P-adic Properties of Time in the Bernoulli Map

Oscar Sotolongo-Costa[1], Jesus San-Martin[2]
[1]- Cátedra de sistemas Complejos "Henri Poincaré", Fac. de Fisica, Universidad de La Habana, Habana 10400, Cuba. (oscarso@ff.oc.uh.cu)
[2]- Dpto. Fisica-Matemática y Fluidos, UNED, a.p. 60141, 28080 Madrid, Spain.

The Bernoulli Map is analyzed with an ultrametric approach, showing the adequacy of the non-Archimedean metric to describe in a simple and direct way the chaotic properties of this map. Lyapunov exponent and Kolmogorov entropy appear to yield a better understanding. In this way, a *p*-adic time emerges as a natural consequence of the ultrametric properties of the map.

PACS: 05.45.+b, 02.90.+p

Keywords: Non-Archimedean, ultrametricity, chaos

Introduction

Time is, perhaps, the most primary notion for all of us. Nevertheless (and maybe because of that) its properties, and specially its metric properties, have not been exhaustively analyzed. Contrary to the metric properties of the space and even of the space-time, where a profusion of research has appeared, there is a void in the metric description concerning time. In our opinion, ultrametricity concepts open a path to this description, and we try to illustrate it by studying the ultrametric properties of time in the case of the Bernoulli map.

In ultrametric spaces, concepts such as exponential separation of neighboring trajectories, and characteristic parameters (Lyapunov exponents and Kolmogorov entropy) seem to yield a simpler understanding than the Euclidean metric.

In the last years, ultrametricity has triggered interest in a wide range of physical phenomena, due to its applications in different fields: spin glasses, mean field theory, turbulence, and nuclear physics. Also optimization theory, evolution, taxonomy, and protein folding benefit from it (for an excellent review see [1]). Wherever a hierarchical concept appears, non-Archimedean analysis is an adequate tool to study the problem.

As an example where Euclidean metric is not very adequate, let us consider Baker's map [4]. The interval $[0,1]\times[0,1]$ is mapped into $[0,1]\times[0,1]$. Therefore, the distance between two points cannot be larger than the distance between two opposite corners in $[0,1]\times[0,1]$.

Nonetheless, Baker's map has a Lyapunov exponent greater than one. Thus, the distance between neighboring points grows exponentially in a finite region of the phase space. In the Euclidean space we would have to define the distance as the Euclidean length of the shortest path lying entirely within the region. Ultrametricity is a promising tool in the theory of branching processes, which, at the same time, has shown promise in the study of self-organized critical processes where a branching representation can be introduced [2,3]. It seems possible to find simpler tools to describe the geometry of these processes.

Here, we illustrate the advantages of a hierarchical representation in the case of the Bernoulli shift. This will enable us, using simple geometric considerations, to determine the magnitudes governing the system, and the advantages of a *p*-adic metric will be stressed over the Euclidean metric. The ultrametric distance will be shown to be consistent with the characteristic behavior of this chaotic one-dimensional map.

In this paper we explore the application of ultrametricity in linking the Bernoulli map with a branching structure, which will reveal the possibilities of assigning an ultrametric measure to processes that, to all appearances, are not linked with a given metric (*e.g*., minority game and related problems) so that an adequate understanding of the ultrametric properties of a given process may lead to deeper understanding [5]. As any nontrivial norm is equivalent to the Euclidean or any of the *p*-adics (Ostrowski's theorem [1]), it would be convenient to measure the distance between points in Baker's map with a *p*-adic metric.

Ultrametricity

An ultrametric space is a space endowed with an ultrametric distance, defined as a distance satisfying the inequality

$$d(A,C) \leq Max\{d(A,B), d(B,C)\} \tag{1}$$

(A, B and C are points of this ultrametric space), instead of the usual triangular inequality, characteristic of Euclidean geometry

$$d(A,C) \leq d(A,B) + d(B,C) \tag{2}$$

A metric space $\mathbf{E}$ is a space for which a distance function $d(x,y)$ is defined for any pair of elements (x,y) belonging to $\mathbf{E}$.

A norm satisfying

$$\|x+y\| \le Max\{\|x\|,\|y\|\} \tag{3}$$

is called a non-Archimedean metric, because equation (3) implies that

$$\|x+x\| \le \|x\| \tag{4}$$

holds, and equation (4) does not satisfy the Archimedes principle:

$$\|x+x\| \ge \|x\| \tag{5}$$

A metric is called non-Archimedean or ultrametric, if (1) holds for any three points (x, y, z)

$$d(x,z) \le Max\{d(x,y), d(y,z)\} \tag{6}$$

A non-Archimedean norm induces a non-Archimedean metric:

$$d(x,z) = \|x-z\| \le Max\{d(x,y), d(y,z)\} \tag{7}$$

It is known that equation (7) implies a lot of surprising facts, *e.g.*, that all triangles are isosceles or equilateral and every point inside a ball is itself at the center of the ball, while the diameter of the ball is equal to its radius.

An example of ultrametric distance is given by the p-adic distance, defined as

$$d_p(x,y) = \|x-y\|_p \tag{8}$$

where the notation defines the p-adic absolute value:

$$\|x\|_p \equiv p^{-r} \tag{9}$$

where p is a fixed prime number, $x \ne 0$ is any integer, and r is the highest power of p dividing x.

Two numbers are p-adically closer as long as r is higher, such that p^r divides $\|x-y\|$. Amazingly, for $p = 5$ the result is that 135 is closer to 10 than 35.

Any positive or negative integer can be represented by a sum

$$x = \sum_{i=0}^{\infty} a_i p^i \tag{10}$$

where

$$0 \le a_i \le p-1 \tag{11}$$

If negative exponents are considered in the sum, rational numbers can also be represented. Such a representation is unique. The set of all sums Q_p is the field of *p*-adic numbers, and contains the field of rational numbers Q but is different from it.

Lyapunov exponent, Kolmogorov entropy and ultrametric time.

With the above description the *p*-adic numbers have a hierarchical structure, whose natural representation is a tree. Let us now use this description to work with the Bernoulli map (See [4]):

$$\begin{aligned} x_{n+1} &= 2x_n \bmod 1 \\ n &= 0,1,2... \end{aligned} \tag{12}$$

Here, we may note that the numbers can be represented as a set of points in a straight line or by a hierarchical structure, depending on the definition of distance (Euclidean or Archimedean) as we will see below.

Let us represent the initial value (state) to be mapped into the unit interval by the sequence $0, a_1 .....a_N .....$ with $a_i = 0$ or 1 to denote the initial value in binary notation.

It is possible to reorder these sequences as a hierarchical tree. To do this, let us perform the following process to represent the result of the application of the Bernoulli map:

We begin at an arbitrary point. We read, consecutively, the values of a_i, from $i=1$ to *N*, of the sequence $a_1.....a_N......$ When a_i takes the value 0 we move to the left, and the same distance down. When a_i takes the value 1 we do the same, but moving to the right. The result is 2^N branches of a hierarchical tree. Any finite path inside this branching structure represents univocally a possible finite sequence $a_1.....a_N.....$

Thus, for instance, the sequence 0,0110 represents: left, right, right, left.

The distance $d(x_i, x_j)$ between two branches (sequences) x_i, x_j in this tree is given by

$$d(x_i, x_j) = \begin{cases} 2^{(m-n)} \rightarrow i \neq j \\ 0 \rightarrow i = j \end{cases} \tag{13}$$

where *m* is the number of levels one must move up the tree to find a common branch linking x_i and x_j, and *N* is the number of levels (the length of the sequence). This is equivalent to

$$d(x_i, x_j) = \begin{cases} 2^{-h} \rightarrow i \neq j \\ 0 \rightarrow i = j \end{cases} \tag{13a}$$

where *h* is the position of the last block a_h in which a_i ($i = 1,....,h$) are common to the two sequences x_i, x_j. This means that the numbers x_i and x_j are near the h^{th} binary place. This distance is ultrametric.

To calculate the Lyapunov exponent it is necessary to know how neighboring points $x_0 + e$ and x_0 evolve during the Bernoulli map. If e be equal to $2^{-h}[1 + 2^{-d_1} + 2^{-d_2} + ...] > 2^{-N}$, then the first different position between $x_0 = 0, a_1 a_2 ... a_{h-1} a_N ...$ and $x_0 + e$ is a_h.

It is then is necessary to move up the tree $N-h+1$ levels from the bottom line to find the common branch in the position a_{h-1} (obviously, the last common figure between x_0 and $x_0 + \boldsymbol{e}$). Thus,

$$d(x_0 + \boldsymbol{e}, x_0) = 2^{-(h+1)} \tag{14}$$

and

$$d(f^n(x_0 + \boldsymbol{e}), f^n(x_0)) = 2^{-h+1+n} \tag{15}$$

because the iteration f^n moves away the common branch n positions from the bottom level.

To calculate the Lyapunov exponent it is necessary to express the exponential growth of the distance between two neighboring points:

$$\lim_{n\to\infty}\lim_{e\to 0} 2^{ln}\boldsymbol{e} = \lim_{n\to\infty}\lim_{e\to 0} d(f^n(x_0 + \boldsymbol{e}), f^n(x_0)) \tag{16}$$

Since the base for measuring the p-adic distance in our space is the number 2, in the preceding equation we have expressed the exponential growth as 2^{ln} instead of e^{ln}.

Replacing $\boldsymbol{e}$ and $d(f^n(x_0 + \boldsymbol{e}), f^n(x_0))$ in the preceding equation we obtain

$$\lim_{n\to\infty}\lim_{h\to\infty} 2^{-h}(1 + 2^{-d_1} + 2^{-d_2} + \ldots)2^{ln} = \lim_{n\to\infty}\lim_{h\to\infty} 2^{-h+1+n} \tag{17}$$

from (17) it can be easily observed that $l = 1$.

As the Lyapunov exponent in the Bernoulli map is ln 2 [4], we recover this result with p-adic metric, since $2 = e^{\ln 2}$. This means that each unit time interval implies a new doubling of branches in each node of the hierarchical tree. Once a unit time interval has elapsed, the number of levels one must move up the tree to find a common branch increases by one. This result will be crucial to understanding how information is lost in the course of time.

In one-dimensional maps like the one considered here, the Kolmogorov entropy coincides with the Lyapunov exponent [4]. The expression for the Kolmogorov entropy is:

$$K = \lim_{n\to\infty} \lim_{t\to 0} \frac{1}{nt} \sum_{i_1 \ldots i_n} p_{i_1 \ldots i_n} \lg_2 p_{i_1 \ldots i_n} \tag{18}$$

where $p_{i_1 \ldots i_n}$ is the probability to reach the i_n-th state of the system in the phase space following a given path $i_1 i_2 \ldots i_n$. It can be seen that in our case this probability only depends on the final state i_n, because for each state there is just one path, *i.e.*, that given by the sequence $i_1 i_2 \ldots i_n$. Moreover, the number of states in the n^{th} level is 2^n, and t is the time elapsed in passing from one state to the next. The probability of occupying one of the 2^n states is $p_n = p_{i_1 i_2 \ldots i_n} = \frac{1}{2^n}$, and this results in

$$K = \lim_{n\to\infty} \lim_{t\to 0} \frac{2}{t\, 2^n} \tag{19}$$

But the distance between two successive states of the n^{th} level is 2^{1-n}, because they are common up to the $(n-1)^{th}$ level. Since the speed v to pass from one sequence to the next is constant in the Bernoulli map, *i.e.*, $v = \frac{2^{1-n}}{t} = 1$, the time t elapsed between these two successive states is $t = 2^{1-n}$. As expected, $k = 1$, coinciding with the Lyapunov exponent. Hence, time has ultrametric properties in this process. Notice that the existence of a *p*-adic proper time is essential to the coincidence of the Kolmogorov entropy and the Lyapunov exponent. The spatial *p*-adic structure is unavoidably joined to the *p*-adic structure of proper time.

Therefore, we can say that this problem possesses a *p*-adic spatial and temporal geometry instead of a sole *p*-adic spatial geometry. To see the importance of the introduction of a *p*-adic time, see [6].

The Kolmogorov entropy measures the loss of information in the process. This loss of information can easily be seen from our representation, since the process of separation of trajectories is such that for any step the increase of the distance between two points duplicates the number of branches through which this increment can be reached. We are loosing information because we do not know exactly the way we are separating two states.

On the other hand, we can see that in the ultrametric space the natural time of the system is also ultrametric. The time of transition between two sequences x_i, x_j satisfies the same expression (13) as the distance between x_i,x_j.

Furthermore, the subsequent behavior of two states that separate at a given point in the ultrametric space depends on the point at which separation occurs, revealing that ultrametricity can be applied to decision processes (like minority games, aging effects, hierarchical processes, *etc*.), where ultrametric concepts have been poorly applied. The application of ultrametricity to the minority game will be treated in future work.

Conclusions

It has been shown that the Bernoulli map leads to a hierarchical structure in the *p*-adic metric. In light of the ultrametric distance, the Lyapunov exponent and the Kolmogorov entropy can be better understood and a direct geometric interpretation is supplied by the hierarchical structure. The *p*-adic metric seems to be the natural metric of this map. The hierarchical structure generates *p*-adic properties for temporal evolution.

This work was partially supported by an "Alma Mater" prize, Havana University, and by the Ministerio de Educacion y Cultura, Spain.

References

[1] Rammal, R., G. Toulouse and M. A. Virasoro, Ultrametricity for Physicists, *Rev. Mod. Phys* **58**, 3, (1986) 765-788.

[2] Garcia-Pelayo, R., Dimension of branching processes and self-organized criticality, *Phys. Rev. E* **49**, 6, (1994) 4903-4906.

[3] Sotolongo-Costa, O., Y. Moreno-Vega, J.J. Lloveras and J. C. Antoranz, Criticality in droplet fragmentation, *Phys. Rev. Lett* **76**, 1, (1996) 42-45.

[4] H. G. Schuster *Deterministic Chaos* Physik-Verlag, (1984) 216pp.

[5] M. Barnsley *Fractals Everywhere* Academic Press (1988) 394pp.

[6] D.K. Arrowsmith and F. Vivaldi, Geometry of p-adic Siegel disks, *Physica D* **71**, (1994) 222-236.