On the Unambiguity of the Solution of the Inhomogeneous Wave Equation

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Some time ago Ref. [1] was published, in which it was stated that the inhomogeneous wave equation can have, under given *initial* conditions, more than one solution. From our viewpoint this wrong statement arises from one point of the theory of partial differential equations which seems to be obvious; but there is no corresponding theorem specifying that point.

However, before analyzing where in [1] this point was missed, we consider the general method of solution of the wave equation presented there. The starting point for solving the wave equation given in [1] is

$$\Delta U - \frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} = f(\mathbf{r},t); \tag{1}$$

with no spatial boundary conditions (we seek a solution in the whole space) and the initial conditions

$$U(\mathbf{r},0) = \vartheta(\mathbf{r}) \quad ; \quad \left.\frac{\partial U(\mathbf{r},0)}{\partial t}\right|_{t=0} = \psi(\mathbf{r}); \tag{2}$$

and the function describing the source has a non-zero value in some local region $|\,\mathbf{r}\,| \le a$. Generally, this function is arbitrary.

The solution of Eq. (1) is well known; it is a convolution of the retarded Green function of the wave equation with the source function. However, the authors of [1] seek another solution of this equation. They do it in the following way:

Solution of Eq. (1) is represented as a sum of two functions,

$$U(\mathbf{r},t) = u(\mathbf{r},t) + V(\mathbf{r},t); \tag{3}$$

Substituting Eq. (3) into Eq. (1), one obtains

$$\Delta u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} + \Delta V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = f(\mathbf{r},t); \tag{4}$$

Now it is assumed that function V obeys the Poisson equation:

$$\Delta V = f(\mathbf{r},t) \quad ; \quad \lim_{r\to\infty} V = 0; \tag{5}$$

The next step made in [1] is transformation of Eq. (4) to the form:

$$\Delta u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = -\Delta V + \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} + f(\mathbf{r},t) = \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2}; \tag{6}$$

Now Eq. (6) is treated as a wave equation for the function u with new initial conditions

$$u(\mathbf{r},0)=\vartheta(\mathbf{r})-V(\mathbf{r},0) \quad ; \quad \left.\frac{\partial u(\mathbf{r},0)}{\partial t}\right|_{t=0}=\psi(\mathbf{r})-\left.\frac{\partial V(\mathbf{r},0)}{\partial t}\right|_{t=0} ; \quad (7)$$

Because formally one is able to solve the equation

$$\Delta u-\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}=\frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} ; \qquad (8)$$

under initial conditions (7), it is stated in [1] that "...Thus, we receive the new solution of equation (1.6), which differs from solution $U(\mathbf{r},t)$. It has the following form:"

$$U(\mathbf{r},t)=u(\mathbf{r},t)+V(\mathbf{r},t)$$

However, this statement is incorrect, as we show below.

First, we consider application of the above method to solution of an ODE with constant coefficients. For simplicity, we apply the method to solve the equation

$$x''+a^2x=f(t); \qquad (9)$$

with initial conditions $x(0)=x_0$ and $x'(0)=x_1$. We use the Laplace operator method to find the solution. Introducing the Laplace transforms

$$X(p)\div x(t); \quad F(p)\div f(t)$$

we find

$$(p^2+a^2)X=F(p)+x_0p+x_1 ; \qquad (10)$$

which has the solution

$$X=\frac{F(p)}{p^2+a^2}+\frac{x_0p+x_1}{p^2+a^2} ; \qquad (11)$$

Now we find solution (11) by the method of [1] but in p-space. Thus we seek the solution X as a sum of two functions $X=Y+Z$ where the original of Y satisfies an analogue of the Poisson equation,

$$y'' = f(t)$$

with the same initial conditions $y(0) = x_0$ and $y'(0) = x_1$. Therefore,

$$Y = \frac{F(p)}{p^2} + \frac{x_0 p + x_1}{p^2}; \tag{12}$$

and we should insert Eq. (12) into Eq. (10). We have

$$\begin{aligned} &\left[p^2 + a^2\right]Z + p^2 \frac{F(p)}{p^2} + \frac{p^2(x_0 p + x_1)}{p^2} + \\ &+ p^2 \frac{F(p)}{p^2} + \frac{p^2(x_0 p + x_1)}{p^2} = F(p) + x_0 + x_1 \end{aligned}; \tag{13}$$

The solution of Eq. (13) is

$$Z = -\frac{a^2 F(p)}{p^2\left(p^2 + a^2\right)} - \frac{a^2(x_0 p + x_1)}{p^2\left(p^2 + a^2\right)}; \tag{14}$$

and finally we have

$$\begin{aligned} &X = Y + Z = \frac{F(p)}{p^2} + \frac{(x_0 p + x_1)}{p^2} - \frac{a^2 F(p)}{p^2\left(p^2 + a^2\right)} - \\ &- \frac{a^2(x_0 p + x_1)}{p^2\left(p^2 + a^2\right)} = \frac{F(p)}{\left(p^2 + a^2\right)} + \frac{x_0 p + x_1}{\left(p^2 + a^2\right)} \end{aligned}; \tag{15}$$

which agrees with Eq. (11).

But now we check if the solutions of the wave equation obtained by direct solving and by the method developed in [1] coincide. For simplicity, we consider the case of an elementary charge uniformly moving along the x-axis, $f(\mathbf{r},t) = q\delta(x - vt)$. Solution of the wave equation (1) in this case is known and, expressed via instantaneous but not retarded variables, it is ([2], Eq. (21.39))

$$U(x,y,z,t)=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\frac{q}{\left[\frac{(x-vt)^2}{1-v^2/c^2}+y^2+z^2\right]^{1/2}}; \qquad (16)$$

Now we seek the solution of Eq. (1) by another method. Solution of the Poisson equation for a uniformly moving charge is

$$V(\mathbf{r},t)=\frac{q}{|\mathbf{r}-\mathbf{v}t|} \qquad \left(=\frac{q}{\sqrt{(x-vt)^2+y^2+z^2}}\right); \qquad (17)$$

Therefore, solution of Eq. (6) is (we go to the co-moving frame, so $\mathrm{X}=X-vt$; $x'=x-vt$):

$$u(\mathrm{X},Y,Z)+\int\frac{\left[qv^2/\sqrt{1-v^2/c^2}\right]dx'\,dydz}{\sqrt{\frac{(\mathrm{X}-x')^2}{1-v^2/c^2}+(Y-y)^2+(Z-z)^2}}\frac{y^2+z^2-2x'^2}{\left[x'^2+y^2+z^2\right]^{5/2}}; \qquad (18)$$

Because the integral of Eq. (18) is difficult to calculate in explicit form, we compare values of the solutions U and $V+u$ calculated on the x-axis where $Y=Z=0$. Then we have for $U(X, t)$

$$U(X,t)=\frac{q}{X-vt}$$

and for $V(X, t)$

$$V(X,t)=\frac{q}{X-vt}$$

So

$$U(X,t)-u(X,t)-V(X,t)=-u(X-vt)$$
$$=-\int\frac{v^2\left[\partial^2V(x',y',z')/\partial x'^2\right]dx'\,dy'\,dz'}{\sqrt{(X-vt-x')^2+(1-v^2/c^2)(y'^2+z'^2)}}; \qquad (19)$$

Calculation of the integral of Eq. (19) is given in an Appendix. We have

$$U(X,t)-u(X,t)-V(X,t)=-\frac{4\pi q v^2\sqrt{1-v^2/c^2}}{X-vt}$$

This result looks strange because it is known that the wave equation can be solved in an unambiguous way. So one can conclude that in fact the method of [1] is not a valid way of solving the wave equation.

Actually, the difference in the solution methods arises when the authors of [1] introduce, without any reasons, an additional condition, *"As we have entered two new unknown functions we should add an appropriate condition."* But in the general case, if one seeks the solution of some equation as a sum of two unknown functions, it does not mean that an additional condition must be imposed on these unknowns. The general solution can be represented as the sum of a general solution of the homogeneous wave equation and of any particular solution of the inhomogeneous wave equation. In this case, an additional condition does not appear. The general solution of the inhomogeneous wave equation can be sought as a sum of two solutions if the source can be presented as a sum of two functions.

Introduction of an additional condition into the scheme developed in [1] means that solving the wave equation is replaced by solving a system of two differential equations; by the way, in certain consequence. One can see that the initial equation (1) is presented as a sum (Eq. (20))

$$\left[\Delta u-\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}=\frac{1}{c^2}\frac{\partial^2 V}{\partial t^2}\right]_{second}+\left[\Delta V+f(\mathbf{r},t)\right]_{first}=0$$

where subscripts *second* and *first* denote the sequence of solving these equations. Namely, the procedure of solving is as follows:

- the first step is solving the equation noted as *first*
- the second step is calculation of the second partial time derivative of the function V
- the third step is solving the equation noted as *second*

So the method of [1] is not a direct solving of the wave equation but is a solving of the system of two differential partial equations (the procedure defined above transforms the original equation into the system of two equations). Here, we have the alternative:

either we solve the wave equation directly or we solve it according to the procedure defined by [1] – but then the original equation transforms to a system of two equations.

Formally, *lhs* and *rhs* "term-by-term" summation of both equations (which arise from the original equation) gives the equation that is fulfilled ($0 = 0$) for solutions of [1]. This is the main argument of the authors of this work. But it does not allow solution of the system (20) as a wave equation.

For the ODE with constant coefficients the solutions obtained by the two ways coincide. The reason is that both "auxiliary" Y and basic X solutions have the same mathematical structure.*

But the wave equation and the Poisson equation have different mathematical structure and, therefore, the direct solution of the wave equation and the solution partly formed from the solution of the Poisson equation must be different, as shown above.

Thus we should finally conclude that the method of derivation of additional solutions presented in [1] cannot be treated as a new way of

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* This point is more transparent in p-space where both

$$Y = \frac{F(p)}{p^2} + \frac{x_0 p + x}{p^2} \qquad \text{and} \qquad X = \frac{F(p)}{p^2 + a^2} + \frac{x_0 p + x}{p^2 + a^2}$$

can be presented in a form of prime fractions.

solving the wave equation. So the basic statement of the authors of [1] that the wave equation has no unique solution is incorrect.

Appendix. Calculation of the integral (19)

Here, we calculate the integral of Eq. (19)

$$u(X-vt)=\int \frac{v^2\left[\partial^2 V(x',y',z')/\partial x'^2\right]dx'\,dy'\,dz'}{\sqrt{(X-vt-x')^2+\left(1-v^2/c^2\right)\left(y'^2+z'^2\right)}}\;; \qquad (A.1)$$

This integral can be calculated in the cylindrical coordinates x', r, f ; $\left(V=\sqrt{r}\right)$

$$I=\int_0^{\infty} d\varsigma \int_{-\infty}^{+\infty} \frac{dx'}{\left[x'^2+\varsigma\right]^{5/2}} \frac{\left[2x'^2-\varsigma\right]}{\sqrt{\gamma^2(\mathrm{X}-x')^2+\varsigma}}\;; \qquad (A.2)$$

where

$$\gamma=\frac{1}{\sqrt{1-v^2/c^2}} \quad ; \quad \varsigma=\sqrt{y'^2+z'^2} \quad ; \quad \mathrm{X}=X-vt$$

First we calculate the integral over ζ-variable. The integral (A.1) can be split onto two integrals ($I=I_1-I_2$):

$$I_1=\int_{-\infty}^{+\infty} 3x'^2\,dx' \int_0^{+\infty} \frac{d\varsigma}{\left[x'^2+\varsigma\right]^2} \frac{1}{\sqrt{\gamma^2 x'^2(\mathrm{X}-x')^2+\left[x'^2+\gamma^2(\mathrm{X}-x')^2\right]\varsigma+\varsigma^2}}$$

$$I_2=\int_{-\infty}^{+\infty} dx' \int_0^{+\infty} \frac{d\varsigma}{\left[x'^2+\varsigma\right]} \frac{1}{\sqrt{\gamma^2 x'^2(\mathrm{X}-x')^2+\left[x'^2+\gamma^2(\mathrm{X}-x')^2\right]\varsigma+\varsigma^2}}$$

The integral I_1 can be calculated as the integral 2.281 of [3]

$$\int_0^\infty \frac{d\varsigma}{(\varsigma+p)^2\sqrt{a+b\varsigma+c\varsigma^2}} =$$
$$= -\int_{1/p}^0 \frac{\tau d\tau}{\sqrt{c+(b-2pc)\tau+\left(a-bp+cp^2\right)\tau}} \quad ; \quad \left[\tau = \frac{1}{\varsigma+p}\right]; \tag{A.3}$$

and the integral I_2 can be calculated as

$$\int_0^\infty \frac{d\varsigma}{(\varsigma+p)\sqrt{a+b\varsigma+c\varsigma^2}} =$$
$$= -\int_{1/p}^0 \frac{d\tau}{\sqrt{c+(b-2pc)\tau+\left(a-bp+cp^2\right)\tau}} \quad ; \quad \left[\tau = \frac{1}{\varsigma+p}\right]; \tag{A.4}$$

Because in our case,

$$p = x'^2 \quad ; \quad c=1 \quad ; \quad a = \gamma^2 x'^2 \left(\mathrm{X}-x'\right)^2 \quad ; \quad b = x'^2+\gamma^2\left(\mathrm{X}-x'\right)^2$$

we have for the coefficients in the integrands of Eqs. (A.3) and (A.4)

$$R = A + B\tau + C\tau^2,$$

where

$$A = c = 1 \quad ;$$
$$B = b\text{-}2pc = x'^2+\gamma^2\left(\mathrm{X}-x'\right)^2 - 2x'^2 = \gamma^2\left(\mathrm{X}-x'\right)^2 - x'^2 \quad ;$$
$$C = a-bp+cp^2 = \gamma^2 x'^2 \left(\mathrm{X}-x'\right)^2 - \left[\gamma^2\left(\mathrm{X}-x'\right)^2 + x'^2\right]x'^2 + x'^4 = 0$$

So the integrals I_1 and I_2 reduce to:

$$I_1 = \int_{-\infty}^{+\infty} 3x'^2\, dx' \int_0^{1/x'^2} \frac{\tau d\tau}{\sqrt{1+\left[\gamma^2\left(\mathrm{X}-x'\right)^2 - x'^2\right]\varsigma}}$$

$$I_2 = \int_{-\infty}^{+\infty} dx' \int_0^{1/x'^2} \frac{d\tau}{\sqrt{1+\left[\gamma^2\left(\mathrm{X}-x'\right)^2 - x'^2\right]\varsigma}}$$

To calculate the integrals over t variable, we use the integrals 2.222.1 and 2.222.2 of [3]. Using notation $D = [g^2(X - x`)^2 - (x`)^2]$ we have

$$I_1 - I_2 = \int_{-\infty}^{+\infty} dx' \left\{ 3x'^2 \left[\frac{1 + D\tau}{3} - 1 \right] \frac{2\sqrt{1 + D\tau}}{D^2} - \frac{2\sqrt{1 + D\tau}}{D} \right\} \Bigg|_0^{1/x'^2}$$

By substituting the upper and lower limits for t and taking into account that

$$\sqrt{1 + D\frac{1}{x'^2}} = \sqrt{\frac{x'^2 + \gamma^2 (\mathrm{X} - x')^2 - x'^2}{x'^2}} = \frac{\gamma |\mathrm{X} - x'|}{|x'|}$$

we have

$$I_1 - I_2 = 2\int_{-\infty}^{+\infty} \frac{dx' (\gamma|\mathrm{X} - x'| - |x'|)^2}{[\gamma(\mathrm{X} - x') + x']^2 [\gamma(\mathrm{X} - x') - x']^2}; \qquad \text{(A.5)}$$

If the nominator of the integrand in Eq. (A.5) had the form $[g(X - x') - (x')]^2$, the value of the integral would be equal to zero. But the presence of modules requires to consider three intervals of integration of the x'-variable, *i.e.* $x' \in (-\infty; 0)$, $x' \in (0; \mathrm{X})$ and $x' \in (\mathrm{X}, +\infty)$. Therefore, we have three integrals

$$I' = 2\int_{-\infty}^{0} \frac{dx'}{[\gamma(\mathrm{X} - x') - x']^2} = \frac{2}{(\gamma + 1)\gamma \mathrm{X}}$$

$$I'' = 2\int_{0}^{\mathrm{X}} \frac{dx'}{[\gamma(\mathrm{X} - x') + x']^2} = \frac{2}{(\gamma - 1)} \left[\frac{1}{\mathrm{X}} - \frac{1}{\gamma \mathrm{X}} \right]$$

$$I''' = 2\int_{\mathrm{X}}^{+\infty} \frac{dx'}{[\gamma(x' - \mathrm{X}) + x']^2} = \frac{2}{(\gamma + 1)\mathrm{X}}$$

So finally we have

$$I_1 - I_2 = I' + I'' + I''' = \frac{4}{\gamma X}$$

References

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