an absolute tensor, then the last term in Eq. (9) would be absent (see, also [27]). As shown in [10], there is a difference in the material derivatives of a contravariant and a covariant tensor, and the choice was left open to additional mechanical considerations. In fact, this is a much deeper question, and goes beyond the scope of the present letter. The issues connected with the choice between the upper convected and lower convected derivative are still debated in the literature and the verdict seems to be that the choice has to be decided by the particular application. The upper convected Oldroyd derivative appears to be relevant in most of the cases. This means that the contravariant stress tensor must be used in the constitutive relationships. We refer the reader to [28] for details.

For the purpose of present work, it suffices to adopt the argument from [25], namely, that the electric field behaves as a contravariant tensor density. Whether the covariant formulation should be preferred over the contravariant one, must be decided after comparing the different formulations with the known properties of the electromagnetic field. As shown later in this work, the upper convected formulation fits precisely within the model, explaining the continuity equation for the charge in a moving frame while it can be demonstrated that the lower convected derivative cannot accomplish this result.

3. Material Invariant Maxwell-Lorentz Electrodynamics

In compressible fluid mechanics, if one neglects the convective part of the acceleration (accounting for the material invariance) one gets a similarly linearized system for the propagation of acoustic waves that is not Galilean invariant. Leaving intact the advective terms shows that the original system is Galilean invariant and that the speed of sound does not depend on the velocity of the moving frame. The constancy of the characteristic speed (light or sound alike) means that there is a medium independent of the emitter and receivers of the waves and once emitted, the phase speed of waves cannot be changed. The only thing that can be changed is the frequency because the next infinitesimally close moment of time the emitter (and/or receiver) change their positions with respect to the absolute medium.

Guided by this analogy, we find that the way to naturally formulate an electrodynamics that is invariant under the change to another laboratory frame, is to replace the partial time derivative of Maxwell's displacement current with the material invariant time derivative. We propose that the Maxwell-Ampere law be formulated as follows:

$$\mu_0 \epsilon_0 \frac{\mathfrak{d} \boldsymbol{E}}{\mathfrak{d} t} = \nabla \times \boldsymbol{B} - \mu_0 \boldsymbol{j}. \tag{10}$$

Note that we use standard nomenclature to denote $\boldsymbol{D} = \epsilon_0 \boldsymbol{E}$ and $\boldsymbol{H} = \frac{1}{\mu_0} \boldsymbol{B}$ (in the absence of internal magnetic moments in the field) which is the natural assumption *en vacuo*. Respectively, $\boldsymbol{E}$ is the electric field, and $\boldsymbol{B}$ is the magnetic induction. The speed of light is then given by $c = (\epsilon_0 \mu_0)^{-\frac{1}{2}}$.

Finally, the material invariant formulation of the equations of electrodynamics reads,

$$\frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{B} = -\nabla \times \boldsymbol{E}, \tag{11}$$

$$\frac{\partial \boldsymbol{E}}{\partial t} + \boldsymbol{v} \cdot \nabla \boldsymbol{E} - \boldsymbol{E} \cdot \nabla \boldsymbol{v} + (\nabla \cdot \boldsymbol{v}) \boldsymbol{E} = c^2 \nabla \times \boldsymbol{B} - \frac{\boldsymbol{j}}{\epsilon_0}. \tag{12}$$

$$\nabla \cdot \boldsymbol{B} = 0. \tag{13}$$

Note, that in Eq. (11), the invariance is ensured by the usual material derivative, while Eq. (12) involves the Oldroyd upper convected derivative. A similar situation is observed in viscoelastic fluids, where the momentum equations involve the usual material derivative, while the rheology is based on the upper convected derivative.

The formulation proposed here is also instrumental in deriving a material invariant continuity equation for the charge. To see this we take the divergence of Eq. (12), and after the cancellation of similar terms (not possible in the case of the lower convected derivative), we get

$$\begin{aligned}
&\nabla \cdot [\boldsymbol{E}_t + \boldsymbol{v} \cdot \nabla \boldsymbol{E} - \boldsymbol{E} \cdot \nabla \boldsymbol{v} + (\nabla \cdot \boldsymbol{v})\boldsymbol{E}] \\
&\qquad = (\nabla \cdot \boldsymbol{E})_t + \boldsymbol{v} \cdot \nabla(\nabla \cdot \boldsymbol{E}) + \nabla \boldsymbol{v} \colon \nabla \boldsymbol{E} - \nabla \boldsymbol{E} \colon \nabla \boldsymbol{v} \\
&\qquad\quad - \boldsymbol{E} \cdot (\nabla \cdot \boldsymbol{v}) + \boldsymbol{E} \cdot (\nabla \cdot \boldsymbol{v}) + (\nabla \cdot \boldsymbol{v})(\nabla \cdot \boldsymbol{E}) \\
&\qquad\quad = (\nabla \cdot \boldsymbol{E})_t + \boldsymbol{v} \cdot \nabla(\nabla \cdot \boldsymbol{E}) + (\nabla \cdot \boldsymbol{v})(\nabla \cdot \boldsymbol{E}) \\
&\qquad = (\nabla \cdot \boldsymbol{E})_t + \nabla \cdot [(\nabla \cdot \boldsymbol{E})(\nabla \cdot \boldsymbol{v})] = {\epsilon_0}^{-1} \left[\rho_t + \nabla \cdot (\rho \boldsymbol{v})\right],
\end{aligned}$$

where the last equality is obtained after the expression for charge density, $\rho = \epsilon_0 \nabla \cdot \boldsymbol{E}$, is substituted. Consequently, this gives the following equation for the charge density

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\boldsymbol{j} + \rho \boldsymbol{v}) = 0, \tag{14}$$

which is the accepted form of the continuity equation in a moving (laboratory) frame [9, 8]. While a naive approach to material invariance would have been to take just the usual material derivative, in doing so one would not obtain the proper equation of conservation of charge. Taking the full fledged invariant derivative is the *only* way to make the full system of electrodynamics material invariant.

It is clear that without the last term in Eq. (14), one cannot explain any electromagnetic phenomena in a moving frame. The main difference here is that we do not arbitrarily add the convective term. Rather, it appears as an integral part of the model, just in the same way as the Lorentz force does.

In closing this section we mention that Eqs. (11), (12), (14) are all Galilean invariant in the sense that when changing to a frame moving with a constant velocity, $\boldsymbol{V}$, the form of the equations remains unchanged. Indeed, in a moving frame one can introduce the new variables $\hat{\boldsymbol{x}} = \boldsymbol{x} - \boldsymbol{V}t$, $\hat{\boldsymbol{v}} = \boldsymbol{v} - \boldsymbol{V}$, $\hat{\boldsymbol{j}} = \boldsymbol{j} + \rho\boldsymbol{V}$, and if $\widehat{\nabla}$ is the nabla vector, and $\hat{\boldsymbol{E}}$ and $\hat{\boldsymbol{B}}$ are the electric field and magnetic induction in the new coordinates, then the governing system has exactly the same form as Eqs. (11), (12), (14), namely

$$\begin{gathered}
\frac{\partial \hat{\boldsymbol{B}}}{\partial t} + \hat{\boldsymbol{v}} \cdot \widehat{\nabla} \hat{\boldsymbol{B}} = \widehat{\nabla} \times \hat{\boldsymbol{E}}, \\
\frac{\partial \hat{\boldsymbol{E}}}{\partial t} + \hat{\boldsymbol{v}} \cdot \widehat{\nabla} \hat{\boldsymbol{E}} - \hat{\boldsymbol{E}} \cdot \widehat{\nabla} \hat{\boldsymbol{v}} + (\widehat{\nabla} \cdot \hat{\boldsymbol{v}}) \hat{\boldsymbol{E}} = c^2 \widehat{\nabla} \times \hat{\boldsymbol{B}} - \frac{\hat{\boldsymbol{j}}}{\epsilon_0} \\
\frac{\partial \rho}{\partial t} + \widehat{\nabla} \cdot (\hat{\boldsymbol{j}} + \rho \hat{\boldsymbol{v}}) = 0.
\end{gathered} \tag{15}$$

One of the consequences of Galilean invariance is that the speed of propagation of small disturbances (speed of light) will be the same in any inertial frame. Hence, there is no need to impose the absolutivity of the speed of light as an additional postulate. As well known from fluid mechanics, the characteristic speed of small disturbances (e.g., sound) is an absolute property of the medium and does not change in a moving frame. Consequently, the material invariant electrodynamics formulated here resembles the modern formulation of Maxwell's theory of elastic liquids.

4. Observable Manifestations of Absolute Continuum

4.1. Material Invariance and Doppler Effect

The fact that the electrodynamics can be material invariant does not necessarily contradict the principle of relativity. In fact the material invariance upholds the principle of relativity as stated by Galileo, namely that the laws of nature will be perceived in the same way (i.e., described by the same equations) by the observers in two different inertial moving frames. This is exactly what reveals the juxtaposition of the system Eqs. (11), (12), (13) and the system Eqs.(15). Unfortunately, in the last hundred years, relativity and "Lorentz invariance of electrodynamics" became synonymous which is clearly wrong. Lorentz contraction is something very real and physical, while an abstract principle that electrodynamics must be Lorentz invariant is hardly justifiable. Yet, for many practical proposes, the predictions of what is called now "theory of relativity" will still be applicable, especially where just the mere space contraction is the important effect. However, there will be many disagreements, of course, especially when accelerating charges are considered. The first principal disagreement to pop up is the Doppler effect. In a material invariant electrodynamics, the only Doppler effect is the classic one. There is no place for the concept of relativistic Doppler effect (whatever it might mean), because the absolutivity of the speed of light is an inherent feature of the model, hence, no need for "relativistic addition" of velocities when considering the propagation of waves. The scale factors for the relativistic, R_d, and classic, C_d, Doppler effects can be written as [29]

$$R_d = \sqrt{\frac{1+v/c}{1-v/c}}, \quad C_d = \sqrt{1-v^2/c^2}, \quad \rightarrow \quad R_d = C_d + O(v^2/c^2).$$

What kind of Doppler effect (classic or relativistic) is present in Nature can be easily found experimentally using interplanetary spaceships because their velocity can be inferred by the rate of change of their spatial position, while, on the other hand, the Doppler shift is independently measured. If the formula for the relativistic Doppler effect is wrong, then discrepancies between the two kind of measurements of the speed of a craft must arise that are of order of v^2/c^2. The superb measurements performed by Pioneer 10 and 11 space probes can be used to this end. There is an apparent blue shift of the Doppler data if compared with the velocity as computed from the trajectory. The magnitude of the blue shift is of order of $10^{-8} = O(v^2/c^2)$ where v is the relative velocity of the craft with respect to Earth (see [30] and the works cited therein). This discrepancy is believed now to have been caused by some kind of unexpected acceleration. The latter is called "anomalous" because no cause can be identified for it coming into being. In our opinion, it is premature to implicate an acceleration or any other physical effect for the discrepancy. First, the results have to be reexamined using the classic Doppler formula instead of the relativistic one as used in the mentioned works. In fact, this idea was voiced out in [31] who actually computed the difference of predictions based on relativistic and classical Doppler effects and showed that this disagreement is numerically very close to the alleged "anomaly". The detailed investigation of this matter is not possible without having the raw data and for this reason it is not done in the present paper. Yet, the fact that the order of magnitude is so close to the difference between the two Doppler formulas should be given the proper consideration by the people who are in possession of the raw data.

4.2. Wave Equation of Electrodynamics

Another important consequence of considering the electrodynamics as the manifestation of a material *metacontinuum* is that the validity of Ohm's law can be extended *en vacuo*. Just as stresses can cause strain in a liquid, the electric field can cause an *intrinsic* kind of current in the material points of *metacontinuum*. Ohm's law can be assumed to be valid *en vacuo*, namely

$$\boldsymbol{j} = \sigma\epsilon_0\boldsymbol{E}, \tag{16}$$

where σ is understood as properly scaled conductivity of the medium. Substituting this expression in Eq. (12) gives

$$c^2\nabla\times\boldsymbol{B} = \frac{\partial\boldsymbol{E}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{E} - \boldsymbol{E}\cdot\nabla\boldsymbol{v} + (\nabla\cdot\boldsymbol{v})\boldsymbol{E} + \sigma\boldsymbol{E} \approx \frac{\partial\boldsymbol{E}}{\partial t} + \sigma\boldsymbol{E}, \tag{17}$$

where the nonlinear terms are neglected in the last term. Here is to be mentioned that it is since long time accepted that Ohm's law, Eq. (16) should be valid *en vacuo* too, see, e.g. the authoritative monograph [32]. In the framework of the present paper, this conjecture finds its natural explanation, and now the Maxwell electrodynamics becomes fully analogous to Maxwell's theory of viscoelastic gases with σ assuming the meaning of viscosity coefficient. In other words, Eq. (17) is the *constitutive* law for a viscoelastic fluid, if the electric fields is considered as the stress vector in the medium. It is quite straightforward to change if need be to a stress tensor description [33, part II] in order to make the analogy between the electrodynamics and the viscoelastic fluids clearer.

The coefficient of viscosity of *metacontinuum* is much smaller than the elasticity coefficient, This means that for fast oscillatory motions the medium behaves almost as an elastic body with very small energy lost due to small viscosity. In the other extreme, when slow quasi-stationary loading is considered, the time derivative vanishes and the rheological law Eq. (17) represents a liquid with very large viscosity c^2/σ. Summarizing the above statements, one can say that *metacontinuum* behaves like a jello.

If the nonlinear terms are neglected, one can eliminate the electric field between Eq. (11) and Eq. (17) to obtain the following wave equation for the magnetic field (after acknowledging the fact that $\nabla \cdot \boldsymbol{B} = 0$)

$$\frac{\partial^2 \boldsymbol{B}}{\partial t^2} + \gamma \frac{\partial \boldsymbol{B}}{\partial t} = c^2 \Delta \boldsymbol{B}, \tag{18}$$

where $\gamma = \sigma c^2$ has the meaning of attenuation coefficient. The wave equation, Eq. (18), contains attenuation term $\gamma \partial_t \boldsymbol{B}$. For γ small but finite, the attenuation term can have a profound impact on the propagation of the electromagnetic waves at long times or large distances. Note that γ cannot be estimated from the ubiquitous Ohm law for currents in matter because the notion of charge has to be first introduced in order to find the coefficient of proportionality.

One effect of attenuation will be that the distant stars appear dimmer than they would have appeared if $\gamma = 0$. Consequently, the distances estimated on the base of luminosity would appear larger than they are in reality. As a result of the overestimated distances, the density (and the mass, for that matter) of the Universe would appear much smaller than the projected mass based on the orbital speeds of stars and the galaxies. A way out of this situation is to assume the presence of dark matter [34]. From the point of view of the present work, the discrepancy between the estimates for the mass of the Universe with and without attenuation can be used to evaluate the attenuation factor γ. The details are very elaborate and go beyond the scope of the present work, but there are no principal difficulties in doing this. The relationship between the actual distance, d, and the distance d^* as perceived under assumption that $\gamma = 0$, is as follows

$$d^* = de^{\frac{\gamma d}{c}}. \tag{19}$$

The number $\bar{d} = \frac{c}{\gamma}$ has the dimension of a distance. The value of $\bar{d}$ must be very large because $\gamma \ll 1$ and $c \gg 1$. When the actual distance $d \approx \bar{d}$, the perceived distance d^* will be approximately three times larger. This will make the apparent density of matter in a cube of length $\bar{d}$ to appear e^3 times smaller than its actual value. Using this relation one can estimate the order of magnitude of $\bar{d}$ (or which is the same, γ.)

4.3. Motion with respect to Background Radiation

Galileo stated the principle that velocity of material objects can only be relative. First Poincare and then Einstein attempted to extend the validity of this principle to electrodynamics. According to Poincare, no mechanical or electromagnetic experiment can discriminate between a state of uniform motion and a state of rest. Although most of the physicists of the 20th and 21st Centuries are infatuated with this statement, it might not be true if a material medium exists in which the electromagnetic phenomena take place. In other words, there exist experiments that can tell if the laboratory frame is moving with respect to the absolute continuum.

As argued above, Maxwell electrodynamics was actually made partially material invariant when the Lorentz force was added, and with the final touches concerned with the displacement current presented here, it is fully material invariant. This means that there must be a way to discover if the laboratory frame is moving with respect to the absolute medium. And this is to examine the Doppler shift of the background microwave radiation. If there is a difference in the redshift of the Cosmic Background Microwave Radiation (CBMR) in different directions, then the local frame is moving with respect to the absolute continuum where the CBMR propagates.

The discovery that there is anisotropy of the Doppler shift of the frequency of the cosmic blackbody radiation was made as early as in 1976 in [35] and confirmed in 1977 in [36]. Since then it has been verified many times (see [37]). The anisotropy of the Doppler shift was clearly observed to follow cosine rule with axis pointed approximately towards constellation Leo. The velocity corresponding to this anisotropy placed at $270 \pm 60 km/sec$ in [35], and at $390 \pm 60 km/sec$ in [36]. In the viewpoint of the present paper, this must be the velocity of Earth with respect to the metacontinuum at rest. The authors used a quotation from Peebles to call this relative motion "new aether drift". It is symptomatic about the deep roots of the official dogma about "relativity" that the authors were forced to use Aesop's language instead of stating that these results clearly and unequivocally reject the principle of relativity as enshrined by Poincare and popularized by Einstein.

The proper conclusion of the cited works on the anisotropy of the background radiation is that there is a way to tell if the observer is in a moving frame or in a frame at rest.

It is easy to understand why Galileo's principle of relativity is correct for point particles and incorrect for fields, such as electromagnetic field. A particle (at least in Galilean-Newtonian physics) does not have structure, while a wave has spatial structure and the changes in this spatial structure, e.g. Doppler shift provides the necessary information about the underlying metacontinuum, including an information about the state of motion of the laboratory frame.

5. The Concept of a Wave-Particle

> "Indeed, one of the most important of our fundamental assumptions must be that the ether not only occupies all space between molecules, atoms or electrons, but that it pervades all these particles. We shall add the hypothesis that, though the particles may move, the ether always remains at rest. We can reconcile ourselves with this, at first sight, somewhat startling idea, by thinking of the particles of matter as of some local modification in the state of the ether. These modifications may of course very well travel onward while the volume-elements of the medium in which they exist remain at rest." H. A. Lorentz [18]

In our opinion, the only way to reconcile the absolute medium (as testified by the absolute speed of propagation of shear waves) and the relative motion of so-called particles is to understand the latter as phase patterns. Since the very notion of a particle presumes a localization, we consider the localized deformation patterns of the metacontinuum to be the particles. It is currently well known that in many continuous systems localized waves behave as particles. These wave-particles are called *solitons*. Soliton research has been the most rapidly growing scientific field in the last couple of decades. We refer the reader to the excellent review [38] and the extensive monographs on the subject, e.g. [39]. The soliton presents an example of a moving "modification of the state" of the absolute continuum. Solitons can *propagate* while the material points of the metacontinuum *remain* in the vicinity of their original positions. Solitons can interact with each other upon their collisions and regain the original form when they separate enough after the collision is over. The quasi-particle behavior of solitons is now very well studied and documented.

If we now call the localized waves "particles", then the medium in which they are propagating will appear as something beyond the mere mechanics of the particle. For this reason we use the coinage *metacontinuum* to designate the absolute medium which is the carrier of all kind of waves and wave-particles alike.

5.1. Localized Vortex Patterns in Metacontinuum

It is well known that the model of inviscid liquids admits potential vortex solutions. The vortex flow is irrotational except for the central point where a singularity is observed. In fact when no interaction with the boundaries is presumed, the potential solutions exist even for the Navier-Stokes equations of viscous liquids. The singularity cannot be removed in the model of Newtonian liquids. Yet, point-vortex flow [40] is one of the theoretically best studied solutions of classical hydrodynamics. Lord Kelvin extended the idea of vortex structures to the alleged aether in an attempt to explain atoms. Our point of view is that the vortices of the metacontinuum cannot account for all observable phenomena associated with material particles. As shown in [41] and [33, Part II], a fourth dimension is needed in order to explain Schrödinger's wave mechanics. At the same time, the vortex is a perfect model of what is known as "electric charge" because it possesses a topological charge (circulation).

To elucidate this concept we begin with the stationary case in 2D when Eq. (18) reduces to a single scalar Laplace equation for the third component of the magnetic field, say $\psi = \boldsymbol{B}_z$, namely

$$\Delta\psi = 0.$$

In the case of polar symmetry $\psi = F(r)$, where $r = \sqrt{x^2 + y^2}$, and then $F(r) = \ln(r)$. For the velocity components, we have the following expressions

$$v_x = \frac{y}{r^2}, \quad v_y = -\frac{x}{r^2}. \tag{20}$$

The vector field generated by Eq. (20) is shown in Fig. 1(a). This is the well known potential vortex in 2D fluid dynamics. (see, e.g., [32, Ch.IX]). The topological charge of the vortex solution is defined as the integral over a closed curve C

$$\Gamma = \oint_C \boldsymbol{v}\mathrm{d}\boldsymbol{s} = \oint_C [u_x dx + u_y dy],$$

where d$\boldsymbol{s}$ is the elementary arc length along the curve C and the quantity Γ is the circulation. If we consider now the vortex of the metacontinuum as the electric charge, then the quantitative value of its *charge* is given by the circulation Γ (the topological charge). Thompson's theorem (see [40, 32]) asserts that in an inviscid liquid the circulation is conserved, i.e., $\Gamma = const$, which gives in our model the conservation of charge.

Some deviation from Thompson's theorem is expected because of the additional acceleration in the Maxwell rheological law. However, the stationary propagating vortices should not be affected by it.

5.2. Effect of Rectilinear Motion on Localized Phase Patterns: Lorentz Contraction

The soliton-like vortex solution from the previous subsection is a kind of *torsional dislocation*. The material points of the metacontinuum may move continuously like in the above considered vortex, while the phase pattern is completely at rest. If we consider now the patterns to be the particles, then we must realize that the laws of motion for the center of a localized pattern will appear on macro scale as the laws of motion for a point-particle.

Should the above described "dislocation" be allowed to move, it would not "plow" through the material points of the continuum. Instead, it would propagate as a *phase pattern*, in much the same way a wave propagates over the water surface. The concept that the charge is a phase pattern removes the most substantial objection against the elastic model of the electromagnetic field, namely, that it is too dense for the particles and charges to move through. Also, no "ether wind" is supposed to trail a propagating dislocation. Such a dislocation does not introduce a further disturbance in the metacontinuum apart from the velocity field of the pattern itself.

Consider now a solution for $\boldsymbol{v}$ which is almost stationary (slowly evolving) in a moving frame. Consider for definiteness the frame moving in the y-direction with phase speed c_y. Introducing a local spatial variable $\eta = y - c_y t$ and neglecting the local time derivatives in the moving frame, one gets

$$\frac{\partial \psi}{\partial t} = -c_y \frac{\partial \psi}{\partial \eta}, \quad \frac{\partial^2 \psi}{\partial t^2} = c_y^2 \frac{\partial^2 \psi}{\partial \eta^2}. \tag{21}$$

and the Laplace equation $\Delta \psi = 0$ recasts to

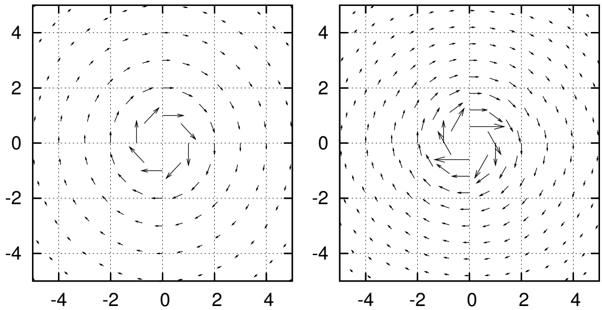
$$\frac{\partial^2 \psi}{\partial x^2} + (1 - c_y^2) \frac{\partial^2 \psi}{\partial \eta^2} = 0, \quad \Rightarrow \quad \frac{1}{z} \frac{\partial}{\partial z} \left(z \frac{\partial \psi}{\partial z} \right) = 0, \tag{22}$$

where $z \equiv \sqrt{x^2 + \eta^2 (1 - c_y^2)^{-1}}$. The solution of Eq. (22) is once again $\psi = \ln z$.

The isolines of ψ are now ellipses that appear contracted in the direction of motion by the Lorentz factor as shown in Fig. 1(b).

Comparison of the two panels of Fig. 1 hints at an analogy between the contraction of the localized wave to the Doppler shortening of harmonic waves ahead of a moving source. Considering the localized wave as a quasi-particle (wave-particle) we conclude that *Lorentz Contraction is the Doppler Effect for Wave-Particles*.

In closing this section, we stress the point that the Galilean invariance of motion inside the metacontinuum reflects its absolute nature, but it does not precludes the wave-particles from



(a) $c_1 = 0, c_2 = 0$ (b) $c_1 = 0, c_2 = 0.8$

Figure 1. Localized torsional dislocation in two dimensions for two different phase velocities of propagation. Left panel: $c_x = 0, c_y = 0$. Right panel: $c_x = 0, c_y = 0.8$

behaving in a more relativistic fashion because they are *phase patterns*, or "quasi-particles". This could be the resolution of long the standing paradox pointed out by Einstein [42, p.21] that one cannot rationally reconcile the absolute speed of light with the apparent relativity of rectilinear motion.

Clearly, the wave-particles are subject to Lorentz contraction, and so are the interparticle forces as a result of the Doppler effect of the waves which are transmitting the long-range interactions. This means that a body of charged wave-particles held together by the internal stresses of the metacontinuum (electromagnetic forces) would become shorter in the direction of motion (better said: "direction of propagation"). This means that the Doppler effect and the Lorentz contraction will cancel each other in any interferometry experiment using a split beam and closed light path. If one assumes the presence of an absolute continuum, then the only strict result from the Michelson and Morley experiment *must* be the nil effect. In other words, the nil effect of the Michelson and Morley experiment can be considered as a strong indication of the existence of an absolute continuum for which the point particles are in fact a coarse-grain description of localized waves.

The above statement is about an idealized version of the Michelson experiment in vacuum and without acknowledging Earth's rotation in the sense that not finding an effect corresponding to $30km/sec$ (or higher) is considered in first approximation as nil effect and supports the assumption of an absolute continuum whose particles (solitons) are contracted in the same fashion as the wave length of a harmonic wave. A most interesting discussion can be found in [22, 23] where the data of Michelson, Morley and Miller is thoroughly reexamined and the slight deviations from the nil effect are ingeniously interpreted to be connected with the speed with respect to the absolute continuum. As a result the speed of Earth with respect to the absolute medium is placed at $420 \pm 30km/sec$ which is in very good quantitative agreement with the CBMR analysis. The small difference are attributed in [22, 23] to gravitation effects which is fully compatible with the theory of *metacontinuum* presented here. It is only natural that gravitation can have effect on the speed of light in the *metacontinuum*.'

6. Conclusions

In the present paper, it is argued that the partial time derivative of the electric field representing Maxwell's displacement current can be replaced by a material invariant time derivative in the same vein as in Oldroyd's [10] reformulation of the constitutive relations for viscoelastic liquids. It is shown that together with the Lorentz force in Faraday's law, the new formulation is material invariant. From this invariant formulation, the continuity equation for the conservation of electric charge is shown to be also invariant.

The material invariance of electrodynamics suggests that there is a material medium (called here the *metacontinuum* in which electromagnetic processes take place. The rheology of the medium is that of a viscoelastic liquid with the displacement current representing the elastic part of the constitutive relation, and Ohm's law representing the viscous part. The material invariance of the model requires that the characteristic speed of propagation of small disturbances (speed of light) is constant which does not depend the velocity of the moving frame, i.e., there is no need to impose the absolutivity of the speed of light as an additional principle. It is an innate feature of material invariant electrodynamics.

The most important consequence of the material invariance of the electrodynamics is that a principle of relativity holds in the sense that the electromagnetic processes will appear in the same form to two different observers in two different inertial frames without the need of transforming the time variable. However, the Poincare formulation of the principle of relativity does not hold because there exists a way to identify the rectilinear motion of the moving frame.

It is proposed to consider the charges as localized waves (solitons) of the field which explains why the absolute continuum is not entrained by the moving particles; they are propagating phase patterns.

As a result, a consistent picture of electrodynamics emerges that is based on the absolutivity of the material field which is the carrier of the electromagnetic interactions and on the relativity of the motion of the centers of the wave-particles when considered as point particles.

Acknowledgments

The author wishes to acknowledge numerous helpful discussions with Dr. P. M. Jordan.

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