The Faraday Induction Law and Field Transformations in Special Relativity

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The non-invariance of the Faraday induction law, revealed in [1] through calculation of an e.m.f. along a mathematical line, is further analyzed for integration over a conducting closed circuit. The principal difference of a conductor from a mathematical line is the appearance of internal electromagnetic fields induced by rearranged conduction electrons. In our analysis we distinguish two general cases: 1- the internal electromagnetic fields from the conduction electrons contribute an induced e.m.f.; 2 - the internal fields do not give such a contribution. Case 2 makes a conducting circuit similar to a mathematical line, where the Faraday law is always correct, while the Einstein relativity principle is violated. However, in such a case the violation of special relativity occurs not for a hypothetical model problem, but in physical reality.

Keywords: Faraday induction law, transformation of electromagnetic field, successive Lorentz transformations

1. Introduction

It has been shown in ref. [1] that the mathematical expression for the Faraday induction law

$$e = -\frac{d}{dt}\int_S \vec{B}d\vec{S}\,, \tag{1}$$

does not follow from the Maxwell's equation

$$\oint_\Gamma \vec{E}d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t}d\vec{S}\,, \tag{2}$$

in particular, due to the inequality

$$\int_S \frac{d\vec{B}}{dt}d\vec{S} \neq \frac{d}{dt}\int_S \vec{B}d\vec{S} \tag{3}$$

for $S=S(t)$, $\Gamma=\Gamma(t)$ (here Γ is the closed line enclosing the area S, and $d\vec{l}$ is the element of the circuit Γ). In this connection the expression (1) was tested in [1] for its Lorentz-invariance. It has been found that the Faraday law is not invariant at least at a formal mathematical level, when e.m.f. is calculated through integration over a mathematical line in space.

For integration over a conductor we have additionally to take into account the effect of rearrangement of the conduction electrons under a presence of external electromagnetic fields. The rearranged electrons create their own electric and magnetic fields, which can be negligible outside the conductor, but significant in its inner volume. It is clear that an influence of these fields cannot be analyzed in a general form due to their dependence on many factors (geometry of conductors, configuration of external fields, etc.). Nevertheless, in further consideration we will distinguish two different general cases:

1 – the electromagnetic fields, being created by such rearranged electrons, contribute the force in integrand of Eq. (4):

$$e = \oint_{\Gamma(t)} \left(\vec{E}(\vec{r},t) + \vec{v}(\vec{r}) \times \vec{B}(\vec{r},t) \right) d\vec{l} \ , \qquad (4)$$

which represents a general definition of e.m.f. [2];

2 – the electromagnetic fields from rearranged electrons give a negligible contribution to the total force in Eq. (4). These cases are consecutively analyzed in Sections 2 and 3. The case 2 is especially interesting, because it is similar to integration over a mathematical line, where a violation of the Einstein relativity principle has been found [1]. However, on the contrary to a hypothetical model problem of [1], here we seek a contradiction of the Faraday law and the special relativity for physical reality.

2. The Faraday induction law: the internal electromagnetic fields of conductor contribute an e.m.f. in a circuit

This case is simply realized under substitution of the closed mathematical line in ref. [1] by a conducting rectangular loop A-B-C-D to be placed inside the charged condenser FC (Fig. 1). Then we may imagine that the side AB is electrically connected with the sides U-C and U1-D by means of sliding contacts.

First calculate e.m.f. in the loop A-B-C-D for a laboratory observer (inertial frame K). Conduction electrons in resting parts of the loop BC, CD and DA are rearranged by such a way, so that to give a resultant vanishing electric field E_R inside the conductor:

$$\vec{E}_R = \vec{E}_{\text{ext}} + \vec{E}_{\text{int}} = 0 ,$$

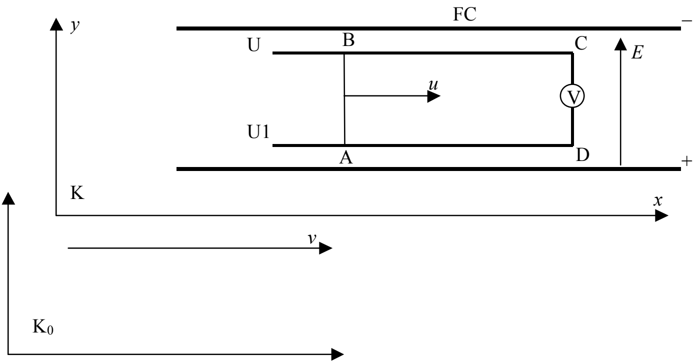


Fig. 1. Conductive loop U-C-D-U1 is short-circuited by a moving conductive bridge AB inside the charged flat condenser FC. V is the voltmeter.

where E_{ext} is the field of external source (FC), and E_{int} stands for the field, created by re-distributed conduction electrons. Since the external magnetic field is absent in the laboratory frame K, that we obtain E_R, B_R=0 inside the segments BC, CD and DA. Due to a homogeneity of field transformations, the same equality $E_r', B_r' = 0$ for the segments BC, CD and DA remains valid for any other inertial observer, and these segments do not contribute an e.m.f. in any inertial frame.

In the moving bridge AB the rearranged conduction electrons induce the electric field

$$(E_{\text{int}})_y = -g_u E \;\; \left(g_u = 1\Big/\sqrt{1-u^2/c^2}\right),$$

as well as the magnetic field along the axis z

$$(B_{\text{int}})_z = -g_u u E/c^2 ,$$

which prevents further rearrangement of the conduction electrons, in order to reach E_R=0. Hence, a resultant force, acting along the axis y per unit conduction electron inside of AB, is

$$F_R = E - g_u E + g_u Eu^2/c^2 = E(1 - 1/g_u).$$

From there the e.m.f. in the circuit A-B-C-D is

$$e_0 = El(1 - 1/g_u) \approx El\,u^2/2c^2, \quad (5)$$

where l is the length of AB (hereinafter we adopt the accuracy of calculations to the order c^{-2}). The magnetic flux across the area ABCD is defined by weak magnetic field, created by moving rearranged conduction electrons in AB outside this segment. If the distance between the segments AB and CD is large, that the magnetic flux across the area ABCD does not change with time under motion of AB. Hence, the Faraday induction law is not correct in the frame K, because of non-vanishing e.m.f. in Eq. (5). The result clearly indicates that, in general, the internal electromagnetic fields contribute an e.m.f., and this effect is dropped in the Faraday law. However, the revealed deflection from the Faraday induction law is impractical, because even under $u \approx 300$ m/s (speed of sound), $El \approx 10^4$ V (potential difference between the plates of FC), $e_0 = 5 \cdot 10^{-9}$ V, that is a negligible value.

Now let us compute the e.m.f. in an inertial frame K_0, wherein the frame K moves at the constant velocity v along the axis x (Fig. 1). In this case for the segments BC, CD and DA, as we mentioned above, $E', B' = 0$, and they can be excluded from further integration.

The electric and magnetic fields inside the segment AB can be found via the field transformations from K to K_0

$$(E'_R)_y = g_v\left[(E_R)_y + v(B_R)_z\right],$$

$$(B'_R)_z = g_v\left[(B_R)_z + v(E_R)_y/c^2\right], \left(g_v = 1\Big/\sqrt{1 - v^2/c^2}\right) \quad (6)$$

taking into account that in K

$$\left(E_R\right)_y = E - g_u E\,,\ \left(B_R\right)_z = -g_u uE/c^2\,. \tag{7}$$

Substituting Eqs. (7) into Eqs. (6), one gets:

$$\left(E'_R\right)_y = g_v E - g_v g_u E - g_v g_u E\,uv/c^2\,,$$

$$\left(B'_R\right)_z = -g_v g_u E\,u/c^2 + g_v E\,v/c^2 - g_v g_u E\,v/c^2\,.$$

The segment AB moves in the frame K_0 at the constant velocity

$$u' = \frac{u+v}{1+uv/c^2}\,. \tag{8}$$

Hence, the e.m.f. in the loop A-B-C-D is equal to

$$\begin{aligned} e &= l\left[\left(E'_r\right)_y - u'\left(B'_r\right)_z\right] = \\ &= g_v lE\left(1-u'v/c^2\right) + g_v g_u lEu\left(u'-v\right)/c^2 - g_v g_u lE\left(1-u'v/c^2\right) \end{aligned} \tag{9}$$

Using the value of u' from Eq. (8), we derive

$$1-u'v/c^2 = \frac{1}{g_v^{\,2}\left(1+uv/c^2\right)},\ u'-v = \frac{u}{g_v^{\,2}\left(1+uv/c^2\right)}.$$

Substituting these equalities into Eq. (9), we obtain

$$e = \frac{lE\left(1-1/g_u\right)}{g_v\left(1+uv/c^2\right)} \approx lE\,u^2/2c^2 = e_0\,. \tag{10}$$

Thus, the e.m.f. in the frames K and K_0 is the same to the adopted accuracy of calculations, that is in agreement with relativistic transformation of e.m.f. [2].

Simultaneously we notice that in the frame K_0 the magnetic flux across the area ABCD, as for mathematical line, is equal to

$$\frac{d}{dt}\Phi = -\frac{uv\sqrt{1-v^2/c^2}\,El}{c^2\left(1+uv/c^2\right)} \approx -lE\frac{uv}{c^2}$$

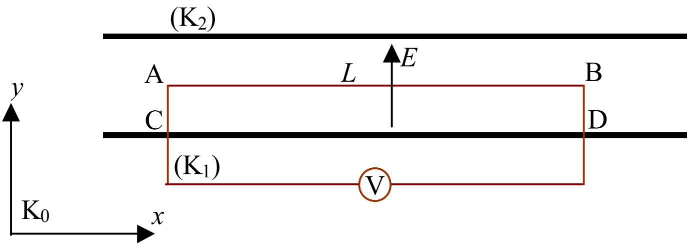


Fig. 2. The inertial frame K_1 is attached to the square conductive loop, while the inertial frame K_2 is attached to the flat condenser. The upper lead AB of the loop lies inside the condenser. The profile leads of loop pass across the tiny holes C and D in the lower plate of condenser.

(see, Eq. (18) of Ref. [1]), which being taken with the opposite sign, differs from Eq. (10). Thus, we reveal that the Faraday induction law is incorrect in K_0. Physical reason for such a violation of this law, as mentioned above, is a contribution of the internal electromagnetic fields of conductor to the induced e.m.f.

3. The Faraday induction law: the internal electromagnetic fields of conductor do not influence an e.m.f. in a circuit

In this section we consider a physical problem as follows. Let there is a conducting rectangular loop with the elongated segment AB inside a flat charged condenser FC (Fig. 2). The thin vertical wires of the loop enter into the condenser via the tiny holes C and D in its lower plate, so that a distortion of electric field E inside the condenser is negligible. An inertial frame K_1 is attached to the loop, while an inertial frame K_2 is attached the FC. There is some external inertial reference frame K_0, wherein the frame K_1 moves at the constant velocity v along the axis x, and the frame K_2 moves at the constant

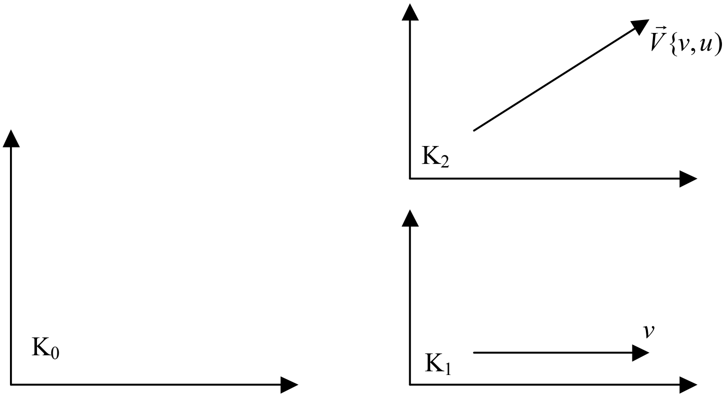


Fig. 3. The motion diagram of inertial reference frames K_1 and K_2 in the third inertial frame K_0.

velocity $\vec{V}\{v,u\}$ in the *xy*-plane (Fig. 3). For such a motion diagram, the frame K_2 moves only along the axis y of K_1. One requires to find an e.m.f. in the loop (indication of the voltmeter V).

One can see that the internal electric field, being induced by re-distributed conduction electrons in the presence of electric field of FC, do not influence the integral (4) along the axis x (segment AB). Besides, the velocity of this segment in K_0 is parallel to its axis A-B, and any internal (or external) magnetic fields do not create a force along this segment. The magnetic forces, being induced by the internal magnetic fields in the sides AC and BD, compensate each other due to equal velocity of these sides in any inertial frame. (Strongly speaking, such a compensation is true to the adopted order of approximation c^{-2}). Hence, an e.m.f. in the circuit is fully determined by the external electromagnetic fields of moving condenser, that makes the loop similar to a mathematical line. At the same time, we have already proved in [1] that through integration over a mathematical line, the Faraday induction law is correct and the Einstein relativity principle is violated. In this connection the problem under consideration looks non-trivially and especially interesting, because a magnetic flux and its time derivative have a property to exist/disappear for different observers (see below). Hence, according to the Faraday law, an e.m.f. in the loop should also exist or disappear in different inertial frames, that means a contradiction with the Einstein relativity principle. Our remaining problem is to demonstrate this conclusion by concrete calculations.

Let us determine the electric and magnetic fields in the frame K_0. We take into account that in the frame of FC (K_2) $B_x = B_y = B_z = 0$, $E_x = E_z = 0$, $E_y = E$, where E is the electric field in space region between the plates of FC. In intermediate calculations, we introduce into consideration the inertial frames K_{2r}, K_{0r}, whose axes x are parallel to $\vec{V}$. Then in K_{2r} $B_x = B_y = B_z = 0$, $E_x = E\sin a, E_y = E\cos a$, E_z=0, where a is the angle of $\vec{V}$ with the axis x of K_0. Substituting these values into the field transformation from K_{2r} to K_{0r},

$$E'_x = E_x; E'_y = g_V\left(E_y + VB_z\right); E'_z = g_V\left(E_z - VB_y\right);$$
$$B'_x = B_x; B'_y = g_V\left[B_y - \left(V/c^2\right)E_z\right]; B'_z = g_V\left[B_z + \left(V/c^2\right)E_y\right], \tag{11}$$

$(g_V = 1/\sqrt{1 - V^2/c^2}\,)$,

we get

$$E_{0rx} = E\sin a\ , B_{0rx} = 0;$$

$$E_{0ry} = g_V E\cos a\ , B_{0ry} = 0;$$

$$E_{0rz} = 0\,,\, B_{0rz} = g_V\left(V/c^2\right)E\cos a\ . \tag{12}$$

Then the electric and magnetic fields in the frame K_0 are (to the order of approximation c^{-2}):

$$E_{0x} = E_{0rx}\cos a - E_{0ry}\sin a =$$
$$E\sin a\cos a - g_V E\cos a\sin a \approx \tag{13a}$$
$$-E\sin a\cos a\,\frac{V^2}{2c^2} = -E\frac{uv}{2c^2};$$

$$E_{0y} = E_{0rx}\sin a + E_{0ry}\cos a =$$
$$E\sin^2 a + g_V E\cos^2 a \approx \tag{13b}$$
$$E + E\frac{V^2}{2c^2}\cos^2 a = E + E\frac{v^2}{2c^2};$$

$$E_{0z} = 0; \tag{13c}$$

$$B_{0x} = B_{0rx}\cos a - B_{0ry}\sin a = 0; \tag{13d}$$

$$B_{0y} = B_{0rx}\sin a + B_{0ry}\cos a = 0; \tag{13e}$$

$$B_{0z} = B_{0rz} = g_V\left(V/c^2\right)E\cos a \approx \frac{vE}{c^2}. \tag{13f}$$

Thus, in K_0 the magnetic field inside the condenser is not equal to zero, and its non-vanishing *z*-component is defined by Eq. (13f). Simultaneously one can see that under motion of FC at the velocity $\vec{V}\{v,u\}$, and motion of loop at the velocity *v* along the axis *x*, the area ABDC between the lower plate of FC and upper line of loop (the gray area in Fig. 4, where the magnetic field B_{0z} exists) decreases with time. Therefore, in the frame K_0 the total time derivative of magnetic flux across the area ABCD decreases with time, too. One can easy find that this time derivative is equal to

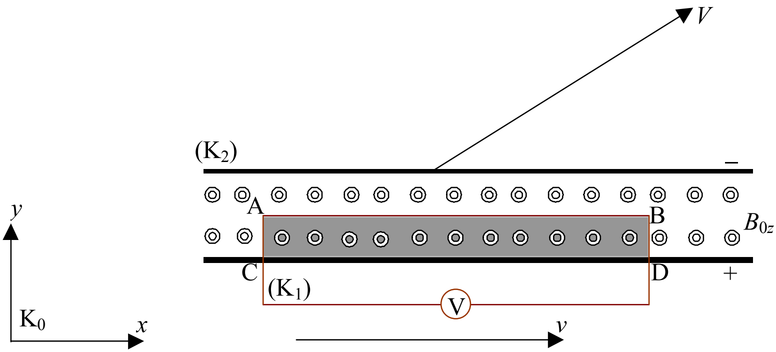
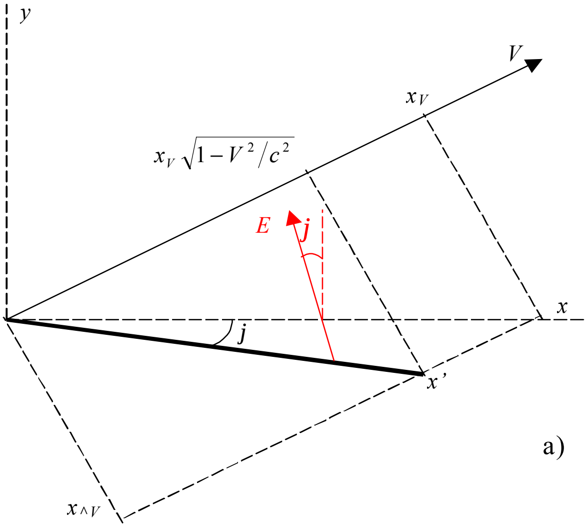


Fig. 4. Observer in the frame K_0 sees that under motion of the frames K_1 and K_2, the gray area ABDC decreases with time and hence, the magnetic flux across the conducting loop also decreases.

$$\frac{d\Phi}{dt} = B_{0z}\frac{dS_{\mathrm{ABDC}}}{dt} \approx -\frac{uvE}{c^2}L\,, \tag{14}$$

where L is the length of the side AB. (In the adopted accuracy of calculations a contraction of this length in K_0 is not significant). Hence, the Faraday induction law requires the appearance of e.m.f. in the loop. Under calculation of e.m.f. we assume that the electric and magnetic fields below the lower plate of FC are negligible. Then we may write the counter-clockwise integral (4) as

$$\begin{aligned}&\oint_{\Gamma}\left(\vec{E}+\vec{v}\times\vec{B}\right)d\vec{l} = \\ &= \int_{DB}\left(E_y + vB_{0z}\right)dy + \int_{BA} E_x dx + \int_{AC}\left(-E_y - vB_{0z}\right)dy\end{aligned}. \tag{15}$$



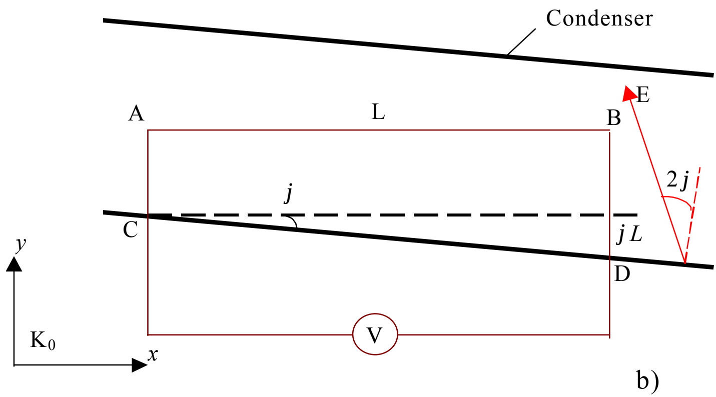


Fig. 5: a – the axis *x* of the frame K_2 constitutes the angle j with the axis *x* of K_0 due to the scale contraction effect in the frame K_0; b – due to this effect, an observer in the frame K_0 fixes that the plates of condenser constitute the angle j with the axis *x* (and with the line AB).

Here we take into account that for the vector $\vec{v}$ to be parallel to the axis x, the integral $\int\limits_{BA}\left(\vec{v}\times\vec{B}\right)d\vec{l}$ is equal to zero. Further, using Eq. (13a), we get:

$$\int\limits_{BA} E_x dx = \frac{uvE}{2c^2}L. \tag{16}$$

Then

$$e = \frac{uvE}{2c^2}L + \left(E + vB_{0z}\right)\left(DB - AC\right). \tag{17}$$

We notice that the segments DB and AC are not equal to each other at any fixed time moment (chosen as integration moment) of the frame K_0. The reason is that the axis x of K_2 is not parallel to the axis x of K_0 due to the scale contraction effect. Indeed, the projection of axis x of K_2 onto the direction to be collinear to the vector $\vec{V}$ (x_V) contracts by g_V times, while the projection of this axis onto the direction to be orthogonal to the vector $\vec{V}$ ($x_{\perp V}$), remains unchanged (see, Fig. 5,a). As a result, the axis x of K_2 is turned out with respect to the axis x of K_0, and the turn angle can be easy calculated:

$$j \approx -uv/2c^2.$$

Thus, the moving condenser in the frame K_0 turns out at the negative angle j, as depicted in Fig. 5,b. As a result, the length of segment DB is longer than the length of segment AC by the value $jL \approx uvL/2c^2$. Hence, integrating over the loop we obtain

$$\begin{aligned} e &= \frac{uvE}{2c^2}L + \left(E + vB_{0z}\right)\left(DB - AC\right) = \\ &\frac{uvE}{2c^2}L + \left(E + \frac{vE}{c^2}\right)\frac{uv}{2c^2}L \approx \frac{uvE}{c^2}L \end{aligned} \tag{18}$$

(here we substitute the magnetic field B_{0z} from Eq. (13f)).

Comparing Eqs. (14), (18), we find that in the frame K_0

$$e = -d\Phi/dt .$$

As we expected, the Faraday induction law is correct for the considered circuit.

We pay attention to the fact that the vector $\vec{E}$ is not orthogonal to the surface of (conducting) plates of FC. Indeed, one can see from Fig. 5 (red fragments) that the angle of electric field with the normal to the plates of FC for an observer in K_0 is

$$j' = j + E_x/E \approx -uv/c^2 = 2j .$$

This result is natural in K_0, because the conduction electrons on the internal surfaces of FC are subject to an action of magnetic force, and its component onto the surfaces is equal to $uB_{0z}\cos j \approx Euv/c^2$. Hence, an equilibrium state of conduction electrons is only possible under non-zero projection of the electric field to these surfaces, which is equal to $-Euv/c^2 = -2jE$. It is just the case of Fig. 5,b.

Further, let us write a transformation from K_0 to K_1:

$$\begin{aligned} &E_{x1} = E_{x0}; E_{y1} = g_v\left(E_{y0} - vB_{z0}\right); E_{z1} = g_v\left(E_{z0} + vB_{y0}\right); \\ &B_{x1} = B_{x0}; B_{y1} = g_v\left(B_{y0} + \left(v/c^2\right)E_{z0}\right); B_{z1} = g_v\left(B_{z0} - \left(v/c^2\right)E_{y0}\right). \end{aligned} \tag{19}$$

Substituting Eqs. (13) into Eqs. (19), one gets:

$$E_{x1} = -E\frac{uv}{2c^2} \tag{20a}$$

$$E_{y1} = g_v\left(E + E\frac{v^2}{2c^2} - \frac{g_V v^2 E}{c^2}\right) \approx E\left(1 - \frac{v^2}{2c^2}\right), \tag{20b}$$

$$E_{z1} = B_{x1} = B_{y1} = 0\,, \tag{20c}$$

$$B_{z1} = g_v \left[\left(v/c^2 \right) E - \left(v/c^2 \right) \left(E + E \frac{v^2}{2c^2} \right) \right] \approx 0\,. \tag{20d}$$

Thus, the magnetic field is the frame K_1 disappears, while the electric field $\vec{E}$ has a non-zero projection onto the axis *x* (Eq. (20a)). A turn of the vector $\vec{E}$ at the angle $j \approx -uv/2c^2$ has a simple physical meaning in special relativity, if we take into account the Thomas-Wigner rotation of the axes of K_1 and K_2 frames for the motion diagram in Fig. 3. The angle of this rotation is [3] $\Omega \approx uv/2c^2 = -j$. It means that the vector $\vec{E}$ is orthogonal to the axis *x* of K_2, and an observer in K_1 frame sees a simple space turn of FC, as depicted in Fig. 6. At the same time, as known in electrostatics, any turn of a charged condenser does not induce an e.m.f. in a closed loop passing through the condenser.

Thus, we have found that in the frame K_0 an e.m.f. in the loop exists, while in the frame K_1 e.m.f. disappears. It occurs in a full accordance with the Faraday induction law: in the frame K_0 the magnetic flux across the area ABCD exists and changes with time, while in the frame K_1 the magnetic flux disappears. However, a presence of e.m.f. in the frame K_0, and its absence in the frame K_1 obviously contradict to the Einstein relativity principle. In another words, a conception about equivalence of all inertial reference frames comes into a deep contradiction with causality: a current in the loop A-B-C-D cannot exist in one inertial frame and be absent in another inertial frame. As a result, we have to recognize that the Faraday induction law, discovered many decades before creation of relativity and being non-invariant in its nature, already disproved this theory.

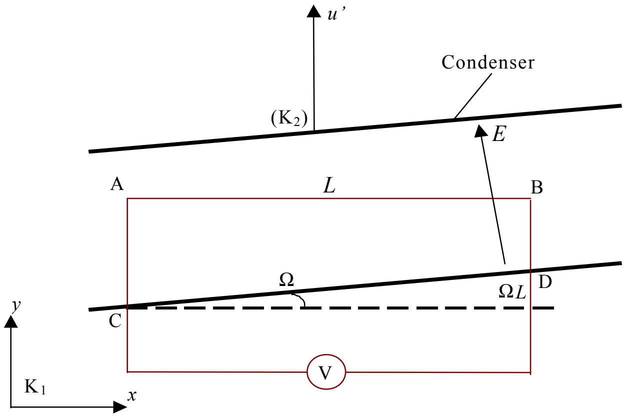


Fig. 6. Due to the Thomas-Wigner rotation between the frames K_1 and K_2, an observer in K_1 sees the space turn of condenser at the angle $\Omega=-j$.

Next problem is to explain the non-invariance of the Faraday induction law in the ether theories, adopting an existence of an "absolute space." Consistent analysis of this problem will be done in a separate paper.

4. Conclusions

The non-invariance of the Faraday induction law with respect to field transformations in special relativity, revealed earlier for e.m.f. along a mathematical line, was further analyzed through integration over a conducting closed circuit. The principal difference of a conductor from a mathematical line is the appearance of internal electromagnetic fields induced by rearranged conduction electrons. In the case where such internal fields contribute an e.m.f. in a conducting circuit, the Faraday induction law is violated, while the Einstein relativity principle remains valid. In these conditions it is especially interesting to analyze conducting circuits in which the rearranged conduction electrons do not contribute an e.m.f. In this case the circuit becomes similar to a mathematical line, where the Faraday induction law is true, while the Einstein relativity principle is violated. A physical problem of just this kind has been found, and actual violation of relativity has been confirmed.

We stress that the latter result does not reveal any mathematical "imperfection" of the relativity theory. It reflects a simple fact that the empirically discovered Faraday induction law is not Lorentz-invariant.

As a result, we conclude that the special theory of relativity, in its application to electromagnetism, was disproved by Faraday as long as several decades before its creation.

Acknowledgements

The author warmly thanks Walter Potzel, Thomas E. Phipps, George Galeczki, Oleg Missevitch and Vladimir Onoochin and Victor Evdokimov for helpful discussions.

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