Black Holes, Other Exotic Stars and Conventional Wisdom

S. E. Bloomer and J. Dunning-Davies,
Department of Physics,
University of Hull,
Hull HU6 7RX, England.
E-mail: j.dunning-davies@hull.ac.uk

The notion of 'conventional wisdom' is considered and its possible role in the determination of the direction of scientific research examined and questioned. Here the influence of 'conventional wisdom' on work in the area of exotic stars, in particular black holes, is scrutinised carefully.
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Introduction

At the end of the nineteenth century, it appeared that many felt little remained to be done to clear up the problems remaining in the field of physics. Seemingly, all the necessary techniques were in place and it was merely a matter of time before all outstanding questions were answered. However, three major problems remaining were those associated with the passage of light through moving media, the advance of the perihelion of the planet Mercury - an advance which, apparently, could not be predicted accurately using Newtonian mechanics, and the interaction of matter with radiation. In the end, these three problems were cleared up in the early years of the twentieth century by the theories of special relativity, general relativity and quantum mechanics respectively. Although approximately one hundred years has elapsed, rightly or wrongly, many still doubt special and general relativity and aspects of quantum theory are also regarded as puzzling by some of its most powerful and successful users. As far as the theories of relativity are concerned, what were regarded as the three basic tests of general relativity - the shift of the perihelion of Mercury, the bending of light rays in the vicinity of a massive object, and the existence of a red-shift due to gravitational effects - have all been solved without recourse to the methods of general relativity [1,2]. It is interesting to note also that Planck's energy distribution for a black-body radiation field has been derived using Maxwellian statistics [3]. A further point of interest, as far as this latter issue is concerned, is that, in the quoted article, it is noted that others had attempted such a derivation but had seemingly stopped short of the final solution while following the correct procedure. The added interest here lies in the fact that an analogous situation appears to occur with the background of special relativity.

It is often claimed that one reason for attempting to find a new theory which superseded Newton was the need for Maxwell's electromagnetic equations to remain invariant under an appropriate coordinate transformation. However, the usually quoted Maxwell equations are derived for a medium at rest. In the book, "*The Classical Theory of Electricity and Magnetism*" by Abraham and Becker [4], the problem of deriving the Maxwell electromagnetic equations for a moving medium is broached. However, a final solution to the problem is not advanced since it is claimed that 'to derive the exact form of the equations for arbitrary values of $\boldsymbol{u}$ (where

u is the velocity of the moving medium), we require the theory of electrons and the theory of relativity'. However, it has been shown recently [5] that, if the mean flow is assumed steady and uniform and, therefore, both homentropic and irrotational, ***u*** will be constant and the familiar Maxwell electromagnetic equations, when no charge is present, will reappear but with the partial derivatives with respect to time replaced by Euler derivatives; that is,

$$\nabla\times\boldsymbol{E} = -\frac{\mu}{c}\frac{D\boldsymbol{E}}{Dt} = -\frac{\mu}{c}\left[\frac{\partial\boldsymbol{H}}{\partial t} + (\boldsymbol{u}.\nabla)\boldsymbol{H}\right],$$

$$\nabla.\boldsymbol{E} = 0,\ \ \nabla.\boldsymbol{H} = 0,$$

$$\nabla\times\boldsymbol{H} = \frac{\varepsilon}{\mu}\frac{D\boldsymbol{E}}{Dt} = \frac{\varepsilon}{\mu}\left[\frac{\partial\boldsymbol{E}}{\partial t} + (\boldsymbol{u}.\nabla)\boldsymbol{E}\right],$$

where

$$D/Dt = \partial/\partial t + (\boldsymbol{u}\cdot\nabla)$$

These four modified Maxwell equations then lead to ***E*** and ***H*** satisfying the so-called progressive wave equations:

$$\nabla^2\boldsymbol{E} = \frac{\varepsilon\mu}{c^2}\frac{D^2\boldsymbol{E}}{Dt^2} \text{ and } \nabla^2\boldsymbol{H} = \frac{\varepsilon\mu}{c^2}\frac{D^2\boldsymbol{H}}{Dt^2}$$

These results actually bring electromagnetism into line with all other areas of continuum mechanics, where use of the Euler derivative and the progressive wave equation is quite usual. However, the modified equations prove to be invariant under Galilean transformation and the need for another transformation, such as the Lorentz transformation, disappears. It might be noted also that these modified Maxwell equations have an important consequence as far as the origin of planetary magnetic fields is concerned. The mechanism generally favoured as providing the best explanation for the origin of these fields is the so-called dynamo mechanism, although the main reason for its adoption appears to have been the failure of the alternatives to provide a consistent explanation. Unfortunately, in 1934, Cowling [6] proved what is essentially an anti-dynamo theorem, showing that there is a limit to the degree of symmetry encountered in a steady dynamo mechanism. In turn, this result shows that the steady maintenance of a poloidal field is not possible and this has caused problems over the origin of the planetary magnetic fields ever since. However, Cowling's proof depended on the usual Maxwell equations and, if the modified Maxwell equations are used, the proof of the theorem does not follow and a major difficulty associated with the origin of planetary magnetic fields is removed [5].

It does seem, therefore, that a genuine need exists to examine afresh some of the basic building blocks of physics and not remain bound by notions of conventional wisdom imposed from elsewhere. One major area which would benefit from such an approach has to be the field of black holes and other exotic stars associated with the end-point in the lives of stars. With its science fiction appeal, this area is one which has attracted much attention over the years and appears protected from critical discussion and examination by some sort of impenetrable aura. This area seems to provide an excellent example to consider in the context of examining the question as to when physics will be allowed to progress freely - free, that is, of the random constraints imposed by so-called 'conventional wisdom'?

Black holes

In the extensive literature on the subject, the notion of a black hole arises in two totally different ways, but these are made to appear one and the same for not totally unreasonable reasons. Possibly the first mention of the notion of what we now call a black hole came with the work of John Michell who, in 1794 [7], essentially derived an expression for the ratio of the mass to radius of a star with a speed equal to, or greater than, the speed of light as its escape speed. In deriving this, Michell seems to have had in mind a genuine three-dimensional physical object and the actual stellar object always remained simply that - a three-dimensional physical object! Although discussed by Laplace at approximately the same time, the idea remained an almost forgotten curiosity until it resurfaced in a somewhat different form via the Schwarzschild solution of the field equations of general relativity. In this solution it was noticed that a singularity existed when the mass and radius of the event horizon (rather than the physical radius) were related in exactly the same way as in Michell's result mentioned above. This could only be viewed as a coincidence. However, apparently without knowledge of Michell's result, this singularity became associated with the end-point in the life of a massive star; this to help resolve the problem of what happened to a star so massive that it was too big to end its life as either a white dwarf or a neutron star. More recently, it has been suggested that there could be other stars composed of other forms of degenerate matter, not just the white dwarfs (degenerate electrons) and neutron stars (degenerate neutrons), but also possibly bodies composed largely of degenerate quarks or even degenerate sub-quarks [8]. If this turned out to be the case, the end points of stars could form a whole series of bodies composed of different forms of degenerate matter and it is conceivable that such a series could have black holes as a limiting case which might, or might not, be achievable. At present, the actual existence of black holes appears to be accepted by many and there is a determination abroad to identify one positively, but, so far, no candidate has satisfied the simple test posed by the result originally due to Michell, that is, that

$$M/R \geq 6.7 \times 10^{26} \text{ kg/m}$$

for a Schwartzschild black hole. For a rotating black hole, the figure in this inequality would be increased a little, depending on the speed of rotation.

One major area of worry where black holes are concerned is thermodynamics. Ever since Bekenstein noticed, in 1972 [9], that the non-decreasing nature of the surface area of a black hole was similar in mathematical form to the second law of thermodynamics in the form relating to the non-decreasing nature of the entropy, the thermodynamics of black holes has provided a major area of hidden controversy. Bekenstein went on to propose that this surface area of a black hole, multiplied by a suitable constant, be interpreted as the actual entropy of the black hole. It appears that, after this identification was agreed to by Hawking [10], it became part of scientific conventional wisdom. However, doubts about it must remain since, if valid, it leads to the possibility of violations of the second law of thermodynamics, as has been pointed out on numerous occasions [11].

The other very worrying aspect of black hole thermodynamics is the excessively cavalier usage of well-known thermodynamic results. It was pointed out several years ago that many of the commonly used results do not apply for black holes since their accepted entropy expression is not extensive. This means that there is no Euler relation linking the various thermodynamic variables or a Gibbs-Duhem relation. More importantly, though, the fact that the entropy expression is not concave means that the heat capacity is negative. Hence, such a system could not come to thermal equilibrium with its surroundings and could, in fact, conceivably violate the second law! There is a definite tendency to think that validity of the second law is sacrosanct and that any theory leading to its violation must be wrong. However, the second law is really only a 'fact of experience', having been postulated following a great many observations of a wide variety of physical systems. It is now well established as a 'law of physics' and is often regarded as a foundation stone for our physical theories. Nevertheless, it is still only a 'fact of experience' and, as such, need not be universally true. Hence, the present theory of the thermodynamics of black holes could herald limitations to the validity of the second law. If so, the possibility should be stated quite openly and the issue examined in detail. It seems highly unlikely that the second law does break down, but at least that possibility should be investigated.

The present situation concerning the thermodynamics of black holes is totally unsatisfactory and the position is well illustrated by an article entitled "Black Holes and Thermodynamics", due to Wald, in a volume dedicated to the memory of Subrahmanyan Chandrasekhar [12]. In this article, the first law of thermodynamics is claimed to be concerned with differences in entropy, among other things. (This notion is repeated in the following article in the same book.) However, entropy is a quantity which is only introduced into the subject via the second law. The equation concerned,

$$dE = TdS + \text{'work terms'},$$

is actually an equation representing the combination of the first and second laws, and, in it, the symbol E represents the internal energy of the system ***not*** its total energy. This latter, seemingly minor, point is one which has caused problems previously, notably in connection with use of the virial theorem but, somewhat ironically, has been clarified by Chandrasekhar [13]. It would seem also that, if the black hole entropy is proportional to surface area, the resulting 'TdS' would not be correct dimensionally to fit into the usual thermodynamic equations, unless the supposed 'T' expression compensates for it. If that is the case, the expression representing the temperature of a black hole would certainly not have the correct dimensions for a temperature. Hence, if the black hole *TdS* term is correct dimensionally, it is conceivable that a redefinition of the individual temperature and entropy terms could lead to the removal of some of the present confusion. After all, if a black hole is, in fact, a physical entity rather than a purely mathematical one, and if its physical surface area never decreases, then surely its physical volume will never decrease either? If this is the case, a re-examination of the complete expression for *TdS* for a black hole might overcome the present difficulties with its thermodynamics.

The above is concerned solely with the theory behind the idea of black holes. The situation as far as observers who search for actual black holes is not too different. The starting point always appears to be that black holes definitely exist. This is certainly the impression conveyed to a public which eventually foots the bill for research. Articles claiming to have found black holes appear regularly but most, if not all, fail to produce completely convincing evidence to support their claims. Frequently the observed object might well be a black hole but so far no-one has produced evidence for the existence of that defining factor for a black hole - an event horizon. Also, and this is the surprising point, no candidate paraded so far satisfies the simple criterion first derived by Michell in 1784; for none of the candidates produced does the ratio of mass to radius equal, or exceed, 6.7×10^{26}kg/m. What is in some ways even more surprising is that, when these claims for black hole discovery appear in the semi-popular scientific journals, those same journals refuse to print letters or notes pointing out that this criterion has not been met and, therefore, in no case has the said object been identified as a black hole beyond all reasonable doubt. It is almost frightening to note that, in many cases, the articles and even their titles indicate that the starting point for an investigation has been the seemingly unchallengeable assumption that a particular object *is* a black hole. One idea falling into this category is the assertion that there is a massive black hole at the centre of each galaxy, in particular at the centre of the Milky Way. It is undoubtedly true that, if the centre of our galaxy was the site of a massive black hole, it would explain certain observed phenomena. However, other feasible explanations for the observed phenomena do exist. For example, people have observed stars near the centre of our galaxy moving at extremely high speeds [14]. The explanation proffered depended on the presence of a massive, compact central object. It was claimed further that a mass of more than a million solar masses was confined to a region of radius less than one tenth of a light year. This was, in total, said to have been strong evidence for a massive black hole to be at the centre of our galaxy. This is not an unreasonable explanation, but it is by no means conclusive. However, the figures quoted do not lead to the Michell inequality being satisfied and, as has been shown previously, the speeds referred to may be attained in a model where the central region is occupied by a million stars of solar mass [15]. The picture painted by this example is not untypical of the situation surrounding most of the claims of 'positive' black hole identifications. The fact remains that no object has been observed which can be claimed to be a black hole beyond all reasonable doubt.

Interestingly, it might be noted that Einstein himself devoted a complete paper [16] to arriving at the conclusion that 'the "Schwarzschild singularities" do not exist in physical reality'. He pointed out that 'this investigation arose out of discussions …. on the mathematical and physical significance of the Schwarzschild singularity. The problem quite naturally leads to the question, answered by this paper in the negative, as to whether physical models are capable of exhibiting such a singularity.' In this, Einstein was surely drawing attention to the fact that the idea of a black hole as deduced from general relativity is only a mathematical notion;

whether or not it corresponds to physical reality may only be determined by observation and/or experiment. Hence, to request that the existence of such an exotic object be displayed *beyond all reasonable doubt* is surely simply a matter of common sense. However, a huge 'industry' has grown on the back of the black hole idea. Hundreds, if not thousands, of academic papers have appeared, devoted to various aspects of the topic. In many cases, physical reality appears to have been forgotten but 'conventional wisdom' not. At the same time, science fiction encroaches more and more on scientific reality and inadvertently increases the pressure for the retention of the 'conventional wisdom' surrounding these truly exotic, if mythical (?), objects.

Boson and other exotic stars

In recent years, as well as white dwarfs and neutron stars, other exotic objects have been suggested as providing possible end-points to the life cycle of the larger stars. Quark stars and even sub-quark stars have been discussed quite widely [17]. Moreover, images of suspected quark stars have begun to appear [18]. As for sub-quark stars, there is little solid observational evidence to support the existence of sub-quark particles. However, there have been measurements of the inclusive differential cross section for jet production in proton - antiproton collisions [19] in which experimental results deviate from predictions. Here the presence of a sub-quark structure could offer an explanation for the higher than expected cross section. As is often the case, other feasible explanations exist and further experimentation is necessary. However, would quarks split into individual components under immense pressure? One other suggestion put forward [20] is that high density quark matter might respond to external stress by forming a Bose condensate of kaons. This raises another as yet unanswered major question, though, of how fermions may be converted into bosons.

Actually, interest in the ideal Bose gas revived in the late seventies, possibly stemming in part from a discussion of the cosmological implications of a massive primordial photon gas by Kuzmin and Shaposhnikov [21]. Subsequently, a review of ideal relativistic Bose condensation was presented by Landsberg [22]. Recently, it has been suggested that kaon condensation could have a role to play in the understanding of neutron stars [23], although the appearance of a Bose-Einstein condensate of charged pions in dense nuclear matter has been under discussion since the early seventies. In the simple case of an ideal relativistic Bose gas, values of admissible temperatures and corresponding number densities may be found by referring to an article dating back to 1965 [24]. From this latter article, it is seen that, in the two limiting cases of a non-relativistic Bose gas and an extreme relativistic Bose gas, the relevant condensation temperatures are given by

$$kT_c = \frac{h^2}{2\pi m}\left[\frac{N}{V\,\varsigma\left(3/2\right)}\right]^{2/3}, \text{ (non-relativistic)}$$

and

$$kT_c = \left[\frac{h^3 c^3 N}{8\pi V\,\varsigma\left(3\right)}\right]^{1/3}, \text{ (extreme relativistic)}$$

respectively.

As is shown in detail in reference [24], if condensation temperature is plotted against mass for each of the above limiting cases, the two resulting straight lines intersect on another straight line whose equation is given by eliminating N/V between the above two equations. The result of this elimination is the equation

$$T_c = \frac{mc^2}{2k}\left\{\pi\left[\frac{\varsigma(3/2)}{\varsigma(3)}\right]\right\}^{1/3} = 8\times10^{36}\,\mathrm{m}\ \ (^{\mathrm{o}}\mathrm{K})$$

Above this line, relativistic effects become important. It might be noted also that the higher the concentration, the higher the rest mass at which these effects begin to appear. For the kaons mentioned in reference [23], the mass of each individual particle is approximately 8.9×10^{-25}gms. and this latter equation gives a value for the relevant condensation temperature of 7.13×10^{12} $^{\mathrm{o}}$K. It follows from either of the earlier two equations that, for these kaons, relativistic effects will come into play for values of the concentration above approximately 3×10^{39}. However, for a number density of 10^{30}, for example, no relativistic considerations would come into play and it is seen, using the equation above which relates to the non-relativistic limit, that the condensation temperature of such kaons would be approximately 3×10^{6} $^{\mathrm{o}}$K. At first sight at least, these figures, obtained for a non-relativistic scenario, do not seem unreasonable. However, these considerations involve the very simple case of an ideal gas. Nevertheless, recourse to more complicated models is not always necessary to gain a basic understanding of what is happening; so often order-of-magnitude calculations can lead to a real understanding of processes under consideration with no need for truly detailed calculations. That might well be the case here. All of the above, apart from the application to kaons, is well documented in [24], from which further details may be extracted.

Bose condensation is normally discussed in three spatial dimensions. This seems eminently reasonable when it is remembered that the most widespread application of the theory has been to the problem of liquid helium, which is obviously a substance existing in three spatial dimensions. However, the properties of the ideal Bose gas, and indeed the ideal quantum gases in general, have been examined in d-dimensions previously. It appears worthwhile drawing attention to this fact since the topic is one of current interest [25].

In the review mentioned above [22], the normal theory is generalised to arbitrary discrete spectra before specialising to a specific density of states function. A discussion of the ideal relativistic Bose gas in d-dimensions follows and it is shown that the condensation phenomenon becomes more pronounced both as the extreme relativistic limit is approached and when higher dimensions are considered. For particles of small rest mass m_o, no condensation is found for $d = 2$ unless $m_o = 0$. Also, no discontinuity is found, at the condensation temperature, in the constant volume heat capacity when $d = 3$ or 4, again unless $m_o = 0$. Again, in this review the idea of examining the properties of the ideal relativistic quantum gases by using an approximate density of states was first proposed. This idea was followed through subsequently [26], where it was shown that, due to the form of the proposed approximate density of states

$$D(\eta,u,T)\,d\eta = A_d\left[\eta^{d-1} + \tfrac{1}{2}(2u)^{d/2}\eta^{d/2-1}\right]d\eta,$$

where

$$A_d = \frac{2\pi^{d/2}\omega V_d}{\Gamma(d/2)}\left(\frac{kT}{hc}\right)^d,$$

V_d being the d-dimensional volume, $\eta kT = e$, $ukT = e_o = mc^2$, the rest energy, and ω the degeneracy factor, the expression for any thermodynamic function evaluated in either the extreme relativistic or non-relativistic limits will agree with the results derived using the exact density of states. It was shown there that:

(i) for $d = 1$, there is no condensation;

(ii) for $d = 2$, there is condensation only for $m_o = 0$, but, in that case, there is no discontinuity in the constant volume heat capacity;

(iii) for $d = 3$ or 4, there is condensation for $m_o = 0$ with a discontinuity in the heat capacity, and for $m_o > 0$ but with no heat capacity discontinuity;

(iv) for $d \geq 5$, there is condensation for $m_o \geq 0$ with a discontinuity in the heat capacity.

These conclusions, using the approximate density of states, are in complete agreement with those derived previously using the exact expression [22]. However, it should be noted that, in general, while results obtained via use of this approximate density of states are in excellent agreement with the exact results in the physically realistic case of three dimensions, as the number of dimensions increases, the agreement becomes progressively worse. Also, it was shown that, when applied to the case of a Fermi gas, results obtained using this approximate density of states differ markedly from those obtained by use of the exact expression, except in the two limiting cases as might reasonably be expected.

Skyrmion stars

Other models for stars have been proposed over the years. One such model relies on an equation of state for dense matter [27] which is based on Skyrme's concept of strong interactions which represents baryons as solitons of classic pion fields. In 1961, Skyrme [28] constructed a model of pion interactions consisting of a conventional model of weak meson interactions and found that, in this model, the meson fields contained points in space where there was a 'topological knot'. These structures were interpreted as being solitons of finite extent but whose number is conserved always and were identified by

Skyrme as baryons. However, the star model resulting, named a skyrmion star [29], turned out to be intrinsically heavier than other models of compact stars using other equations of state. These skyrmion stars could be as heavy as approximately 2.8 solar masses and it is this that has led to the speculation that such stars might exist in nature.

However, the entire discussion of skyrmion stars seems totally different from that of other forms of star, particularly when considering those structures which are felt to develop at the end-point of the life of a star. Firstly, the concept seems to have been derived from purely mathematical considerations and, that being the case, the direct relevance of the proposal to an actual physical situation has to be established. Also, there is no suggestion here of such a star coming about as a result of contraction of an original star leading to the establishment of a stable configuration consisting of degenerate matter of some form; e.g. degenerate electrons in a white dwarf; degenerate neutrons in a neutron star. As Ouyed points out [29], the skyrmion stars considered are 'not boson/soliton stars where the soliton is a global structure over the scale of the star, but rather form their constituent baryons as topological solitons using pion fields'. Again as pointed out by Ouyed, this is fundamentally different from the situation affecting other so-called 'exotic' stars, which also follow on from solutions to a non-linear theory of strong forces. Hence, it seems that the idea of skyrmion stars is one to be considered, but with caution.

From the point of view of models considered here, however, as far as figures proposed so far are concerned, the skyrmion stars, like all other examples, seem to have masses and corresponding radii such that the ratio of mass to radius is less than what might be termed the Michell limit and so, they do not satisfy that important criterion for a black hole. Of course, the relation here considered is based on the star's actual radius, not the radius of the event horizon. While that is acceptable for the traditional Michell limit, when general relativistic effects are taken into account, it is the radius of the event horizon which would come into play. Once again, it may be noted that, while no star has been found with a mass and radius satisfying the Michell limit, there has been no claim so far of anyone identifying the actual event horizon of any object.

Conclusion

It seems that black holes are often regarded as being the ultimate state for stellar matter. Physically, such objects, if they exist in nature, would be truly exotic; in science fiction such objects are a constant source of wonder. Unfortunately, the two areas of interest often seem to overlap. Here a few models of astrophysical bodies relating to the end stages in the life of a star are considered. It is seen once again that, although many of these objects may have masses and corresponding radii which lead to ratios approaching the orthodox Michell limit, none has been found which actually achieves that limit. It might also be re-emphasised that, in all 'positive' claims of identification of black holes at the centre of our galaxy, the Milky Way, or at the centres of other galaxies, none has so far satisfied this simple criterion. The semi-popular science journals continue to highlight claims of black hole identification at the centres of galaxies and omit suggestions of caution about such claims. Again, 'conventional wisdom' seems to hold sway. It is to be hoped that information such as that presented above will aid a return to caution regarding claims about black holes as well as about other topics in various areas of science - not just astrophysics - and that a return to truly open-minded scientific investigation, unhampered by notions of so-called 'conventional wisdom', will follow quickly.

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