The Electromagnetic Form of the Dirac Electron Theory

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In the present paper it is shown that the Dirac electron theory can be represented with the special form of the Maxwell theory, if the Dirac wave function is identified with the plane electromagnetic wave in some specific way. This representation allows us to see new possibilities in the connection of classical and quantum electrodynamics.

Keywords: quantum electrodynamics, classical electrodynamics, quantum mechanics interpretation.

1.0. Introduction

W.J. Archibald [1] was the first who paid attention to the possibility of representing the Schrödinger equation in electromagnetic form. For the Dirac electron equations this possibility was mentioned in the book [2]. The other forms (quaternion, biquaternion, etc.) of the Dirac equation and the separate mathematical aspects of this theme were considered in many articles [3]. But an understanding of the physical meaning of these representations has been lacking until now.

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The Dirac equation has many particularities. In the modern interpretations these particularities are considered mathematical features that do not have a physical meaning. For example [4] (section 34-4), "it can prove that all the physical consequences of Dirac's equation do not depend on the special choice of Dirac's matrices. They would be the same if a different set of four 4x4 matrices... had been chosen. In particular it is possible to interchange the roles of the four matrices by unitary transformation. So, their differences are only apparent."

The mathematical properties of the Dirac matrices are well known: they are anti-commutative and Hermitian; they compose a group of 16 matrices; the bilinear forms of these matrices have defined transformation properties; it has been repeatedly pointed out that in classical physics these matrices describe the vector rotations, etc.

Below we will show that all the Dirac electron equation particularities can be entirely connected with the particularities of the equations of classical electrodynamics. We will also show why "the physical consequences of Dirac's equation do not depend on the special choice of Dirac's matrices."

2.0. Electrodynamics form of the Dirac equation

The Dirac bispinor contains four functions, while an electromagnetic field in the general case contains six functions. As it is known, the number of the functions in these theories is not connected in any way to the space dimension (see e.g. [5]). Obviously, in this case it is impossible to reduce one another. We will show that such a possibility exists only in a specific case, when Dirac's spinor is identified with the fields of the plane electromagnetic wave.

2.1. The spinor form of the Dirac equation

As it is known, there are two mathematical description forms of the representation of the Dirac electron equation: spinor and bispinor.

The Dirac equations in the spinor form [2-6] are the following:

$$\begin{cases} \hat{e}j + c\hat{\vec{s}}\,\hat{\vec{p}}c + mc^2 j = 0, \\ \hat{e}c + c\hat{\vec{s}}\,\hat{\vec{p}}j - mc^2 c = 0, \end{cases} \tag{2.1}$$

where $\hat{\vec{s}}$ are Pauli matrices, and j and c are the so-called spinors, represented by the following matrices:

$$j = \begin{pmatrix} j_1 \\ j_2 \end{pmatrix}, \quad c = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \tag{2.2}$$

2.2. The bispinor form of the Dirac equation

More often the Dirac equation is described in the bispinor form. Entering the function:

$$y = \begin{pmatrix} j \\ c \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \tag{2.3}$$

called bispinor, the equations (2.1) can be written in one equation.

There are two bispinor Dirac equation forms [6]:

$$\left[\left(\hat{a}_o\hat{e} + c\hat{\vec{a}}\,\hat{\vec{p}}\right) + \hat{b}\,mc^2\right] y = 0, \tag{2.4}$$

$$y^{+}\left[\left(\hat{a}_o\hat{e} - c\hat{\vec{a}}\,\hat{\vec{p}}\right) - \hat{b}\,mc^2\right] = 0, \tag{2.5}$$

which correspond to the two signs of the relativistic expression of the energy of the electron:

$$e = \pm\sqrt{c^2 \vec{p}^2 + m^2 c^4}\ , \tag{2.6}$$

Here $\hat{e} = i\hbar \frac{¶}{¶\ t}$, $\hat{\vec{p}} = -i\hbar\vec{\nabla}$ are the operators of the energy and momentum, e, $\vec{p}$ are the electron energy and momentum, c is the light velocity, m is the electron mass, y is the wave function (y^+ is the Hermitian-conjugate wave function) named bispinor and $\hat{a}_o = \hat{1}$, $\hat{\vec{a}}$, $\hat{a}_4 \equiv \hat{b}$ are the Dirac matrices:

$$\begin{gathered}
\hat{a}_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \hat{a}_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \\
\hat{a}_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \vec{a}_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.
\end{gathered} \tag{2.7}$$

It is also known that for each sign of the equation (2.6) there are two Hermitian-conjugate Dirac equations.

We will consider the electrodynamics meaning of all these equations.

2.3. Electrodynamics form of the Dirac equation without mass

Let us consider, for example, the plane electromagnetic wave moving on y - axis:

$$\begin{cases} \vec{E} = \vec{E}_o e^{-i(wt \pm ky)}, \\ \vec{H} = \vec{H}_o e^{-i(wt \pm ky)}, \end{cases} \tag{2.8}$$

In the general case it has two polarizations and contains the following field vectors:

$$E_x, E_z, H_x, H_z \quad (E_y = H_y = 0) \tag{2.9}.$$

The direction of the plane electromagnetic wave is defined by the Poynting vector [7]:

$$\vec{S} = \frac{c}{4p}\left[\vec{E} \times \vec{H}\right] = \frac{c}{4p}\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ E_x & E_y & E_z \\ H_x & H_y & H_z \end{pmatrix}, \tag{2.10}$$

where $\vec{i}, \vec{j}, \vec{k}$ are the unit vectors of x, y, z - axes. For the wave, which moves along y - axis we have:

$$\vec{S}_y = -\vec{j}(E_x H_z - E_z H_x), \tag{2.11}$$

Let us enter the Dirac spinors as electromagnetic waves in the following way:

$$j = \begin{pmatrix} E_x \\ E_z \end{pmatrix}, \quad c = i\begin{pmatrix} H_x \\ H_z \end{pmatrix} \tag{2.12}$$

where $j = j(y)$ and $c = c(y)$.

In this case the bispinor $y = y\ (y)$ will have the following form:

$$y=\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}=\begin{pmatrix} E_x \\ E_z \\ iH_x \\ iH_z \end{pmatrix}, y^{+}=\begin{pmatrix} E_x & E_z & -iH_x & -iH_z \end{pmatrix}, \quad (2.13)$$

Using (2.13), we can write the equation of the electromagnetic wave, moving along the y - axis, in the form:

$$\left(\hat{e}^{2}-c^{2}\hat{\vec{p}}^{2}\right)y=0, \quad (2.14)$$

The equation (2.14) can also be written in the following form:

$$\left[\left(\hat{a}_o\hat{e}\right)^2-c^2\left(\hat{\vec{a}}\,\hat{\vec{p}}\right)^2\right]y=0, \quad (2.15)$$

In fact, taking into account that

$$\left(\hat{a}_o\hat{e}\right)^2=\hat{e}^2, \quad \left(\hat{\vec{a}}\,\hat{\vec{p}}\right)^2=\hat{\vec{p}}^2, \quad (2.16)$$

we see that the equations (2.14) and (2.15) are equivalent.

Factorizing (2.15) and multiplying it from the left on the Hermitian-conjugate function y^{+} we get:

$$y^{+}\left(\hat{a}_o\hat{e}-c\hat{\vec{a}}\,\hat{\vec{p}}\right)\left(\hat{a}_o\hat{e}+c\hat{\vec{a}}\,\hat{\vec{p}}\right)y=0, \quad (2.17)$$

The equation (2.17) may be disintegrated on two Dirac equations without mass:

$$y^{+}\left(\hat{a}_o\hat{e}-c\hat{\vec{a}}\,\hat{\vec{p}}\right)=0, \quad (2.18)$$

$$\left(\hat{a}_o\hat{e}+c\hat{\vec{a}}\,\hat{\vec{p}}\right)y=0, \quad (2.19)$$

It is not difficult to show (using (2.13) that the equations (2.18) and (2.19) are the Maxwell equations of the electromagnetic waves: retarded and advanced.

2.4. Electrodynamics form of Dirac equation with mass

Let us consider first two Hermitian-conjugate equations, corresponding to the minus sign of the expression (2.6):

$$\left[\left(\hat{a}_o\hat{e} + c\hat{\vec{a}}\ \hat{\vec{p}}\right) + \hat{b}\ mc^2\right] y = 0\,, \qquad (2.20')$$

$$y^{+}\left[\left(\hat{a}_o\hat{e} + c\hat{\vec{a}}\ \hat{\vec{p}}\right) + \hat{b}\ mc^2\right] = 0\,, \qquad (2.20'')$$

Using (2.13) from (2.20') and (2.20'') we obtain:

$$\begin{cases} \frac{1}{c}\frac{\P\ E_x}{\P\ t} - \frac{\P\ H_z}{\P\ y} + i\frac{w}{c}E_x = 0, \\ \frac{1}{c}\frac{\P\ E_z}{\P\ t} + \frac{\P\ H_x}{\P\ y} + i\frac{w}{c}E_z = 0, \\ \frac{1}{c}\frac{\P\ H_x}{\P\ t} + \frac{\P\ E_z}{\P\ y} - i\frac{w}{c}H_x = 0, \\ \frac{1}{c}\frac{\P\ H_z}{\P\ t} - \frac{\P\ E_x}{\P\ y} - i\frac{w}{c}H_z = 0, \end{cases} \qquad (2.21),$$

$$\begin{cases} \frac{1}{c}\frac{\P E_x}{\P t} - \frac{\P H_z}{\P y} - i\frac{w}{c}E_x = 0, \\ \frac{1}{c}\frac{\P E_z}{\P t} + \frac{\P H_x}{\P y} - i\frac{w}{c}E_z = 0, \\ \frac{1}{c}\frac{\P H_x}{\P t} + \frac{\P E_z}{\P y} + i\frac{w}{c}H_x = 0, \\ \frac{1}{c}\frac{\P H_z}{\P t} - \frac{\P E_x}{\P y} + i\frac{w}{c}H_z = 0, \end{cases} \tag{2.22}$$

where $w = \frac{mc^2}{\hbar}$. The equations (2.21) and (2.22) are Maxwell equations with complex currents. (It is interesting that along with the electrical current, the magnetic current also exists here. This current is equal to zero by Maxwell's theory, but its existence according to Dirac does not contradict the quantum theory).

As we see, the equations (2.21) and (2.22) differ by the current directions. We could foresee this result before the calculations, since the functions y^+ and y differ by the argument signs: $y^+ = y_0 e^{-iwt}$ and $y = y_0 e^{iwt}$.

Let us compare now the equations that correspond to both plus and minus signs of (2.6).

For the plus sign of (2.5) we have the following two equations:

$$\left[\left(\hat{a}_o\hat{e} - c\hat{\vec{a}}\ \hat{\vec{p}}\right) - \hat{b}\,mc^2\right] y = 0, \tag{2.23}$$

$$y^+\left[\left(\hat{a}_o\hat{e} - c\hat{\vec{a}}\ \hat{\vec{p}}\right) - \hat{b}\,mc^2\right] = 0, \tag{2.24}$$

The electromagnetic form of the equation (2.23) is:

$$\begin{cases} \dfrac{1}{c}\dfrac{\partial E_x}{\partial t} + \dfrac{\partial H_z}{\partial y} + i\dfrac{\omega}{c}E_x = 0, \\ \dfrac{1}{c}\dfrac{\partial E_z}{\partial t} - \dfrac{\partial H_x}{\partial y} + i\dfrac{\omega}{c}E_z = 0, \\ \dfrac{1}{c}\dfrac{\partial H_x}{\partial t} - \dfrac{\partial E_z}{\partial y} - i\dfrac{\omega}{c}H_x = 0, \\ \dfrac{1}{c}\dfrac{\partial H_z}{\partial t} + \dfrac{\partial E_x}{\partial y} - i\dfrac{\omega}{c}H_z = 0, \end{cases} \qquad (2.25)$$

Obviously, the electromagnetic form of the equation (2.24) will have the opposite signs of the currents comparatively to (2.25).

Comparing (2.25) and (2.21) we can see that the equation (2.25) can be considered as the Maxwell equation of the retarded wave. If we don't want to use the retarded wave, we can transform the wave function of the retarded wave to the form:

$$\psi_{ret} = \begin{pmatrix} E_x \\ -E_z \\ iH_x \\ -iH_z \end{pmatrix}, \qquad (2.26)$$

Then, contrary to the system (2.25) we get the system (2.22).

The transformation of the function ψ_{ret} to the function ψ_{adv} is called, in quantum mechanics, the charge conjugation operation.

2.5. Electrodynamics sense of bilinear forms

Enumerate the main Dirac matrices [6]: 1) $\hat{\alpha}_4 \equiv \hat{\beta}$ is the scalar,

2) $\hat{a}_m = \left\{ \hat{a}_0 , \hat{\vec{a}} \right\} \equiv \{ \hat{a}_0, \hat{a}_1, \hat{a}_2, \hat{a}_3 \}$ is the 4-vector, 3) $\hat{a}_5 = \hat{a}_1 \cdot \hat{a}_2 \cdot \hat{a}_3 \cdot \hat{a}_4$ is the pseudoscalar.

Using (2.13) and taking into account that $y = y\left(y\right)$ it is easy to obtain the electrodynamics expressions of the bispinors, corresponding to these matrices:

1) $y^{+}\hat{a}_4 y = \left(E_x^{\,2} + E_z^{\,2}\right) - \left(H_x^{\,2} + H_z^{\,2}\right) = \vec{E}^2 - \vec{H}^2 = 8p\ I_1$, where I_1 is the first scalar of the Maxwell theory;

2) $y^{+}\hat{a}_o y = \vec{E}^2 + \vec{H}^2 = 8p\ U$ and $y^{+}\hat{a}_y y = 8p\ c\vec{g}_y$. Thus, the electrodynamics form of the 4-vector bispinor value is the energy-momentum 4-vector of the Maxwell theory: $\left\{ \frac{1}{c} U, \vec{g} \right\}$.

3) $y^{+}\hat{a}_5 y = 2\left(E_x H_x + E_z H_z\right) = 2\left(\vec{E}\cdot\vec{H}\right)$ is the pseudoscalar and $\left(\vec{E}\cdot\vec{H}\right)^2 = I_2$ is the second scalar of the electromagnetic field theory.

As it is known, from the Dirac equation the probability continuity equation can be obtained:

$$\frac{\P\ P_{pr}\left(\vec{r},t\right)}{\P\ t} + div\ \vec{S}_{pr}\left(\vec{r},t\right) = 0, \tag{2.27}$$

Here $P_{pr}\left(\vec{r},t\right) = y^{+}\hat{a}_0 y$ is the probability density, and $\vec{S}_{pr}\left(\vec{r},t\right) = -cy^{+}\hat{\vec{a}}\, y$ is the probability flux density. Using the above results we can obtain: $P_{pr}\left(\vec{r},t\right) = 8p\ U$ and $\vec{S}_{pr} = c^2\vec{g} = 8p\ \vec{S}$. Then in the electromagnetic form the equation (2.27) has the form:

$$\frac{\P\ U}{\P\ t} + div\ \vec{S} = 0\,, \tag{2.28}$$

which is the energy conservation law of the electromagnetic field.

3.0. The electrodynamics sense of the matrices choice

3.1. The electrodynamics sense of the transposition of the matrices

As we saw above, the matrix sequence $(\hat{a}_1, \hat{a}_2, \hat{a}_3)$ agrees with the electromagnetic wave, which has $-y$-direction. But herewith only the $\hat{a}_2$-matrix is "working", and the other two matrices do not give the terms in the equation. The verification of this fact is the Poynting vector calculation: the bilinear forms of $\hat{a}_1, \hat{a}_3$-matrices are equal to zero, and only the matrix $\hat{a}_2$ gives the right non-zero component of the Poynting vector.

A question arises: how to describe the waves, which have x and z-directions. It is not difficult to see that the matrices' sequence is not determined by some special requirements. In fact, this matrices sequence can be changed without breaking any quantum electrodynamics results [4, 6].

Introducing the axes' indexes, which indicate the electromagnetic wave direction, we can write three groups of the matrices, each of which corresponds to one and only one wave direction:

$$(\hat{a}_{1x}, \hat{a}_{2y}, a_{3z}),\ (\hat{a}_{2x}, \hat{a}_{3y}, \hat{a}_{1z},),\ (\hat{a}_{2z}, \hat{a}_{1y}, \hat{a}_{3x})\ .$$

Let us choose now the wave function forms, which give the correct Maxwell equations. We will take as the initial form that of the $-y$ -direction, which we have already used. From it, by means of the indexes' transposition around the circle, we will get the forms of the x and y - directions.

Since in the initial case the Poynting vector has the minus sign, we can suppose that the transposition must be counterclockwise. Let us examine the supposition, checking the Poynting vector values:

1) For $(\hat{a}_{1x}, \hat{a}_{2y}, a_{3z})$ we have

$$y = y(y),\ y = \begin{pmatrix} E_x \\ E_z \\ iH_x \\ iH_z \end{pmatrix},\ y^{+} = \left(E_x\ E_z\ -iH_x\ -iH_z \right)$$

$$y^{+}\hat{a}_{1_x} y = \left(E_x\ E_z\ -iH_x\ -iH_z \right) \begin{pmatrix} iH_z \\ iH_x \\ E_z \\ E_x \end{pmatrix} = 0,$$

$$y^{+}\hat{a}_{2_y} y = -2(E_z H_x - E_x H_z) = -2\left[\vec{E} \times \vec{H} \right]_y,\ y^{+}\hat{a}_{3_z} y = 0, \tag{3.1}$$

2) For $(\hat{a}_{2x}, \hat{a}_{3y}, \hat{a}_{1z},)$ we have

$$y = y(x),\ y = \begin{pmatrix} E_z \\ E_y \\ iH_z \\ iH_y \end{pmatrix},\ y^{+} = \left(E_z\ E_y\ -iH_z\ -iH_y \right);$$

$$y^{+}\hat{a}_{2_x} y = -2\left[\vec{E} \times \vec{H} \right]_x,\ y^{+}\hat{a}_{3_y} y = 0,\ y^{+}\hat{a}_{1_z} y = 0, \tag{3.2}$$

3) For $(\hat{a}_{2z}, \hat{a}_{1y}, \hat{a}_{3x})$ we have

$$y = y(z),\ y = \begin{pmatrix} E_y \\ E_x \\ iH_y \\ iH_x \end{pmatrix},\ y^{+} = \left(E_y\ E_x - iH_y - iH_x \right);$$

$$y^{+}\hat{a}_{3_x} y = 0,\ y^{+}\hat{a}_{1_y} y = 0,\ y^{+}\hat{a}_{2_z} y = -2\left[\vec{E}\times\vec{H}\right]_z, \qquad (3.3)$$

As we see, we took the correct result: by the counterclockwise indexes' transposition the wave functions describe the electromagnetic waves, which move in a negative direction with regard to the corresponding co-ordinate axes.

We can suppose that, by the clockwise indexes' transposition, the wave functions will describe the electromagnetic waves, which move in a positive direction along the co-ordinate axes. Let us
prove this:

1) For $(\hat{a}_{1x}, \hat{a}_{2y}, a_{3z})$ we have

$$y = y(y),,y = \begin{pmatrix} E_z \\ E_x \\ iH_z \\ iH_x \end{pmatrix},\ y^{+} = \left(E_z\ E_x - iH_z - iH_x \right);$$

$$y^{+}\hat{a}_{1_x} y = 0,\ y^{+}\hat{a}_{2_y} y = 2\left[\vec{E}\times\vec{H}\right]_y,\ y^{+}\hat{a}_{3_z} y = 0, \qquad (3.4)$$

2) For $(\hat{a}_{2x}, \hat{a}_{3y}, \hat{a}_{1z},)$ we have

$$y = y(x)\,,\; y = \begin{pmatrix} E_y \\ E_z \\ iH_y \\ iH_z \end{pmatrix},\; y^{+} = \left(E_y \;\; E_z \; -iH_y \; -iH_z \right);$$

$$y^{+}\hat{a}_{2_x} y = 2\left[\vec{E}\times\vec{H} \right]_x,\; y^{+}\hat{a}_{3_y} y = 0\,,\; y^{+}\hat{a}_{1_z} y = 0\,, \tag{3.5}$$

3) For $(\hat{a}_{2_z}, \hat{a}_{1_y}, \hat{a}_{3_x})$ we have

$$y = y(z)\,,\; y = \begin{pmatrix} E_x \\ E_y \\ iH_x \\ iH_y \end{pmatrix},\; y^{+} = \left(E_x \;\; E_y \; -iH_x \; -iH_y \right);$$

$$y^{+}\hat{a}_{3_x} y = 0\,,\; y^{+}\hat{a}_{1_y} y = 0\,,\; y^{+}\hat{a}_{2_z} y = 2\left[\vec{E}\times\vec{H} \right]_z, \tag{3.6}$$

As we see, once again we get the correct results.

Now we will prove that the above choice of the matrices gives the correct electromagnetic equation forms. Using the bispinor Dirac equation (2.23) as an example and transposing the indexes clockwise we obtain for the positive direction of the electromagnetic wave the following results for the x, y, z-directions correspondingly:

$$\begin{cases} \frac{1}{c}\frac{\P E_y}{\P t}+\left(\frac{\partial H_z}{\partial x}\right)=-i\frac{w}{c}E_y, \\ \frac{1}{c}\frac{\P E_z}{\P t}-\left(\frac{\partial H_y}{\partial x}\right)=-i\frac{w}{c}E_z, \\ \frac{1}{c}\frac{\partial H_y}{\partial t}-\left(\frac{\partial E_z}{\partial x}\right)=i\frac{w}{c}H_y, \\ \frac{1}{c}\frac{\P H_z}{\P t}+\left(\frac{\partial E_y}{\partial x}\right)=i\frac{w}{c}H_z, \end{cases} \quad \begin{cases} \frac{1}{c}\frac{\P E_z}{\P t}+\left(\frac{\partial H_x}{\partial y}\right)=-i\frac{w}{c}E_z, \\ \frac{1}{c}\frac{\P E_x}{\P t}-\left(\frac{\partial H_z}{\partial y}\right)=-i\frac{w}{c}E_x, \\ \frac{1}{c}\frac{\partial H_z}{\partial t}-\left(\frac{\partial E_x}{\partial y}\right)=i\frac{w}{c}H_z, \\ \frac{1}{c}\frac{\partial H_x}{\partial t}+\left(\frac{\partial E_z}{\partial y}\right)=i\frac{w}{c}H_x, \end{cases}$$

$$\begin{cases} \frac{1}{c}\frac{\P E_x}{\P t}+\left(\frac{\partial H_y}{\partial z}\right)=-i\frac{w}{c}E_x, \\ \frac{1}{c}\frac{\P E_y}{\P t}-\left(\frac{\partial H_x}{\partial z}\right)=-i\frac{w}{c}E_y, \\ \frac{1}{c}\frac{\P H_x}{\P t}-\left(\frac{\partial E_y}{\partial z}\right)=i\frac{w}{c}H_x, \\ \frac{1}{c}\frac{\partial H_y}{\partial t}+\left(\frac{\partial E_x}{\partial z}\right)=i\frac{w}{c}H_y. \end{cases} \tag{3.7}$$

As we can see, we have obtained three equation groups, each of which contains four equations, as is necessary for the description of all electromagnetic wave directions. In the same way for all other forms of the Dirac equation analogue results can be obtained.

Obviously, it is possible via canonical transformations to choose the Dirac matrices in such a way that the electromagnetic wave will have any direction.

3.2. The electrodynamics sense of canonical transformations of Dirac's matrices and bispinors

As is known [2,8], the transition from some independent variables to others is made by means of the unitary operator, which is called the canonical transformation operator.

Actually the choice (2.7) of the matrices a, made by us is not unique. In this case there is a free transformation of a kind:

$$a = S\, a' S', \tag{3.8}$$

where S is a unitary matrix. The last one corresponds to functions $y\,'$ transformation:

$$y = S\, y\,', \tag{3.9}$$

If we choose matrices $a\,'$ as:

$$\hat{a}\,'_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},\ \hat{a}\,'_2 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix},$$
$$\hat{a}\,'_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},\ \vec{a}\,'_4 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \tag{3.10}$$

then the functions y will be connected to functions $y\,'$ according to the relationship:

$$y = \frac{y\,'_1 - y\,'_4}{\sqrt{2}},\ y = \frac{y\,'_2 + y\,'_3}{\sqrt{2}},\ y = \frac{y\,'_1 + y\,'_4}{\sqrt{2}},\ y = \frac{y\,'_2 - y\,'_3}{\sqrt{2}}, \tag{3.11}$$

The unitary matrix S, which corresponds to this transformation, is equal to:

$$S = \frac{1}{\sqrt{2}} \begin{Bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{Bmatrix}, \tag{3.12}$$

It is not difficult to check that by means of this transformation we will also receive the equations of the Maxwell theory. Actually, using (2.13) and (3.11) it is easy to receive:

$$\frac{y'_1 - y'_4}{\sqrt{2}} = E_x, \frac{y'_2 + y'_3}{\sqrt{2}} = E_z, \frac{y'_1 + y'_4}{\sqrt{2}} = iH_x, \frac{y'_2 - y'_3}{\sqrt{2}} = iH_z, \tag{3.13}$$

whence:

$$y' = \frac{\sqrt{2}}{2} \begin{pmatrix} (E_x + iH_x) \\ (E_z + iH_z) \\ (E_z - iH_z) \\ -(E_x - iH_x) \end{pmatrix}, \tag{3.14}$$

Substituting these functions in the Dirac equation we will receive the correct Maxwell equations for the electromagnetic waves (in double quantity). It is possible to assume, that the functions y' correspond to the electromagnetic wave, moving under the angle of 45 degrees to both coordinate axes.

Thus, we see that actually any choice of the Dirac matrices changes only the direction of the electromagnetic wave .

4.0. The electromagnetic form of the electron theory Lagrangian

As it is known [7], the Lagrangian of the Maxwell theory in the case of the electromagnetic waves is:

$$L_M = \frac{1}{8p}\left(\vec{E}^2 - \vec{H}^2\right), \tag{4.1}$$

and as a Lagrangian of the Dirac theory can take the expression [6]:

$$L_D = y^+\left(\hat{e} + c\hat{\vec{a}}\ \hat{\vec{p}} + \hat{b}\ mc^2\right)\ y\ , \tag{4.2}$$

For the electromagnetic wave moving along the $-y$-axis the equation (4.2) can be written:

$$L_D = \frac{1}{c}y^+\frac{¶\ y}{¶\ t} - y^+\hat{a}_y\frac{¶\ y}{¶\ y} - i\frac{mc}{\hbar}y^+\hat{b}\ y\ , \tag{4.3}$$

Transferring each term of (4.3) in the electrodynamics form we obtain the electromagnetic form of the Dirac theory Lagrangian:

$$L_s = \frac{¶\ U}{¶\ t} + div\ \vec{S} - i\frac{w}{4p}\left(\vec{E}^2 - \vec{H}^2\right), \tag{4.4}$$

(Let us note that in the case of the variation procedure we must distinguish the complex conjugate field vectors $\vec{E}^*, \vec{H}^*$ and $\vec{E}, \vec{H}$).

The equation (4.4) can also be written in another form. Using the complex electrical and "magnetic" currents:

$j_t^{\ e} = i\dfrac{w}{2p}\vec{E}$ and $j_t^{\ m} = i\dfrac{w}{2p}\vec{H}$ we take:

$$L_s = \frac{¶\ U}{¶\ t} + div\ \vec{S} - \left(\vec{j}_t^{\ e}\vec{E} - \vec{j}_t^{\ m}\vec{H}\right), \tag{4.5}$$

It is interesting that since $L_s = 0$ thanks to (2.4), we can take the equation:

$$\frac{\partial U}{\partial t} + div\ \vec{S} - \left(\vec{j}_t^{\ e} \vec{E} - \vec{j}_t^{\ m} \vec{H} \right) = 0\,, \tag{4.6}$$

which has the form of the energy-momentum conservation law for the Maxwell equation with current.

Conclusion

The above results show that the Dirac theory can be written in the electromagnetic form as consistently as in the usual spinor form. Such representation makes the new interpretation of the quantum electrodynamics possible [9].

References

1. W.J. Archibald. *Canad. Journ. Phys*., 33, 565 (1955)
2. A.I. Akhiezer, W.B. Berestetskii. *Quantum electrodynamics*. Moscow, 1959 (Interscience publ., New York,1965).
3. T. Koga. *Int. J. Theor. Phys*., 13, No 6, p.377-385 (1975); A.A. Campolattoro. *Int. J. Theor. Phys*. 19, No 2, p.99-126 (1980); W.A. Rodrigues, Jr. arXiv: math-ph/0212034 v1.
4. E. Fermi. *Notes on quantum mechanics*. (The University of Chicago press, 1960).
5. H.A. Bethe. *Intermediate Quantum Mechanics*. 1964.
6. L.T. Schiff. *Quantum Mechanics*. (2nd edition, McGraw-Hill Book Company, Inc, New York, 1955).
7. M.-A. Tonnelat. *Les Principes de la Theorie Electromagnetique et de la Relativite*. (Masson et C. Editeurs, Paris, 1959).
8. V. Fock. *The principles of the quantum mechanics*. Leningrad, 1932
9. A.G. Kyriakos. http://arXiv.org/abs/quant-ph/0204037/.