Further Difficulties with the Klein-Gordon Equation

E. Comay
School of Physics and Astronomy
Raymond and Beverly Sackler Faculty of Exact Sciences
Tel Aviv University
Tel Aviv 69978, Israel
Email: elic@tauphy.tau.ac.il

Herein, the Dirac equation is compared with the Klein-Gordon equation. In contrast to the Dirac case, it is proved that the Klein-Gordon equation has difficulties with the Hamiltonian differential operator of relativistic quantum mechanics and with the definition of an inner product of wave functions, which is a requirement for a construction of a Hilbert space. An added discussion of the Pauli-Weisskopf article and that of Feshbach-Villars proves that their theories lack a self-consistent expression for the Hamiltonian. Related difficulties are pointed out.

Keywords: Relativistic Quantum Mechanics, Dirac Equation, Klein-Gordon Equation, Hamiltonian

1. Introduction

The Klein-Gordon (KG) equation and the Dirac equation were published in the very early days of quantum mechanics (see [1], bottom of pp. 25, 34). The KG equation is regarded as the relativistic quantum mechanical equation of a spin-0 massive particle and the Dirac equation describes a spin-1/2 massive particle. The Dirac equation of an electrically charged particle can be found in any textbook on relativistic quantum mechanics or quantum electrodynamics. This equation is regarded as a correct description of a system belonging to the domain of validity [2] of relativistic quantum mechanics. Thus, for example, the Dirac equation can be used for the hydrogen atom if one is ready to ignore small effects like the Lamb shift.

Unlike the Dirac equation, the KG equation is not free of objections. Problems concerning a definition of a positive definite density were recognized very soon (see [3], pp. 7,8; [4], pp. 27-29). Note also that Dirac maintained his negative opinion on this equation throughout his life [5]. On the other hand, claims stating that Dirac's opinion of the KG equation is wrong were published (see [1], second column of p. 24).

New difficulties with the KG equation were published recently [6]. Thus, new arguments proving that the KG wave function cannot describe probability are given; it is proved that a KG particle cannot interact with electromagnetic fields; the classical limit of the Yukawa interaction is inconsistent with special relativity and some other claims.

The 4-current of a particle represents specific properties of its state, namely its density and its 3-current. The KG electromagnetic interaction discussed in [6] relies on the (self-evident) requirement stating that the 4-current of a KG particle (like that of any other particle) should not depend on field variables of *external particles*. In the discussion carried out below, this requirement is removed and the ensuing consequences are analyzed. This work examines the structure of the Hamiltonian operator of the system in relativistic quantum mechanics and the orthonormal basis for the Hilbert space of solutions of quantum mechanical equations. The significance of the corresponding Lagrangian density is pointed out.

Units where $\hbar = c = 1$ are used. The Lorentz metric $g_{\mu\nu}$ is diagonal and its entries are (1,-1,-1,-1). Greek indices run from 0 to 3 and Latin ones run from 1 to 3 (unless stated otherwise). The summation convention holds for a pair of upper and lower indices. The lower case symbol $_{,\mu}$ denotes the partial differentiation with respect to x^μ. In particular, $\phi_{,0} \equiv \partial\phi/\partial t$.

The second Section contains an analysis of the Dirac equation. The Pauli-Weisskopf (PW) and the Feshbach-Villars (FV) theories of the KG equation are discussed in the third and the fourth Sections, respectively. The last Section contains a discussion of the findings.

2. The Dirac Equation

Let us examine the theoretical structure of a Dirac field interacting with an electromagnetic field. This subject is useful not only for its own sake but also as an example which may be compared with the corresponding analysis of the KG equation. The matter part of the Lagrangian density is (see [7], p. 84)

$$\mathcal{L} = \bar{\psi}[\gamma^\mu(i\partial_\mu - eA_\mu) - m]\psi, \tag{1}$$

where γ^μ denotes a set of four Dirac γ matrices, ψ is the Dirac wave function, $\bar{\psi} = \psi^\dagger \gamma^0$ and $\psi^\dagger$ is the Hermitian conjugate of ψ. The definition $\gamma^0 = \beta, \quad \gamma^i = \beta\alpha^i$ relates the Dirac γ matrices and the α^i, β matrices. The components of the 4-potential are the electric potential V and the vector potential $\mathbf{A}$. Thus, $A^\mu = (V, \mathbf{A})$.

A variation of (1) with respect to $\bar{\psi}$ yields the Dirac equation (see [7], p. 84)

$$\gamma^\mu (i\partial_\mu - eA_\mu)\psi = m\psi. \tag{2}$$

An important quantity is the 4-current of the Dirac particle

$$j^\mu = \bar{\psi}\gamma^\mu\psi, \tag{3}$$

which satisfies the conservation law

$$j^\mu_{,\mu} = 0. \tag{4}$$

The validity of this relation is independent of the external electromagnetic field (see [8], p. 119). The 0-component of the 4-current (3) represents the density of the Dirac particle

$$\rho = \bar{\psi}\gamma^0\psi = \psi^\dagger\psi. \tag{5}$$

The matter part of the Hamiltonian density is derived from the Lagrangian density (1) by the well known relation (see [7], p. 87)

$$\begin{aligned} \mathcal{H} &= \sum \psi_{,0} \frac{\partial \mathcal{L}}{\partial \psi_{,0}} - \mathcal{L} \\ &= \psi^\dagger [\boldsymbol{\alpha} \cdot (-i\nabla - e\mathbf{A}) + \beta m + eV]\psi, \end{aligned} \tag{6}$$

where the summation runs on $\psi_{,0}$ and $\bar{\psi}_{,0}$. (As a matter of fact, only $\psi_{,0}$ is found in (1)). Here quantities should be written in terms of coordinates and conjugate momenta. However, this point is not essential for the discussion below. Hence, it is omitted throughout this work.

Using the expression for the density (5), one readily extracts from the Hamiltonian density (6) an expression for the Hamiltonian differential operator used in the Schroedinger picture of relativistic quantum mechanics

$$H = \boldsymbol{\alpha} \cdot (-i\nabla - e\mathbf{A}) + \beta m + eV \tag{7}$$

(Here the term "Hamiltonian differential operator" denotes a Hamiltonian like (7) which contains differential operators and other terms, and is distinguished from the matrix form of the Hamiltonian.)

It is well known that the Hamiltonian operator H plays a cardinal role in the Schroedinger picture of quantum mechanics, because it defines the time evolution and the energy states of the system (see [7], p. 6)

$$H\psi = i\frac{\partial\psi}{\partial t}. \tag{8}$$

Now, due to the principle of superposition, quantum mechanics uses equations that are linear in ψ. For this reason, the Hamiltonian operator H of (8) should not depend on ψ. This requirement is satisfied by the Dirac Hamiltonian (7).

By substituting the Hamiltonian operator (7) into the quantum mechanical relation (8), one obtains the Hamiltonian form of the Dirac equation

$$[\boldsymbol{\alpha} \cdot (-i\nabla - e\mathbf{A}) + \beta m + eV]\psi = i\frac{\partial\psi}{\partial t}. \tag{9}$$

Multiplying the Euler-Lagrange equation (2) by $\gamma^0 = \beta$ and putting the time derivative on the right-hand side, one realizes that (2) is equivalent to (9). The complete agreement between (9) and the Dirac equation (2), derived as the Euler-Lagrange equation of the Lagrangian density (1), indicates the self-consistence of the theory.

It is interesting to note the relativistic properties of the Hamiltonian density (6) and of the Hamiltonian operator (7). Examining the first line of (6) and remembering that the Lagrangian density $\mathcal{L}$ is a Lorentz scalar, one realizes that (6) is a tensorial component T^{00} of the second rank tensor

$$T^{\mu\nu} = \sum \psi^{,\mu} \frac{\partial \mathcal{L}}{\partial \psi_{,\nu}} - \mathcal{L} g^{\mu\nu}. \tag{10}$$

This is the required covariance property of energy density. In classical physics, energy density is the T^{00} component of the energy-momentum tensor $T^{\mu\nu}$ (see [9], p. 77). Now, since the probability density ρ of (5) is a 0-component of a 4-vector, one concludes that also the Hamiltonian operator H of (7) is a 0-component of a 4-vector. Evidently, this property is essential for satisfying covariance of the fundamental quantum mechanical relation (8). This discussion shows just one reason for the usefulness of constructing the theory on the basis of a Lagrangian density. This point is used below in the analysis of the FV Hamiltonian.

It can be concluded that the following properties hold for the Dirac theory:

1. The conserved 4-current depends on ψ and on the corresponding $\bar{\psi}$, and is independent of the external field A_μ.

Hence, one can use the positive definite density $\psi^\dagger\psi$ and construct an orthonormal basis for the Hilbert space of solutions. This basis is not affected by changes of external quantities.

2. Since the Dirac Lagrangian density (1) is *linear* in the time-derivative $\partial\psi/\partial t$, the corresponding Hamiltonian density (6) does not contain derivatives of ψ with respect to time. The same is true for the Hamiltonian differential operator. Hence, in the case of a Dirac particle, the fundamental quantum mechanical relation (8) takes the standard form of an explicit first-order partial differential equation. Here a derivative with respect to time is equated to an expression which is free of time derivatives. This property does not hold for Hamiltonians that depend on time derivative operators.

3. The differential operator representing the Dirac Hamiltonian (7) is easily extracted from the Hamiltonian density (6) and is free of ψ, $\bar{\psi}$ and their derivatives. An examination of (8) proves that this property is consistent with the linearity of quantum mechanics and with the superposition principle as well.

4. The equation (9) obtained from the substitution of the Dirac Hamiltonian operator (7) into the quantum mechanical relation (8), agrees completely with the Dirac equation (2) obtained as the Euler-Lagrange equation of the Lagrangian density (1). This property means that the Euler-Lagrange equation (2) does not impose additional

restrictions on the Hamiltonian's eigenfunctions and on their corresponding eigenvalues.

5. The term eA^μ correctly represents electromagnetic interactions.

These points indicate the self-consistency of the Dirac theory. It is proved below that difficulties arise if one carries out an analogous analysis of the KG equation.

3. The Pauli-Weisskopf Theory of the KG Equation

Let us turn to the PW theory of a charged KG particle (see Section 3 of [10]). These authors use the following Lagrangian density (see eq. (37) therein)

$$\mathcal{L} = (\phi^*_{,0} - ieV\phi^*)(\phi_{,0} + ieV\phi) - \sum_{k=1}^{3}(\phi^*_{,k} + ieA_k\phi^*)(\phi_{,k} - ieA_k\phi) - m^2\phi^*\phi. \quad (11)$$

Note that here and later on, minor changes are made in the form of quoted equations. Thus, units where $\hbar = c = 1$ are introduced; the symbol ϕ denotes the KG wave function and the electromagnetic 4-potential is $A^\mu = (V, \mathbf{A})$. On the other hand, the Lorentz metric of quoted equations is that of the original articles.

The Hamiltonian density associated with (11) is found next to this equation (see eq. (37a) therein)

$$\mathcal{H} = (\phi^*_{,0} - ieV\phi^*)(\phi_{,0} + ieV\phi) + \sum_{k=1}^{3}(\phi^*_{,k} + ieA_k\phi^*)(\phi_{,k} - ieA_k\phi) + m^2\phi^*\phi. \quad (12)$$

The Lagrangian density (11) is used in a derivation of the second-order KG equation of motion of a charged KG particle (see eq. (39) therein)

$$(\frac{\partial}{\partial t} - ieV)(\frac{\partial}{\partial t} - ieV)\phi = \sum_{k=1}^{3}(\frac{\partial}{\partial x^k} + ieA_k)(\frac{\partial}{\partial x^k} + ieA_k)\phi + m^2\phi. \tag{13}$$

The conserved 4-current of the KG particle is derived too. The 0-component of this quantity, namely the density, is (see eq. (42) therein)

$$\rho = i(\phi^*\phi_{,0} - \phi^*_{,0}\phi) - 2eV\phi^*\phi. \tag{14}$$

and the corresponding current is

$$\boldsymbol{j} = i((\nabla\phi^*)\phi - \phi^*\nabla\phi) - 2e\boldsymbol{A}\phi^*\phi. \tag{15}$$

The density and current satisfy the continuity equation

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\boldsymbol{j} = 0. \tag{16}$$

Unlike the case of a Dirac particle, here the 4-current of a KG particle depends on derivatives of ϕ *and* on external electromagnetic quantities. Moreover, the density (14) is not positive definite and is interpreted as charge density (see [1], pp. 25, 26). Difficulties associated with these strange and counterintuitive properties of the KG particle are discussed later in this Section.

Before proceeding with the analysis, let us write down the canonical Hamiltonian obtained from the application of the first line of (6) to the Lagrangian density (11)

$$\mathcal{H} = \phi^*_{,0}\phi_{,0} - e^2V^2\phi^*\phi + \sum_{k=1}^{3}(\phi^*_{,k} + ieA_k\phi^*)(\phi_{,k} - ieA_k\phi) + m^2\phi^*\phi. \tag{17}$$

As mentioned (see [11], p. 68) this expression is not gauge invariant.

The Hamiltonian density (12) is obtained from (17) in a way analogous to that which casts the canonical energy-momentum tensor of electromagnetic fields into a symmetric form (see [9], pp. 77-83). Here the divergenceless tensor is

$$U^{\mu\nu} = eA^\mu j^\nu, \tag{18}$$

where j^ν is the 4-dimensional notation of (14) and (15).

Let us now examine the issue of the Hamiltonian differential operator required for the Schroedinger picture of the fundamental quantum mechanical relation (8). (It is shown above how easily this task is accomplished for the Dirac Hamiltonian. In this case one just removes the Dirac density factor $\psi^\dagger\psi$ from the Hamiltonian density (6) and extracts the required expression (7)). This quantity is not given in [10].

The following argument proves that this task can be accomplished neither for the Hamiltonian density (12) nor for that of (17). For this purpose it is enough to examine the terms containing highest order time derivatives of the Hamiltonian densities and of the charge density, namely $\phi^*_{,0}\phi_{,0}$ and $i(\phi^*\phi_{,0} - \phi^*_{,0}\phi)$, respectively. Let $\hat{H}$ denote the required operator. Evidently, due to the superposition principle and the linearity of quantum mechanics, $\hat{H}$ should neither depend on ϕ, ϕ^* nor on their derivatives. Hence, under these restrictions on the structure of $\hat{H}$, it is clear that $\hat{H}$ cannot exist because $\phi^*_{,0}\phi_{,0}$ is symmetric with respect to ϕ and ϕ^*, whereas $(\phi^*\phi_{,0} - \phi^*_{,0}\phi)$ is antisymmetric with respect to these functions. This proof does not rely on terms containing the electric charge e. Hence, it applies also to the case of an uncharged KG particle described by a complex field.

At this point one realizes that the Hamiltonian density (12) cannot be used for a derivation of the Hamiltonian operator H of (8), which is a partial differential equation. Hence, it may be asked whether one can, at least, construct a Hamiltonian matrix by taking the spatial integral of the Hamiltonian density (12) (see [10], eq. (37.a))

$$\begin{aligned} H_{ij} \;=\; & \int [((\phi_i^*)_{,0} - ieV\phi_i^*)((\phi_j)_{,0} + ieV\phi_j) + \\ & \sum_{k=1}^{3}((\phi_i^*)_{,k} + ieA_k\phi_i^*)((\phi_j)_{,k} - ieA_k\phi_j) + \\ & m^2\phi_i^*\phi_j]d^3x. \end{aligned} \tag{19}$$

Here the indices $_{ij}$ denote elements of a basis of an assumed Hilbert space of solutions. The following lines prove that this expression is not free of serious discrepancies.

As used in quantum mechanics, the Hamiltonian (19) has a meaning if it is obtained by an application of an orthonormal basis of functions spanning the Hilbert space of solutions. This requirement is essential because the KG equation (13) is homogeneous. Hence, if ϕ solves (13) then let λ be a constant number and $\phi' = \lambda\phi$ solves it too. Thus, multiplying ϕ_i and ϕ_j by λ one obtains $H'_{ij} = \lambda^2 H_{ij}$. Now, the Hamiltonian's eigenvalues should be well defined because they represent energy of states. It follows that one must find a way for restricting the freedom of usage of the multiplicative factor λ. This goal is attained by an application of an orthonormal basis for the Hilbert space of solutions.

Now, as shown for a KG particle [6], the dimension of the product $\phi^*\phi$ is $[L^{-2}]$, whereas a normalizable function should have the dimension $[L^{-3}]$ for this product. Furthermore, the conserved density (14) *is not positive definite* and is interpreted as a charge density which can take either positive or negative values (see [1], pp. 25, 26). Hence, it is not clear how can one construct an orthonormal basis for the Hilbert space of wave functions for the Hamiltonian matrix (19). Indeed, a Hilbert space requires a definition of an inner product where $(\phi, \phi) \geq 0$ (see [12], pp. 73-91).

The goal of constructing an orthonormal basis for the Hilbert space of wave functions of the KG Hamiltonian faces other difficulties too. These difficulties follow the introduction of a charged KG particle. This issue is illustrated by the following experiment. Let an electron impinge on a charged KG particle (like in a Rutherford scattering). Assume that, in spite of the problem mentioned above, one finds a way for constructing an orthonormal basis for the Hilbert space of solutions of the KG equation at time $t \to -\infty$, by using the (charge) density (14). Now, as the electron approaches the KG particle, its electric potential V varies and so does the KG density (14). These changes affect the inner product based on (14). Hence, for two given wave functions, ϕ_i, ϕ_j, this inner product has no unique value, *due to its dependence on the external, time dependent electric potential V.* Therefore, (14) cannot be used for the definition of the inner product required for the Hilbert space. This conclusion means that *there is no Hilbert space for the Hamiltonian* (19). This result is just one example showing that the counter-intuitive formula for the charge density of a KG particle (14) entails difficulties, because the density of a specific KG particle depends on electromagnetic quantities associated with *external* particles.

The problems with the construction of an orthonormal basis for the KG Hamiltonian also affect the calculation of expectation values of other operators. (Note, for example, that $-i\int \phi^*\nabla\phi\, d^3x$ does not yield the momentum expectation value, because the dimension of $\phi^*\phi$ is $[L^{-2}]$.) Thus, in relativistic quantum mechanics of a KG particle, one still has no way for finding expectation values of physical quantities.

The Heisenberg picture faces the same problems. Indeed, in this picture one uses the basis for the Hilbert space of the Schroedinger picture at a certain time, which is defined $t = 0$ (see [7], p. 7). Hence, the Heisenberg picture faces the same problems. Moreover, since the Fock space is related to the Hilbert space of solutions of the Hamiltonian (see [13], pp. 40, 41), one concludes that problems also exist with the quantum field theory of KG particles.

4. The Feshbach-Villars Hamiltonian

Let us next turn to the theory described in the FV article [1]. These authors construct an expression for the Hamiltonian differential operator of a charged KG particle that, in the Schroedinger picture, takes the standard quantum mechanical form (8). For this purpose they use a 2-component wave function

$$\Psi = \begin{pmatrix} \psi \\ \chi \end{pmatrix}, \tag{20}$$

where ψ and χ are linear combinations of the KG wave function ϕ and of its time derivative $\phi_{,0}$ (see eqn. (2.11)-(2.17) therein). These authors present the following Hamiltonian differential operator that can be used in the Schroedinger picture of (8) (see (2.18) therein)

$$H = (\tau_3 + i\tau_2)(1/2m)(\mathbf{p} - e\mathbf{A})^2 + m\tau_3 + eV, \tag{21}$$

where τ_2 and τ_3 are Pauli matrices.

The analysis of FV does not rely on a Lagrangian density. Hence, as explained in Section 2, it is not clear whether or not the Hamiltonian (21) satisfies relativistic covariance. As a matter of fact, a proof of this essential property is not found in [1]. The following analysis explains why it is impossible to construct such a proof.

Let us analyze covariance properties of (21). As stated above, the fundamental quantum mechanical relation (8) indicates that (21) should be a 0-component of a 4-vector. The last term of (21) is eV where e is a Lorentz scalar denoting the charge of the KG particle and V is the 0-component of the electromagnetic 4-potential A_μ. Hence, the last term of (21) is a 0-component of a 4-vector, as required. The second term of (21) is $m\tau_3$. Here m is a scalar denoting the KG particle's self mass. Now, the τ_3 Pauli matrix certainly cannot transform like a 0-component of a 4-vector. Therefore, the second term and the last term of (21) have different covariant properties.

Furthermore, the first term of (21) is also inconsistent with the last one, because it is not a 0-component of a 4-vector. Indeed, the following expression shows the tensorial form of this term

$$(\mathbf{p} - e\mathbf{A})^2 = (E - eV)^2 - (P^\mu - eA^\mu)(P_\mu - eA_\mu)g^{00}. \quad (22)$$

Here the first term on the right-hand side is a product of two energy quantities. Hence, under a Lorentz transformation it behaves like a tensorial component W^{00}. This property holds also for the last term of (22). Here, in principle, one may alter the tensorial rank of each term by using the relativistic metric $g_{\mu\nu}$, the Kronecker delta $\delta^\mu_\nu = g^{\mu\alpha}g_{\alpha\nu}$ and the completely antisymmetric unit tensor of the fourth rank $\varepsilon^{\alpha\beta\gamma\delta}$. Evidently, the rank of each of these tensors is an even number. Thus, one cannot put the first term of (21), which is a component of an even rank tensor W^{00} and the last one, which belongs to an odd rank tensor, A^μ, in the same equation without violating covariance. Obviously, the factor $(\tau_3 + i\tau_2)$ and the Lorentz scalar $1/2m$ cannot settle this contradiction. This discussion proves that (21) violates relativistic covariance and therefore it takes an unacceptable form of the Hamiltonian. Since (21) contains charge independent terms that do not transform as a 0-component of a 4-vector, one concludes that this Hamiltonian is unacceptable for an uncharged KG particle too.

5. Concluding Remarks

The present work is a continuation of an earlier one [6] which discusses other difficulties of the KG equation [14]. The results of this work are described in the following lines. First, the theory derived from the Lagrangian density of a charged Dirac particle is discussed. It is shown that the Hamiltonian density and the

Hamiltonian differential operator of the Schroedinger picture are derived in a straightforward manner and the results are self-consistent. In particular, the Euler-Lagrange equation derived from the Dirac Lagrangian density agrees with the fundamental quantum mechanical equation $i\partial\psi/\partial t = H\psi$. Moreover, a self-consistent orthonormal basis for the Hilbert space of solutions can be constructed and used.

It is shown that an analogous structure does not exist for the KG equation. An expression for the Hamiltonian differential operator of the Schroedinger picture is not given in [10] and it is proved above that this quantity cannot be extracted from the KG Hamiltonian density (12). Note also that an attempt to construct a Hamiltonian differential operator for a KG particle without relying on a Lagrangian density [1] fails too. In this case it is proved that the suggested Hamiltonian (21) violates relativistic covariance and should be rejected. The noncovariant properties of this Hamiltonian can also be inferred from the fact that in a textbook it is relegated to a Section discussing the nonrelativistic limit (see [15], pp. 198- 207).

Since no acceptable Hamiltonian differential operator exists for a charged KG particle, one obviously cannot close the logical cycle and prove that the Hamiltonian equation of motion $i\partial\phi/\partial t = H\phi$ is consistent with the Euler-Lagrange equation obtained from the KG Lagrangian density (11). This is certainly not an easy task, because the KG equation has a second-order time derivative whereas the KG Hamiltonian density and the fundamental quantum mechanical equation $i\partial\phi/\partial t = H\phi$ contain only first-order time derivatives.

It is also proved in Section 3 that if one is ready to accept (14) as an expression for (charge) density, then the inner product of the Hilbert space is destroyed. Hence, (14) is inconsistent with a fundamental basis of quantum mechanics (see [12], p. 86). This outcome confirms an earlier conclusion [6] stating that a KG particle cannot carry an electric charge.

The following discussion compares the structure of the Lagrangian density of the Dirac equation with that of the KG equation and provides a possible explanation of the origin of the difficulties of the latter. The unit system where $\hbar = c = 1$ facilitates this task. Here dimension of every physical quantity is written in terms of one unit, which is taken here to be that of length $[L]$. Thus, energy, momentum and mass have the dimension $[L^{-1}]$.

The action S is dimensionless. Thus, the relation

$$dS = (\int \mathcal{L} d^3x)dt \tag{23}$$

proves that the dimension of the Lagrangian density is $[L^{-4}]$. The first line of (6) proves that this is also the dimension of the Hamiltonian density $\mathcal{H}$. Now, in the case of the Dirac equation, operators take the first power of energy, momentum and mass (henceforth called energy-like operators). Therefore, the dimension of the product $\bar{\psi}\psi$ is $[L^{-3}]$. Thus, in the case of the Dirac equation, terms representing energy-like operators play a general role and take the same form for all states of the Dirac particle. On the other hand ψ and $\bar{\psi}$ represent *specific* information concerning the particle's state. This is the underlying reason for the straightforward extraction of the Dirac Hamiltonian operator (7).

The structure of the KG Lagrangian density (11) (and that of the associated Hamiltonian densities (12) and (17)) differs from that of the Dirac case. Here energy-like operators take the second power. Hence, the dimension of the product $\phi^*\phi$ is $[L^{-2}]$. Now, since the dimension of density is $[L^{-3}]$, one finds that in the case of a complex field of an uncharged particle, the expression for the density is $i(\phi^*\phi_{,0} - \phi^*_{,0}\phi)$. A complication arises in the case of a charged particle (14), where the density of the KG particle also depends on electromagnetic quantities.

Now, in the KG equation of motion (13), energy-like operators yield energy-like terms. On the other hand, the expression for the density (14), *which is not related to energy*, contains energy-like operators. Thus, energy-momentum operators $(i\partial/\partial t, -i\nabla)$ play two different roles in the structure of the KG theory: in the KG equation of motion they represent energy-momentum quantities whereas in (14) one energy operator changes its role and is used for density.

The situation becomes even more unexpected where an electric charge and electromagnetic fields are a part of the system. Here the substitution $P^\mu \rightarrow P^\mu - eA^\mu$ (see [7], p. 84 and [10], eq. (36)) is performed. Thus, eA^μ, which is a companion of energy-momentum, is carried together with the latter and plays a part in the description of the density of a KG particle.

This discussion explains how the KG theory uses energy-momentum operators for two distinct roles: as energy-momentum operators representing energy balance in the KG equation, and as a part of the expression representing a specific property of the solution, namely its 4-current in general and its density in particular. This ambiguity is probably the underlying reason for the inability to extract the KG Hamiltonian operator from the Hamiltonian density (12). Another problem of the KG particle is shown in Section 3. Thus, if one uses the density (14), then the construction of a self-consistent orthonormal basis for the Hilbert space of solutions cannot be accomplished.

The following difficulties of the KG equation are discussed above. Some of these points are related by contrast to the satisfactory properties of the Dirac equation listed at the end of Section 2.

1. The KG theory lacks a positive definite expression for density. The assumed charge density (14) depends on external quantities (the electric potential V). This expression cannot be used for constructing a self-consistent Hilbert space of solutions.

2. The Hamiltonian density depends on time-derivatives of ϕ and ϕ^*. In other words, assuming that one is able to construct a Hamiltonian for the KG particle then the Hamiltonian density depends on the Hamiltonian. This relation cannot be regarded as a desirable one.

3. One cannot use the Hamiltonian density for a construction of the Hamiltonian differential operator. In Section 4 it is proved that another attempt to construct a Hamiltonian differential operator fails. Thus, the theory has no covariant expression for the Hamiltonian differential operator.

4. Assuming that this Hamiltonian is constructed, it is not clear in this case that the *second-order* KG equation is equivalent to the *first-order* fundamental quantum mechanical equation $i\partial\phi/\partial t = H\phi$.

5. One cannot construct electromagnetic interactions of a charged KG particle by using a linear factor eA^μ [6]. An

application of the quadratic term $(p^\mu - eA^\mu)(p_\mu - eA_\mu)$ destroys the structure of the Hilbert space of solutions.

6. No justification is given for the different meaning of the standard energy-momentum operators, namely $(i\partial/\partial t, -i\nabla)$: as energy-momentum operators in the KG equation and as an element in the description of the 4-current.

It should be pointed out that the foregoing difficulties refer to the KG equation that takes a *fundamental dynamical role* and is derived from a Lagrangian density. On the other hand, the KG equation can be used as a mathematical relation. For example, components of solutions of the Dirac equation satisfy the KG equation (see [15], p. 7).

References

[1] H. Feshbach and F. Villars, Rev. Mod. Phys. **30**, 24 (1958).

[2] F. Rohrlich *Classical Charged Particles* (Addison-Wesley, Reading Mass, 1965). pp. 3-6.

[3] S. Weinberg, *The Quantum Theory of Fields* (University Press, Cambridge, 1995). Vol. 1.

[4] L. H. Ryder, *Quantum Field Theory* (University Press, Cambridge, 1996).

[5] P. A. M. Dirac *Mathematical Foundations of Quantum Theory*, Editor A. R. Marlow (Academic, New York, 1978). (See pp. 3,4).

[6] E. Comay, Apeiron, **11**, No. 3, 1 (2004).

[7] J. D. Bjorken and S. D. Drell *Relativistic Quantum Fields* (McGraw, New York, 1965). (See chapter 12.)

[8] V. B. Berestetskii, E. M. Lifshitz and L. P. Pitaevskii, *Quantum Electrodynamics* (Pergamon, Oxford, 1982).

[9] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon, Oxford, 1975).

[10] W. Pauli and V. Weisskopf, Helv. Phys. Acta, **7**, 709 (1934). English translation: A. I. Miller *Early Quantum Electrodynamics* (University Press, Cambridge, 1994). pp. 188-205.

[11] G. Wentzel, *Quantum Theory of Fields* (Interscience, New York, 1949).

[12] A. E. Taylor and D. C. Lay *Introduction to Functional Analysis* (John Wiley, New York, 1980).

[13] G. Sterman, *An Introduction to Quantum Field Theory* (University Press, Cambridge, 1993).

[14] It is claimed in [6] that the action of the KG Lagrangian density does not fit the classical action. This point also holds for the Dirac Hamiltonian density (V. Dvoeglazov, private communication). This issue probably does not create a contradiction because the variation of the quantum expression is done with respect to the fields, which are regarded as generalized coordinates. On the other hand, the variation of the classical problem is done with respect to the coordinates of the classical path.

[15] J. D. Bjorken and S. D. Drell *Relativistic Quantum Mechanics* (McGraw, New York, 1964).