On the Non-Invariance of the Faraday Law of Induction

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The paper confirms a known fact that the Faraday law of induction does not, in general, follow from Maxwell's equations. It represents a separate and independent physical law, being established experimentally. In this connection the invariance properties of this law are tested. It has been concluded that the Faraday law of induction is not invariant with respect to field transformations in an empty space, and hence, it is incompatible with the Einstein relativity principle.

It is well known that Maxwell's equations are Lorentz-invariant. It is also known that these equations can be written in both differential and integral forms. In particular, for the equation

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{1}$$

the integral form is

$$\oint_\Gamma \vec{E} d\vec{l} = -\int_S \frac{\partial \vec{B}}{\partial t} d\vec{S}, \tag{2}$$

where Γ is the closed line enclosing the area S, and $d\vec{l}$ is the element of the circuit Γ. In case of a fixed line Γ and area S, eq. (2) can be written as

$$\boldsymbol{e} = -\frac{\partial}{\partial t} \int_S \vec{B} d\vec{S}, \tag{3}$$

where

$$\boldsymbol{e} = \oint_\Gamma \vec{E} d\vec{l} \tag{4}$$

is the electromotive force (e.m.f.), and $\Phi = \int_S \vec{B} d\vec{S}$ is the magnetic flux across the area S. For fixed Γ and S eq. (3) represents a direct inference of eq. (1), and hence, it is Lorentz-invariant, too. Often eq. (3) is considered to be closely related to the Faraday induction law. However, it is known that the Faraday law is valid not only for fixed Γ and S, but also for Γ, S, depending on time. In the general case ($S=S(t)$, $\Gamma=\Gamma(t)$), the experimentally established Faraday law

$$\boldsymbol{e} = -\frac{d}{dt} \int_{S(t)} \vec{B} d\vec{S}, \tag{5}$$

does not follow from eq. (2) due to the inequality

$$\frac{d}{dt} \int_{S(t)} \vec{B} d\vec{S} \neq \int_{S(t)} \frac{d\vec{B}}{dt} d\vec{S} \tag{6}$$

for S variable with time. Historically the situation was just the opposite: The Faraday law (5) was discovered experimentally in the first half of 19th century, and it suggested to Maxwell his eqs. (1), (2).

However, we see that, in general, eqs. (2) and (5) are mathematically quite different relationships due to the inequality (6). Thus, the Lorentz-invariance of Maxwell's equations does not yet mean the Lorentz-invariance of the Faraday law (5). Hence, we have to check its Lorentz-invariance separately.

We will carry out the test for the case where $S=S(t)$, $\Gamma=\Gamma(t)$, and the magnetic field is constant:

$$\vec{B}(\vec{r},t) = \vec{B}_0 \tag{8}$$

at least near and within the area S. In addition, we assume a flat area S, *i.e.*, its normal $\vec{n}$ is a constant. Then

$$\boldsymbol{e} = -\frac{d}{dt}\int_{S(t)}\vec{B}d\vec{S} = -\vec{B}_0\frac{d}{dt}\int_{S(t)}d\vec{S} = -\vec{B}_0\vec{n}\frac{dS(t)}{dt}. \tag{9}$$

One can show that for single-valued current in a circuit the e.m.f. is transformed as [1]

$$\boldsymbol{e}' = \boldsymbol{e}\sqrt{1-v^2/c^2}\,, \tag{10}$$

where $\vec{v}$ is a relative velocity between two inertial reference frames (here $\boldsymbol{e}$ belongs to a resting (laboratory) frame). Two important inferences follow from transformation (10): 1) the e.m.f. simultaneously vanishes in all inertial frames; 2) if $\boldsymbol{e}\neq 0$, its sign is the same for all observers. In fact, both these properties of e.m.f. are required by the causality principle. However, the *rhs* of Eq. (10) does not satisfy these requirements under space-time transformations. This is already seen from the simple fact that the magnetic field $\vec{B}_0$ can appear in one inertial reference frame and disappear in another inertial frame. For instance, this occurs in the case where the field $\vec{B}_0$ is produced by a system of charged particles, uniformly moving at the constant speed $\vec{v}$ in the laboratory frame. Then, in the frame attached

to the particles, all of them are stationary, and $\vec{B}_0 = 0$. Thus, the Faraday law of induction is not invariant with respect to the field transformations of SRT.

The same conclusion can be derived in another way, proceeding from the definition of e.m.f. as

$$e = \oint_{\Gamma(t)} \vec{F} d\vec{l} \, , \tag{11}$$

where $\vec{F}$ is the force, acting per unit charge in the loop Γ [1]. For fixed Γ, eq. (11) leads to eq. (4). For $\Gamma = \Gamma(t)$ one needs to add the magnetic force [1]:

$$e = \oint_{\Gamma(t)} \left(\vec{E}(\vec{r}, t) + \vec{v}(\vec{r}) \times \vec{B}(\vec{r}, t) \right) d\vec{l} \, , \tag{12}$$

where $\vec{v}(\vec{r})$ is the velocity of a point with the radius-vector $\vec{r}$ within the circuit Γ in the reference frame considered (here we assume $\vec{v}$ independent of time). In the particular case of a constant magnetic field (eq. (8)),

$$\oint_{\Gamma} \vec{E} d\vec{l} = 0 \, ,$$

as follows from eq. (1) and the Stokes theorem. Hence,

$$e = \oint_{\Gamma} \left(\vec{v}(\vec{r}) \times \vec{B}_0 \right) d\vec{l} \, , \tag{13}$$

which again represents a non-invariant equation: the field $\vec{B}_0$ can appear or disappear in different inertial reference frames, as well as change its sign, while e.m.f. should be transformed according to (10).

Further, defining the e.m.f. in a closed circuit, we have to distinguish an integration over a mathematical line and integration over a matter (conductor, insulator, semi-conductor, *etc.*). The
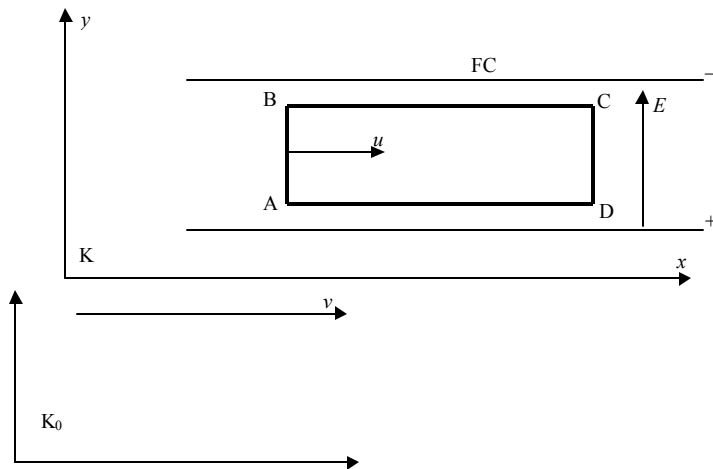
Fig. 1. The square line A-B-C-D with moving side AB inside the charged flat condenser FC.

simplest particular example, demonstrating the general conclusion about non-invariance of the Faraday law of induction, is a closed mathematical line, lying inside the charged condenser.

Let there be a closed square line ABCD in the plane xy with decreasing with time area due to a motion of the left side AB towards to the right side CD with the constant velocity u along the x axis (Fig. 1). Let this loop be placed inside the flat condenser FC, creating the electric field E along the y axis. We assume that the area of the plates of FC is sufficiently large to consider the field $\vec{E}$ as constant in space region near the loop. The condenser FC and the side CD are both at rest in the laboratory frame K. In the frame K the charged plates are stationary, and there is no magnetic field in this frame. Thus, the

magnetic flux across the line A-B-C-D is equal to zero, regardless of the motion of AB, and there is no e.m.f.

Now let us consider the same problem in an inertial frame K_0, wherein the frame K moves along the x axis at the constant velocity v (Fig. 1). In order to find the electric and magnetic fields in K_0, we apply the transformation law for these fields, taking into account that the components in K are:

$$E_x = E_z = 0 \, , \; E_y = E \, , \; B_x = B_y = B_z = 0 \, .$$

Then we obtain the following components of the electromagnetic field in K_0:

$$E_{0y} = \frac{E_y + vB_z}{\sqrt{1 - v^2/c^2}} = \frac{E}{\sqrt{1 - v^2/c^2}} \, , \tag{14}$$

$$B_{0z} = \frac{B_z + (v/c^2)E_y}{\sqrt{1 - v^2/c^2}} = \frac{vE}{c^2\sqrt{1 - v^2/c^2}} \, , \tag{15}$$

$$E_{0x} = E_{0z} = 0 \, , \; B_{0x} = B_{0y} = 0 \, .$$

From a physical viewpoint, the appearance of a non-vanishing component of magnetic field along the z axis of K_0 is explained by a motion of charged plates of FC along the x axis. Thus, the magnetic flux across the area A-B-C-D (S_{ABCD}) is

$$\Phi = \int_{S_{ABCD}} B_{0z} \, dS = \frac{vES_{ABCD}}{c^2\sqrt{1 - v^2/c^2}} \, ,$$

and its time derivative

$$\frac{d}{dt}\Phi = \frac{vE}{c^2\sqrt{1 - v^2/c^2}} \frac{dS_{ABCD}}{dt} \, . \tag{16}$$

The time derivative of S_{ABCD} is proportional to the difference of the velocities of the sides AB and CD along the x axis of the frame K_0. The velocity of CD side is equal to v, while the velocity of AB side is

$$u_0 = \frac{u + v}{1 + uv/c^2} \ .$$

From this the difference of the velocities is derived as

$$\Delta u = \frac{u + v}{1 + uv/c^2} - v = \frac{u\left(1 - v^2/c^2\right)}{1 + uv/c^2} \ .$$

Hence, designating the length of the segment AB as l, we obtain:

$$\frac{dS_{ABCD}}{dt} = -l\Delta u = -\frac{lu\left(1 - v^2/c^2\right)}{1 + uv/c^2}. \tag{17}$$

Substituting eq. (17) into eq. (16), we get the e.m.f. in the circuit A-B-C-D:

$$\boldsymbol{e} = -\frac{d}{dt}\Phi = \frac{uvEl\sqrt{1 - v^2/c^2}}{c^2\left(1 + uv/c^2\right)} \ . \tag{18}$$

Thus, in the frame K_0 the e.m.f. is defined by eq. (18), while it is vanishing in the frame K. This is obvious in contradiction with the Einstein relativity principle.

The same expression (18) can be derived from eq. (13), too. Indeed, in the frame K_0

$$\boldsymbol{e} = \oint_\Gamma \left(\vec{v}(\vec{r}) \times \vec{B}\right) d\vec{l} = u_0 B_z l - v B_z l = \Delta u \frac{vEl}{c^2\sqrt{1 - v^2/c^2}}$$
$$= \frac{uvEl\sqrt{1 - v^2/c^2}}{c^2\left(1 + uv/c^2\right)}$$

,

which coincides with (18).

Thus, we find that the Faraday law of induction is intrinsically incompatible with the Einstein relativity principle. In this short paper we will not discuss possible inferences from this fact, and we will not analyze the case of calculation of e.m.f. under integration over matter. This will be done in later papers. We only mention that the preferential status of SRT among an infinite number of other theories of empty space-time that agree with available experimental observations [2-4] is not warranted.

Acknowledgments

References

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