

A derivation of two homogenous Maxwell equations

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We present a theoretical derivation of two homogenous Maxwell equations, based on Stokes theorem for Minkowski space tensors. A more general equation is also derived for the case of a field-strength tensor which is not antisymmetric. (*Communicated by V. Dvoeglazov. Received on Jan 22, 2004.*)

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Introduction

The Maxwell equations (1)-(2) for the electromagnetic field and the Lorentz 4-force law (3) for a charged particle are generalizations based on the experiments on the forces between electric charges and currents. These equations can be written in the

covariant form [1]

$$\frac{\partial \mathcal{F}_{\alpha\beta}}{x_\alpha} = \frac{4\pi}{c} J_\beta, \quad (1)$$

$$\frac{\partial \mathcal{F}_{\alpha\beta}}{\partial x_\gamma} + \frac{\partial \mathcal{F}_{\beta\gamma}}{\partial x_\alpha} + \frac{\partial \mathcal{F}_{\gamma\alpha}}{\partial x_\beta} = 0, \quad (2)$$

$$F_\alpha = \frac{q}{c} \mathcal{F}_{\alpha\beta} U_\beta, \quad (3)$$

where x is the position 4-vector, J is the 4-current, F is the 4-force, U is the 4-velocity, and \mathcal{F} is the antisymmetric field-strength tensor. We work in the x, y, z, ict Minkowski space, where there is no distinction between covariant and contravariant tensors.

These equations are intimately related with the principles of special relativity. Indeed, it was the consistent treatment of the electrodynamics of moving bodies that led to relativity [2]. Tolman [3] and Jefimenko [4] have derived the Lorentz force law from the Maxwell equations and special relativity. Frisch and Wilets [5] have derived the Maxwell equations and the Lorentz force law by applying the transformations of special relativity to Gauss's law of the flux of the electric field. Dyson [6] reproduces an argument due to Feynman, in which Maxwell equations follow from Newton's law of motion and the quantum mechanics commutation relations. It is remarkable how nonrelativistic assumptions can lead to relativistically invariant equations. Gersten [7] and Dvoeglazov [8] have derived generalized Maxwell equations from first principles, similar to those which have been used to derive the Dirac quantum relativistic electron equation. In this paper we derive two homogenous Maxwell equations by using Stokes theorem, in a fully relativistic manner.

Derivation of two homogenous Maxwell equations

A consequence of the antisymmetry of the field-strength tensor is that the magnitude $im_o c$ of the 4-momentum p is constant. We will consider an extension of the Lorentz 4-force law (3) to the case of a field-strength tensor not necessarily antisymmetric $\mathcal{F}_{\alpha\beta} = \mathcal{F}_{\alpha\beta}^{(a)} + \mathcal{F}_{\alpha\beta}^{(s)}$. Consequently the rest mass m_o of the test particle will no longer be constant (but the electric charge will not be modified). Such a theory which allows for the variation of the rest mass has been under investigation by Galeriu [9]. In this theory the concept of 'material point particle' is rejected, a time symmetric interaction taking place between finite segments along the world-lines of the particles.

We will assume that the rest mass of a particle at a given SpaceTime point, and with a given velocity, does not depend on the history of that particle. This is the underlying physical principle behind the homogenous Maxwell equations (2). The classical theory is obviously a special case of this more general theory, limited to an antisymmetric field-strength tensor.

Consider two SpaceTime events A and B, and a charged test particle moving from A to B on any possible smooth path Γ , restricted only to the condition that the initial and final velocities be given. Since at A and B the direction of the 4-momentum is given, and the magnitude of the 4-momentum is also unique, we conclude that the variation of the 4-momentum between A and B is the same regardless of the path followed. For two different paths, Γ_1 and Γ_2 , we can write

$$p_{B\alpha} - p_{A\alpha} = \int_{\Gamma_1} dp_{\alpha} = \int_{\Gamma_2} dp_{\alpha}. \quad (4)$$

The expression (3) of the Lorentz 4-force allows us to write the

integrals in (4) as the circulation of the field-strength tensor

$$\int_{\Gamma} dp_{\alpha} = \int_{\Gamma} F_{\alpha} d\tau = \int_{\Gamma} \frac{q}{c} \mathcal{F}_{\alpha\beta} \frac{dx_{\beta}}{d\tau} d\tau = \int_{\Gamma} \frac{q}{c} \mathcal{F}_{\alpha\beta} dx_{\beta}. \quad (5)$$

From (4) and (5) we obtain that in general

$$\oint \frac{q}{c} \mathcal{F}_{\alpha\beta} dx_{\beta} = 0 \Rightarrow \oint \mathcal{F}_{\alpha\beta} dx_{\beta} = 0. \quad (6)$$

Stokes theorem, usually used in connection with the null circulation of a vector, will now be applied for the more general case of a tensor. Stokes theorem in the 4-dimensional Minkowski space takes the form [1]

$$\oint \mathcal{F}_{\alpha\beta} dx_{\beta} = \frac{1}{2} \int df_{\beta\gamma} \left(\frac{\partial \mathcal{F}_{\alpha\gamma}}{\partial x_{\beta}} - \frac{\partial \mathcal{F}_{\alpha\beta}}{\partial x_{\gamma}} \right), \quad (7)$$

where $df_{\beta\gamma}$ are projections of a surface element. Due to the arbitrary nature of the paths Γ_1 and Γ_2 , from equations (6)-(7) it follows that

$$\frac{\partial \mathcal{F}_{\alpha\gamma}}{\partial x_{\beta}} - \frac{\partial \mathcal{F}_{\alpha\beta}}{\partial x_{\gamma}} = 0. \quad (8)$$

This is the most general condition that the field-strength tensor must satisfy. We separate the symmetric and the antisymmetric components in (8), obtaining

$$\frac{\partial \mathcal{F}_{\gamma\alpha}^{(s)}}{\partial x_{\beta}} - \frac{\partial \mathcal{F}_{\alpha\beta}^{(s)}}{\partial x_{\gamma}} = \frac{\partial \mathcal{F}_{\gamma\alpha}^{(a)}}{\partial x_{\beta}} + \frac{\partial \mathcal{F}_{\alpha\beta}^{(a)}}{\partial x_{\gamma}}. \quad (9)$$

From (9) we obtain two more equations by cyclic permutations of the indices ($\alpha \rightarrow \beta, \beta \rightarrow \gamma, \gamma \rightarrow \alpha$) and ($\alpha \rightarrow \gamma, \beta \rightarrow \alpha, \gamma \rightarrow \beta$).

By summing up all the three equations the symmetric components cancel, and we end up with two homogenous Maxwell equations

$$\frac{\partial \mathcal{F}_{\alpha\beta}^{(a)}}{\partial x_\gamma} + \frac{\partial \mathcal{F}_{\beta\gamma}^{(a)}}{\partial x_\alpha} + \frac{\partial \mathcal{F}_{\gamma\alpha}^{(a)}}{\partial x_\beta} = 0. \quad (10)$$

By subtracting (9) from (10) we also see that, if no symmetric components are allowed (the left side of (9) is zero), a problematic equation emerges:

$$\frac{\partial \mathcal{F}_{\beta\gamma}^{(a)}}{\partial x_\alpha} = 0. \quad (11)$$

These results, and other problems related to the radiation reaction 4-force [10], suggest that the symmetric part of the field-strength tensor cannot be ignored.

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