A Necessary Algebraic Condition for R4 Embedded into E5

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A necessary algebraic condition is obtained for the intrinsic geometry of any spacetime locally and isometrically embedded into a pseudo-Euclidean 5-space.

Keywords: Local and isometric embedding; embedding of Riemannian 4-spaces.

Introduction

Here we study spacetimes admitting local and isometric embedding into a flat 5-dimensional space, thus a new geometric object arise called second fundamental form **b**. Which enrich the Riemannian structure and offers the possibility [1-4] of reinterpreting physical fields using **b**. Unfortunately to date such hope has not been realized since it has been extremely difficult to establish a natural correspondence between the quantities governing the extrinsic geometry of spacetime and physical fields. In spite of this, one cannot but accept the great value of the embedding process, for it combines harmoniously such themes as the Petrov [5-11] and the Churchill-Plebañski [5,12-16] classifications, exact solutions and their symmetries [6,17-32], the Newman-Penrose formalism [6,11,33-35], and the kinematics of timelike and null congruences [6,17,22,33,36,37]. On the other hand, it offers the option of obtaining exact solutions that cannot be deduced by any other means [6].

In the embedding problem let us recall, the intrinsic geometry of the spacetime (determined through the metric tensor) is assumed given, what is required is the extrinsic geometry of our 4-space with respect to the flat 5-manifold. Thus we have one additional dimension and therefore one second fundamental form which cannot be prescribed arbitrarily since it determines the corresponding extrinsic geometry. For a local and isometric embedding to be realizable, it is necessary and sufficient that the Gauss-Codazzi equations hold [17, 23, 34, 38].

A spacetime (R4) can be embedded into E5 if and only if [5, 6, 16, 35, 39, 40] there exist the second fundamental form $b_{ac} = b_{ca}$ fulfilling the Gauss-Codazzi equations:

$$R_{ijrc} = \varepsilon\left(b_{ir}b_{jc} - b_{ic}b_{jr}\right) \tag{1}$$

$$b_{ij;r} = b_{ir;j} \ , \tag{2}$$

Where $\varepsilon = \pm 1$, R_{abcd} is the R4 curvature tensor and; j denotes the covariant derivative. Such a 4-space is said to be of class one.

Given a metric tensor g_{ac}, perhaps the corresponding R4 does not admit embedding into E5, but since we do not know such fact, we would waste time by looking for a b_{ij} that does not exist. So in order to save work it is useful to know previously if they satisfied a number of (algebraic and/or differential) necessary conditions [30, 41-43] required on the intrinsic geometry of the 4-space; when those conditions are not fulfilled the corresponding space is not of class one. If on the contrary the conditions are fulfilled, we emphasize that this is not a guarantee for the possible embedding of the spacetime into a pseudo-Euclidean 5-space.

In this work we obtain an algebraic necessary condition, by means of (1), which has not been found in the literature; that condition imposes restrictions on the internal geometry of a R4 which can be embedded into E5. Our condition involves the g_{ij}, the R_{abcd}, as well as the Ricci tensor $R_{ic} = R^{r}{}_{icr}$, the scalar curvature $R = R^{c}{}_{c}$ and a Lanczos invariant [44-46].

Algebraic necessary conditions for R4 of class one

It has been published elsewhere several conditions which are necessary for class one spacetimes. For instance, Collinson [41, 43, 47-49] obtained that:

$$*R_{it}{}^{jm} R_{rcjm} = -\frac{k_2}{12}\eta_{itrc} \quad , \tag{3}$$

$$k_1 \equiv *R^{ijrc} R_{ijrc} = 0 \quad , \tag{4}$$

where k_1 and k_2 are the Lanczos scalars [39, 42, 44, 46, 50-52]:

$$k_2 \equiv *R*^{ijrc} R_{ijrc} \quad , \tag{5}$$

and [6, 41, 42, 45, 47, 48] η_{itrc} , $*R_{ijar}$ and $*R*_{arcj}$ are the Levi-Civita tensor, the simple dual and the double dual of the Riemann tensor, respectively; it is clear that (3) implies (4). If an R4 violates (3) then it cannot be embedded into E5. By the way, it is worth to point out that when embedding is feasible then [42, 49, 50, 53] $k_2 = -24\det(b^{i}{}_{j})$.

Another example is the necessary condition obtained by Lovelock [41, 54]:

$$R_{abcd}\left(\frac{R}{2}R^{abcd}+\frac{1}{2}R^{ab}{}_{ij}R^{cdij}-R^{ac}R^{bd}-R^{aicd}R^{b}{}_{i}+R^{aicj}R^{b}{}_{i}{}^{d}{}_{j}\right)=0 \quad ,(6)$$

Both algebraic conditions (3) and (6) are consequence of the Gauss equation (1).

Now let us deduce a new necessary condition for every R4 of class one; in fact with (1) it is simple to prove the following relation not found in the literature:

$$R_{ijc[q}b_{r]a}=R_{qra[i}b_{j]c}\,, \tag{7}$$

where [ ] means antisymmetrization on the involved indices. Recalling the identity [42, 49, 50, 53, 55-57]:

$$pb_{ij}=\frac{k_2}{48}g_{ij}-\frac{1}{2}R_{imnj}G^{mn} \quad , \tag{8}$$

where $G_{ac}=R_{ac}-\dfrac{R}{2}g_{ac}$ is the Einstein tensor and $p=\dfrac{\varepsilon}{3}b^{ac}G_{ac}$, then (7) leads to a new necessary algebraic condition:

$$\left(R_{ijc[q}R_{r]mna}-R_{qra[i}R_{j]mnc}\right)G^{mn}+\frac{k_2}{24}\left(R_{qra[i}g_{j]c}-R_{ijc[q}g_{r]a}\right)=0 \tag{9}$$

This expression (9) is not equivalent to (3) and/or (6), that is, it does not contain the same geometric information as the Collinson [47] and Lovelock [54] conditions do.

In (9) we have many free indices, so that by their contraction, another necessary conditions can be deduced, which may be simpler to apply in some specific situations, but that actually will not give us more information than (9). For instance, when contracting *i* with q in (9), we obtain that every *R4* embedded into *E5* should fulfill:

$$\begin{aligned}&\left(R_{ar}R_{jmnc}-R_{jc}R_{rmna}+R^{i}{}_{jrc}R_{imna}-R^{i}{}_{raj}R_{imnc}\right)G^{mn}+\\&\frac{k_2}{24}\left(R_{jc}g_{ar}-R_{ar}g_{jc}\right)=0\end{aligned}, \qquad (10)$$

and if in (10) j and c are contracted it results another new necessary condition:

$$\begin{aligned}&\left(R_{ar}R_{mn}-RR_{rmna}+R^{i}_{r}R_{imna}-R^{i}{}_{ra}{}^{j}R_{imnj}\right)G^{mn}-\\&\frac{k_2}{6}\left(R_{ar}-\frac{R}{4}g_{ar}\right)=0\end{aligned}, \qquad (11)$$

Conclusions

Therefore, as a suggestion, prior to intending the construction of b_{ij}, the fulfillment of (3), (6), (9), (10) and (11) have to be checked out, this is important since it would be a vain effort to seek for an inexistent b_{ij}. On the other hand, it must be interesting to construct explicitly a metric verifying (3) and (6), but in contradiction with (9), that is, to show that (9) has different geometric information as (3) and (6). Finally, as the embedding problem has great physical importance, then it is very important a deep analysis of the Gauss-Codazzi equations in their algebraic-differential aspects.

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