The Sliding Rods Paradox

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This paper sets out a kinematic analysis of the precession movements and explains a paradoxical situation in the theory of relativity. This study is made within the framework of special relativity and using a simple methodology, without reference to the POLT decomposition theorem.

Keywords: relativity, slope deflection, Thomas precession, kinematics.

1. Introduction

All the issues raised in this paper are based on the study of one kinematic difficulty caused by the "phase difference" term vX/c^2 in the 4th Lorentz Transformation Equation [1][7]. We apply the premises of special relativity to the vertical movement of a horizontal segment AB that remains parallel at all times to the X axis of an inertial reference frame $S(X,Y;T)$. We will assume that $S(X,Y;T)$ moves at a constant velocity $(v,0)$ relative to the x axis of a reference frame $s(x,y;t)$, keeping the xX and yY axes parallel, respectively (fig. 1), so that the origins of the two systems coincide at the instant

$t = T = 0$. Throughout this paper we will refer to the two inertial frames "s" and "S," assuming at all times that one is moving relative to the other under the kinematic conditions that we have just set out, even though we do not expressly mention this circumstance in each specific instance.

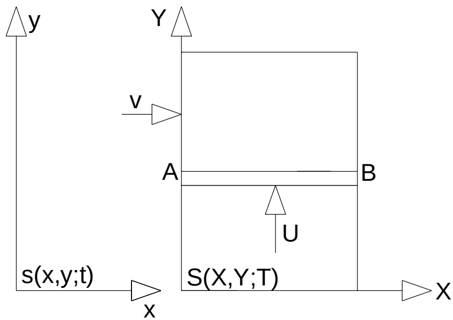

Fig 1: Horizontal segment AB moves vertically at a constant velocity *U* in S(*X,Y;T*).

2. Slope deflection

Taking $X(A) = 0$, the notation of the space-time co-ordinates in S($X,Y;T$) for describing events corresponding to the arrival of the ends A($0,Y$) and B(L,Y) of the segment AB at any height Y will be:

Arrival of A at height Y in "S": $(0, \quad Y; \quad T(A))$

Arrival of B at height Y in "S": $(L, \quad Y; \quad T(A))$ (BS)

Applying the Lorentz transformations in their homogeneous form, we obtain, in the system s($x,y;t$):

Arrival of A at height Y in "s": $(\gamma vT(A), \quad Y; \quad \gamma T(A))$

Arrival of B at height Y in "s": $(\gamma(L + vT(A)), \; Y; \; \gamma(T(A) + vL/c^2)))$

$$\gamma = c/\sqrt{c^2 - v^2}$$

It is to be noted that the values of height Y for the ends A and B, observed from s($x,y;t$), are not reached simultaneously:

$$t(A) = \gamma T(A)$$

$$t(B) = \gamma(T(A) + \frac{vL}{c^2})$$

These times coincide only where the reference frame “S” is not in motion relative to the reference “s.” Thus, segment AB is seen to be inclined in the reference frame s($x,y;t$), since the right end B of the segment takes longer to reach any height $y = Y$ than end A.

If we substitute the inverse of the time transformation $T = \gamma(t - vx/c^2)$ in the expression $Y = UT$ and we use the relation y = Y, we find that:

$$y = U\gamma(t - \frac{vx}{c^2}) \tag{yxt}$$

which is the linear equation corresponding to the sloping straight line of the segment AB in the reference frame s($x,y;t$). We note that this straight line rises in the vertical direction “y” at a constant velocity $U\gamma$, and that the slope $m = -U\gamma v/c^2$ in respect of the “x” axis depends on the relative velocity v between the reference frames “s” and “S” and on the velocity U of vertical ascent in S($X,Y;T$). Thus, for each pair of values for U and v we will have a different value for the slope m of the segment AB seen from s($x,y;t$).

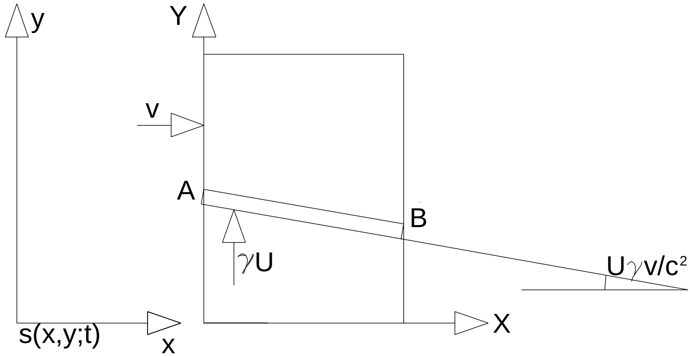

Fig 2: **Deflection diagram of inclination**: in s(*x,y;t*) the segment AB is inclined with a slope $-U\gamma v/c^2$ and rises in s(*x,y;t*) at a constant velocity γU because the condition $y = U\gamma(t - vx/c^2)$ is fulfilled.

Consequently, the inclination *m* observed from "s" will vary and create an effect of rotation or "precession" in respect of the axes of the reference frame s(*x,y;t*), if:

d*U*/d*t* is not equal to 0

d*v*/d*t* is not equal to 0

so that if the moving reference frame S(*X,Y;T*) or the horizontal segment rising vertically in S(*X,Y;T*) are accelerated, the observer in the reference frame s(*x,y;t*) will observe the segment rising and at the same time changing its orientation in the plane *xy* in accordance with a specific value of angular velocity w_p.

3. Thomas precession

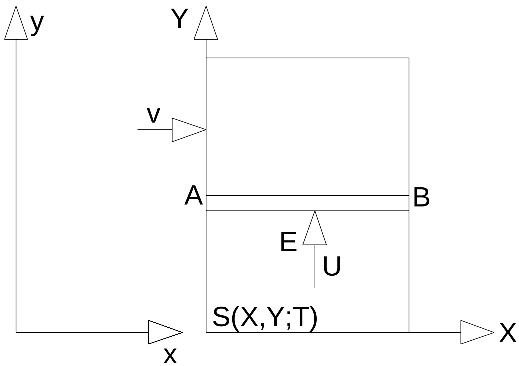

Fig 3: **Thomas precession**: we assume that the horizontal segment AB moves vertically with constant acceleration E and velocity U in S($X,Y;T$). At instant $T = 0$ we find that $U = Y = 0$.

In the specific case in which the horizontal segment rises with constant vertical acceleration E and therefore dU/dt is not equal to zero, it is relatively simple to calculate the angular velocity of precession [6]. If we assume that at instant $T = 0$ we have that $U = Y = 0$, then we can apply the formulation of uniformly accelerated movement $Y = ET^2/2$ and $U = ET$ to obtain:

$$Y = \frac{UT}{2}$$

Replacing T with its relativistic transformation $T = \gamma(t - vx/c^2)$ and Y with $Y = y$, we have that $y = U\gamma(t - vx/c^2)/2$. The coefficient by which x is multiplied is the inclination of the segment AB in "s" and its value is $m = -U\gamma v/2c^2$. If we calculate the time derivative dm/dt and the angle approaches the tangent, we find that the angular velocity of precession is w_p = dm/dt = $-\gamma v$(dU/dt)/2c^2. In turn, dU/dt = dU/dT/dt/dT = $E/\gamma(1 + vU_x/c^2)$ and we have:

$$w_p = -\frac{vE}{2(c^2 + vU_x)} \qquad \text{(w)}$$

with: Ux horizontal component of the velocity of the segment AB in S($X,Y;T$)

As we know that the horizontal segment AB has no horizontal motion in "S", $U_x(AB) = 0$:

$$w_p(AB) = -\frac{vE}{2c^2} \qquad \text{(wAB)}$$

or in vector form ([2], [3], [4], [5], [6]):

$$\vec{w}_p(AB) = \frac{\vec{E} \times \vec{v}}{2c^2}$$

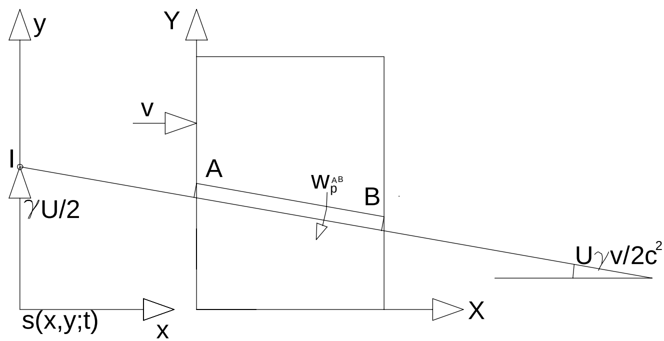

Fig 4: **Thomas precession**: the straight line containing segment AB in Fig. 3 moves in s($x,y;t$) like a boomerang: point I moves vertically **at velocity** $\gamma U/2$ **and** the rest of the points of the line IAB turn clockwise relative to I with a precession velocity $w_p^{AB} = -vE/2c^2$. If the segment AB were to move in the direction of the X axis in "S," its precession velocity w_p in "s" would be different.

Therefore, whenever we have a horizontal segment, rod or vector AB with uniform acceleration perpendicular to the motion of an inertial reference frame that in turn moves with uniform velocity *v* relative to another inertial reference frame, we will find the precession effect [6] that we have just seen and that Llewellyn Hilleth Thomas described for the first time in 1926 [5].

4. The sliding rods paradox

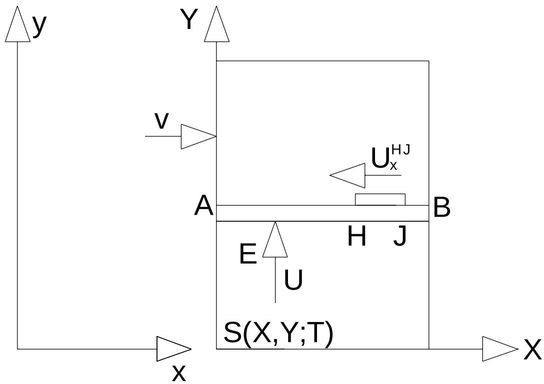

Fig 5: **Sliding rods paradox**: we assume that: 1) horizontal segment AB rises in S(*X,Y;T*) with constant acceleration *E* and velocity *U*. 2) segment HJ rises in S(*X,Y;T*) with constant acceleration *E*, velocity *U* and horizontal component *Ux*.

Returning to the case of the horizontal rod AB moving vertically in S with velocity *U* and acceleration *E*, if another horizontal rod HJ moves to the left along the horizontal segment AB *without breaking contact with it at any time along its full length*, it will move with a horizontal velocity component –*Ux*(HJ) in S(*X*,*Y*;*T*) (see Fig. 5) and will therefore turn in s(*x*,*y*;*t*) in accordance with (w) at an angular velocity of precession different from that of the rod AB (see Fig. 6):

$$w_p(HJ) = -\frac{vE}{2(c^2 - vU_x(HJ))} \quad \text{(wHJ)}$$

This is truly paradoxical, since in this case in s($x,y;t$) we will necessarily *observe that the rods* AB *and* HJ *lose contact with each other* due to the fact that they turn in the plane of *xy* with a different angular velocity of precession value w_p (see Fig. 6) according to the equations (wAB) and (wHJ). How can this be true if both rise in "S" at the same velocity *U* and therefore the slope in "s" of the two rods in both cases is $m = -U\gamma v/2c^2$?

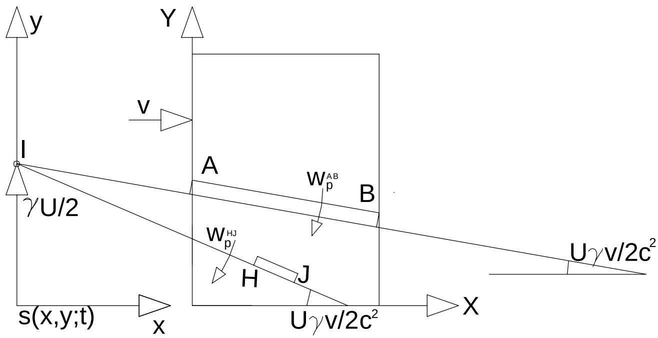

Fig 6: **Sliding rods paradox**: in s($x,y;t$) the two rods AB and HJ in Fig. 5 turn at different precession velocities $w_p = -vE/2(c^2 - vU_x)$ because they have different horizontal velocities Ux in "S"; on the other hand, inclination $m = -U\gamma v2/c^2$ is the same, since AB and HJ have the same vertical velocity *U* in "S." This situation is obviously inconsistent. Furthermore, it is impossible for the two rods to be in contact in "S" and lose contact in "s."

For example, let us assume that the current circulating in a straight horizontal conductor is equivalent to an electrically charged rod moving through the conductor. If the horizontal conductor accelerates vertically up the Y axis and remains parallel at all times to the *X* axis, the charge circulating through the conductor turns in the *xy* plane at a different precession velocity from the conductor containing the charge, which is wholly unsustainable, since in that case the charged rod and the conductor carrying the charge would lose contact with each other. It seems clear that both the conductor and the charge that it carries must conserve the same direction at all times, and furthermore that the contact between them must be considered an "absolute" reality in any inertial system taken as the reference.

Let us examine another example:

A spaceship S($X,Y;T$) moves at a constant velocity (v,0) in the positive direction of the x axis of a space station s($x,y;t$), maintaining its orientation so that the axes x,y are always parallel to the X,Y axes respectively (see Fig. 7). Inside the spaceship S($X,Y;T$) there is a lift moving vertically with velocity (0,U) and acceleration (0,E) in the positive direction of the Y axis of the spaceship. On the floor of the lift there is a worm, G, moving to the left at a constant velocity U_G, so its velocity relative to the spaceship is ($-U_G$,U).

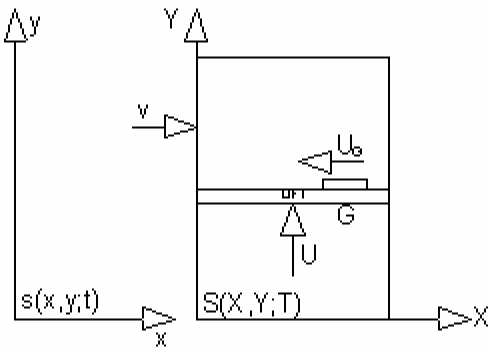

Fig. 7: **Diagram of the "worm" paradox**. The worm G behaves like a rod moving along another rod that is part of the floor of the lift. In this case, the sliding rods paradox is fulfilled.

The lift (segment AB) and the worm G (segment HJ) are observed from s($x,y;t$) to turn with different precession angular velocities. Consequently, once again we encounter a contradictory situation: the worm loses contact with the floor of the lift in "s" since it has a different precession angular velocity.

The difficulty here is caused by the term vX/c^2 of the time transformation, which makes the precession velocity w_p dependent upon the horizontal component U_x of the velocity in "S."

5. Conclusion

We can calculate the frequency of the Thomas precession w_p in s($x,y;t$) and apply it to the two horizontal rods AB and HJ that remain in contact at all times throughout their full length, sliding along each other and moving vertically with the same vertical acceleration component E in the reference frame S($X,Y;T$). We would expect the values of w_p to be equal, so that, observed from the stationary reference frame s($x,y;t$), the contact between the two rods would not be lost. However, this is not the case and we then come to a totally inconsistent situation (Section 4: the sliding rods paradox).

This is caused by the term vX/c^2 of the time transformation, which makes the precession velocity w_p dependent on the component U_x.

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