Addition of velocities and electromagnetic interaction: geometrical derivations using 3D Minkowski diagrams

Călin Galeriu
Dept. of Physics, Worcester Polytechnic Institute
100 Institute Rd., Worcester, MA 01609, USA

This article presents intuitive, geometrical derivations of the relativistic addition of velocities, and of the electromagnetic interaction between two uniformly moving charged particles, based on 2 spatial + 1 temporal dimensional Minkowski diagrams. We calculate the relativistic addition of velocities by projecting the world-line of the particle on the spatio-temporal planes of the reference frames considered. We calculate the real component of the electromagnetic 4-force, in the proper reference frame of the source particle, from the Coulomb force generated by a charged particle at rest. We then obtain the imaginary component of the 4-force, in the same reference frame, from the requirement that the 4-force be orthogonal to the 4-velocity. The 4-force is then projected on a real 3 dimensional space to give the Lorentz force.

Keywords: classical electrodynamics, Minkowski space

©2003 C. Roy Keys Inc.

Introduction

Special Relativity, as presented in today's textbooks, is an advanced mathematical theory. The 1 spatial + 1 temporal dimensional Minkowski diagrams , which initially introduce the Lorentz transformation, the time dilatation and the length contraction, are soon put aside in favor of an approach based on differential calculus and linear algebra. One gets little intuitive understanding of the law of relativistic addition of velocities, and of the fact that "magnetism is a kind of 'second-order' effect arising from relativistic changes in the electric fields of moving charges" [1]. This article goes beyond the usual 2D Minkowski space diagrams [2], and presents intuitive, geometrical derivations of the relativistic addition of velocities, and of the electromagnetic interaction between two uniformly moving charged particles, based on 2 spatial + 1 temporal dimensional Minkowski diagrams. The geometrical approach to special relativity [3] provides us with strong motivation to investigate alternative theories of the electromagnetic interaction, which allow for the variation of the electron's rest mass.

Relativistic addition of velocities

Consider a reference frame K' which is moving with a velocity $\mathbf{V} = V\hat{\mathbf{x}}$ relative to another one K, and a particle moving with a velocity $\mathbf{v}' = v'_x\hat{\mathbf{x}}' + v'_y\hat{\mathbf{y}}' + v'_z\hat{\mathbf{z}}'$ in the reference frame K'. The reference frames are chosen such that their origins and the particle coincide at the space-time point O, as shown in Fig. 1. Notice that $\hat{\mathbf{y}} = \hat{\mathbf{y}}'$ and $\hat{\mathbf{z}} = \hat{\mathbf{z}}'$, because $\mathbf{V}$ has a component only in the x direction. The Oz axis is not plotted, but is similar to the Oy axis. We have to find the velocity $\mathbf{v} = v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}}$

©2003 C. Roy Keys Inc.

of the particle in the reference frame K.

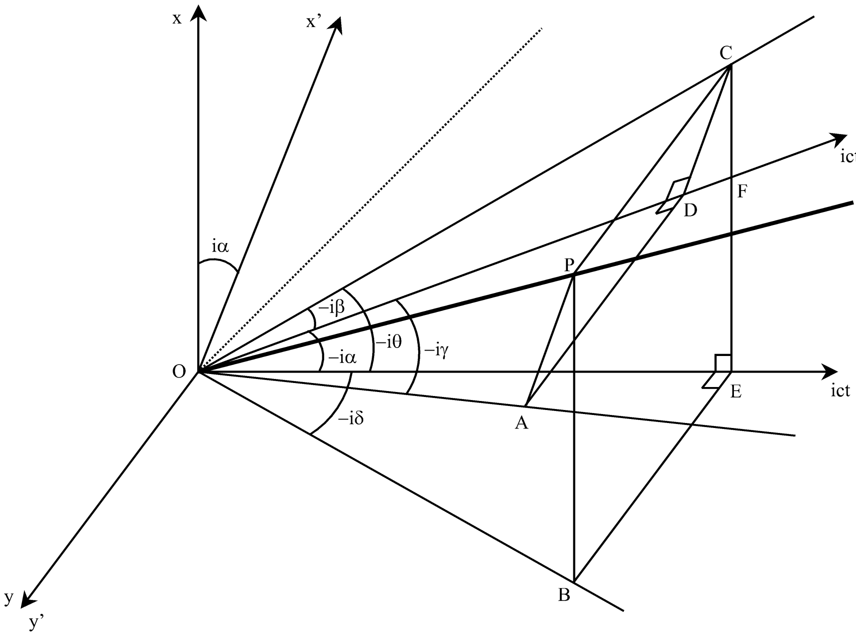

Figure 1. Relativistic addition of velocities. The world-line OP is projected on various spatio-temporal planes. OA is the projection on (y',O,ict'), OB is the projection on (y,O,ict), and OC is the projection on (x,O,ict). The planes (x,O,ict) and (x',O,ict') coincide.

The world-line OP of the particle is projected on the complex planes (x, O, ict), (y, O, ict), (z, O, ict), (x', O, ict'), (y', O, ict'), (z', O, ict'), and the resulting angles from the respective projections give the components of the velocity of the particle in the two reference frames considered. For the situation considered

©2003 C. Roy Keys Inc.

the planes (x, O, ict) and (x', O, ict') coincide. It is seen from Fig. 1 that

$$\frac{EF}{OE} = \frac{V}{i\ c} = \tan(-i\alpha) \Rightarrow \tan(i\alpha) = \frac{i\ V}{c}, \tag{1}$$

$$\frac{DC}{OD} = \frac{v'_x}{i\ c} = \tan(-i\beta) \Rightarrow \tan(i\beta) = \frac{i\ v'_x}{c}, \tag{2}$$

$$\frac{DA}{OD} = \frac{v'_y}{i\ c} = \tan(-i\gamma) \Rightarrow \tan(i\gamma) = \frac{i\ v'_y}{c}, \tag{3}$$

$$\frac{EB}{OE} = \frac{v_y}{i\ c} = \tan(-i\delta) \Rightarrow \tan(i\delta) = \frac{i\ v_y}{c}, \tag{4}$$

$$\frac{EC}{OE} = \frac{v_x}{i\ c} = \tan(-i\theta) \Rightarrow \tan(i\theta) = \frac{i\ v_x}{c}. \tag{5}$$

In order to express the velocities v_x and v_y as functions of V,v'_x, and v'_y we need to express the angles $i\delta$ and $i\theta$ as functions of $i\alpha$,$i\beta$, and $i\gamma$.

In the plane (x, O, ict) of the Lorentz boost the addition of velocities is based on the addition of angles:

$$i\theta = i\alpha + i\beta, \tag{6}$$

$$\tan(i\theta) = \tan(i\alpha + i\beta) = \frac{\tan(i\alpha) + \tan(i\beta)}{1 - \tan(i\alpha)\tan(i\beta)}. \tag{7}$$

From (7), by substitution of the tangents (1)-(5), it follows that

$$v_x = \frac{V + v'_x}{1 + V v'_x/c^2}. \tag{8}$$

©2003 C. Roy Keys Inc.

Two rectangles, $APCD$ and $BPCE$, result from the projection process. It is evident that

$$\frac{CP}{OC} = \frac{EB}{OC} = \frac{EB}{OE}\frac{OE}{OC} = \tan(-i\delta)\cos(-i\theta), \tag{9}$$

$$\frac{CP}{OC} = \frac{DA}{OC} = \frac{DA}{OD}\frac{OD}{OC} = \tan(-i\gamma)\cos(-i\beta). \tag{10}$$

From (9)-(10) it follows that

$$\tan(i\delta) = \frac{\cos(i\beta)\tan(i\gamma)}{\cos(i\alpha + i\beta)} = \frac{\tan(i\gamma)}{\cos(i\alpha)[1 - \tan(i\alpha)\tan(i\beta)]}. \tag{11}$$

By substitution of the tangents (1)-(5) and of $\cos(i\alpha) = [1 + \tan^2(i\alpha)]^{-1/2}$ we get

$$v_y = \frac{v'_y(1 - V^2/c^2)^{1/2}}{1 + Vv'_x/c^2}. \tag{12}$$

A similar expression is obtained for the v_z component.

Electromagnetic interaction between two uniformly moving charged particles

Consider two charged particles (with charges Q_1 and Q_2) at some arbitrary positions, moving with arbitrary, but uniform, velocities. We orient our 3D reference frame in such a way that the first particle (which generates the field) is initially at the origin, moving along the Ox axis with velocity $\mathbf{V} = V\hat{\mathbf{x}}$, and the vector $\mathbf{R} = R\hat{\mathbf{R}} = R[\cos(\theta)\hat{\mathbf{x}} + \sin(\theta)\hat{\mathbf{y}}]$ connecting the two particles lies in the (x, O, y) plane. The angle between $\mathbf{R}$ and the Ox axis is θ. The second particle (subject to the electromagnetic

©2003 C. Roy Keys Inc.

field generated by the first one) is moving with velocity $\mathbf{v} = v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}} + v_z\hat{\mathbf{z}}$. The world-lines of the particles (CO and AG) are plotted in Fig. 2.

Analytical calculation of the Lorentz force The electric field (in Gaussian units) generated by the first particle at the position of the second particle is [4, 5]

$$\mathbf{E} = \frac{Q_1}{R^2}(1 - \frac{V^2}{c^2})[1 - \frac{V^2}{c^2}\sin^2(\theta)]^{-3/2}\hat{\mathbf{R}}. \tag{13}$$

The magnetic field generated by the first particle is

$$\mathbf{H} = \frac{1}{c}\mathbf{V} \times \mathbf{E}. \tag{14}$$

The Lorentz force acting on the second particle is

$$\mathbf{F} = Q_2\mathbf{E} + \frac{Q_2}{c}\mathbf{v} \times \mathbf{H}. \tag{15}$$

From (13)-(15) the Cartesian components of the force are obtained [6]

$$F_x = \frac{Q_1Q_2}{R^2}(1 - \frac{V^2}{c^2})[1 - \frac{V^2}{c^2}\sin^2(\theta)]^{-3/2}[\cos(\theta) + \sin(\theta)\frac{v_yV}{c^2}], \tag{16}$$

$$F_y = \frac{Q_1Q_2}{R^2}(1 - \frac{V^2}{c^2})[1 - \frac{V^2}{c^2}\sin^2(\theta)]^{-3/2}\sin(\theta)(1 - \frac{v_xV}{c^2}), \tag{17}$$

$$F_z = 0. \tag{18}$$

Geometrical derivation of the Lorentz force The force components (16)-(18) can be obtained in a more graphical way, if we start with the Coulomb force generated by a charged particle at rest. One key assumption or experimental fact is that

©2003 C. Roy Keys Inc.

in a frame where all the source charges producing an electric field $\mathbf{E}$ are at rest, the force on a charge q is given by $\mathbf{F} = q\mathbf{E}$ independent of the velocity of the charge in that frame [7].

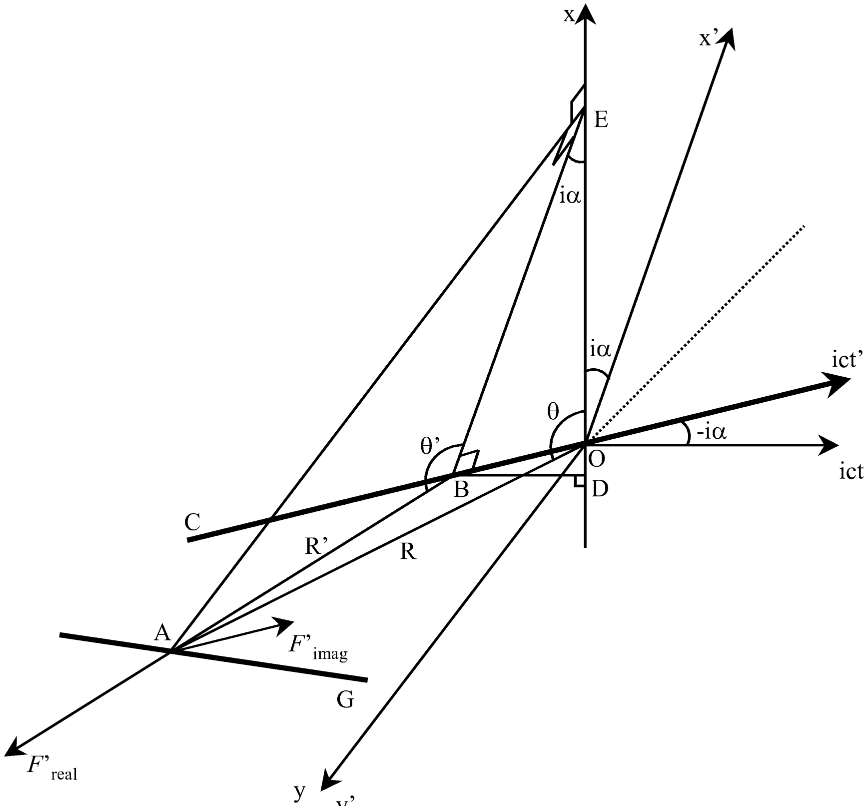

Figure 2. Electromagnetic interaction between two uniformly moving charged particles. CO is the world-line of the source particle, and AG is the world-line of the test particle. In the proper reference frame of the source particle there is a Coulomb force directed along the BA radial direction.

The reference frame K' in which the source particle is at rest is moving with velocity $\mathbf{V}$ relative to the original frame K.

In the reference frame K the particle at A is observed to interact with the particle at O. The distance between particles is R, the length of the segment OA.

In the reference frame K' the particle at A is observed to interact with the particle at B, where the segment BA is a position vector $\mathbf{R}'$ parallel to the plane (x', O, y'). The following construction gives the position of point B: the segment AE is parallel to Oy and intersects the Ox axis at E, whereas the segment EB is parallel to Ox' and intersects the world-line CO at B. BD projects the point B on the Ox axis at D.

Relative to K', the particle at B exerts a radial Coulomb force on the particle at A. This force (in Gaussian units) is

$$\mathbf{F}' = \frac{Q_1 Q_2}{R'^2}\hat{\mathbf{R}}', \tag{19}$$

where $\mathbf{R}' = R'\hat{\mathbf{R}}' = R'[\cos(\theta')\hat{\mathbf{x}}' + \sin(\theta')\hat{\mathbf{y}}']$.

The key point in getting the force $\mathbf{F}$ in the reference frame K is to notice that the force, in any reference frame considered, is given by the projection on the real 3D space of that frame of the 4-force $\mathbb{F}$ (which is a Minkowski-space vector), that is

$$\mathbb{F} = \mathbb{F}_{\text{real}} + \mathbb{F}_{\text{imag}} = \gamma(v)\mathbf{F} + \gamma(v)\frac{P}{c}\hat{\mathbf{i}}, \tag{20}$$

$$\mathbb{F} = \mathbb{F}'_{\text{real}} + \mathbb{F}'_{\text{imag}} = \gamma(v')\mathbf{F}' + \gamma(v')\frac{P'}{c}\hat{\mathbf{i}}', \tag{21}$$

where $\gamma(v) = (1 - v^2/c^2)^{-1/2}$, $P = \mathbf{F}\cdot\mathbf{v}$, and $\hat{\mathbf{i}}$ is an imaginary unit vector ($\hat{\mathbf{i}}\cdot\hat{\mathbf{i}} = -1$) along the time axis.

©2003 C. Roy Keys Inc.

We will obtain the 4-force $\mathbb{F}$ from its real and imaginary components ($\mathbb{F}'_{\text{real}}$ and $\mathbb{F}'_{\text{imag}}$) in the reference frame K', then we will decompose the same 4-force into its real and imaginary components ($\mathbb{F}_{\text{real}}$ and $\mathbb{F}_{\text{imag}}$) in the reference frame K. The Lorentz force we are looking for is just $\mathbf{F} = \mathbb{F}_{\text{real}}/\gamma(v)$.

From (19)-(21) it follows that

$$\mathbb{F}'_{\text{real}} = \gamma(v')\frac{Q_1 Q_2}{R'^2}\hat{\mathbf{R}}'. \tag{22}$$

To get the imaginary component $\mathbb{F}'_{\text{imag}}$ we use the orthogonality between the 4-force and the 4-velocity, $\mathbb{F} \cdot \mathbb{V} = 0$, where the 4-velocity is $\mathbb{V} = \gamma(v')\mathbf{v}' + \gamma(v')c\hat{\mathbf{i}}'$. This orthogonality condition leads to

$$\mathbb{F}'_{\text{imag}} = \gamma(v')\frac{Q_1 Q_2}{R'^2}\frac{v'_{\text{rad}}}{c}\hat{\mathbf{i}}', \tag{23}$$

where the radial component of the velocity is

$$v'_{\text{rad}} = \mathbf{v}' \cdot \hat{\mathbf{R}}' = v'_x \cos(\theta') + v'_y \sin(\theta'). \tag{24}$$

It is possible to give a geometrical interpretation [3] to (22)-(23), within the framework of time-symmetric electrodynamics [8]. The expressions (22)-(23) exhibit an explicit dependence on the velocity of the test particle, and an implicit dependence on the velocity of the source particle (we work in its proper reference frame). But, from a *geometrical* point of view, a point in Minkowski space is just a fixed point, it does not have a velocity! The theory therefore considers interactions between segments of finite length along the world-lines of the particles. A key result of the theory [3] is that, when accelerated motion is considered, the rest mass of the electron is no longer constant,

©2003 C. Roy Keys Inc.

but its averaged variation is still null. Therefore the geometrical approach to special relativity provides us with strong motivation to investigate alternative theories of the electromagnetic interaction, which allow for the variation of the electron's rest mass. Other authors are investigating theoretically [9, 10, 11, 12] or experimentally [13, 14, 15] alternative expressions for the electromagnetic force. See [16] for a review article.

The components of the force $\mathbf{F}$ in the reference frame K are given by the projection of the 4-force $\mathbb{F}$ on the 3D real space of K. An easy way to do this is to notice that we can decompose $\mathbb{F}'_{\text{real}}$ (which has the direction of the segment BA) and $\mathbb{F}'_{\text{imag}}$ (which has the direction of the segment BO) into sums of 4-vectors, each of the 4-vectors being parallel to one of the axes of the reference frame K:

$$\mathbf{BA} = \mathbf{BD} + \mathbf{DE} + \mathbf{EA}, \tag{25}$$

$$\mathbf{BO} = \mathbf{BD} + \mathbf{DO}. \tag{26}$$

Because these expansions do not involve any component along the Oz axis, this simply means that $F_z = 0$. The projections of the 4-force on the Ox and Oy axes are

$$\gamma(v)F_x = F'_{\text{real}}\frac{DE}{BA} + F'_{\text{imag}}\frac{DO}{BO}, \tag{27}$$

$$\gamma(v)F_y = F'_{\text{real}}\frac{EA}{BA}, \tag{28}$$

where F'_{real} and F'_{imag} are the magnitudes of the 4-vectors (22)-(23). The lengths of the various segments needed above are as follows:

$$AO = R, \tag{29}$$

©2003 C. Roy Keys Inc.

$$EA = AO\sin(\theta) = R\sin(\theta), \tag{30}$$

$$OE = AO\cos(\theta) = R\cos(\theta), \tag{31}$$

$$BE = OE\cos(i\alpha) = R\cos(\theta)\cos(i\alpha), \tag{32}$$

$$DE = BE\cos(i\alpha) = R\cos(\theta)\cos^2(i\alpha), \tag{33}$$

$$AB = (AE^2 + BE^2)^{1/2} = R\cos(i\alpha)[1 + \tan^2(i\alpha)\sin^2(\theta)]^{1/2} = R', \tag{34}$$

We also notice that $DO/BO = \sin(-i\alpha)$. The force components in (27)-(28) become

$$F_x = \frac{\gamma(v')}{\gamma(v)}\frac{Q_1Q_2}{R^2}\frac{\cos(\theta)}{\cos(i\alpha)[1 + \tan^2(i\alpha)\sin^2(\theta)]^{3/2}} + i\frac{\gamma(v')}{\gamma(v)}\frac{Q_1Q_2}{R^2}\frac{v'_{\mathrm{rad}}}{c}\frac{\sin(-i\alpha)}{\cos^2(i\alpha)[1 + \tan^2(i\alpha)\sin^2(\theta)]}, \tag{35}$$

$$F_y = \frac{\gamma(v')}{\gamma(v)}\frac{Q_1Q_2}{R^2}\frac{\sin(\theta)}{\cos^3(i\alpha)[1 + \tan^2(i\alpha)\sin^2(\theta)]^{3/2}}. \tag{36}$$

We can also calculate

$$\sin(\theta') = \frac{EA}{AB} = \frac{\sin(\theta)}{\cos(i\alpha)[1 + \tan^2(i\alpha)\sin^2(\theta)]^{1/2}}, \tag{37}$$

$$\cos(\theta') = \frac{BE}{AB} = \frac{\cos(\theta)}{[1 + \tan^2(i\alpha)\sin^2(\theta)]^{1/2}}. \tag{38}$$

If the velocity of the particle at A has the components v_x, v_y, v_z, as measured in the reference frame K, and K is moving with the velocity $\mathbf{V}' = -V\hat{\mathbf{x}}'$ relative to K', then the particle will have the following components of the velocity in the reference frame K':

$$v'_x = \frac{v_x - V}{1 - Vv_x/c^2}, \tag{39}$$

©2003 C. Roy Keys Inc.

$$v'_y = \frac{v_y(1 - V^2/c^2)^{1/2}}{1 - Vv_x/c^2}, \tag{40}$$

$$v'_z = \frac{v_z(1 - V^2/c^2)^{1/2}}{1 - Vv_x/c^2}. \tag{41}$$

With these components we find that

$$\gamma(v') = \left(1 - \frac{{v'_x}^2 + {v'_y}^2 + {v'_z}^2}{c^2}\right)^{-1/2} = \gamma(v)\frac{1 - Vv_x/c^2}{(1 - V^2/c^2)^{1/2}}, \tag{42}$$

and the radial velocity (24), with the help of (37)-(38), becomes

$$v'_{\rm rad} = \frac{(v_x - V)\cos(i\alpha)\cos(\theta) + v_y(1 - V^2/c^2)^{1/2}\sin(\theta)}{(1 - Vv_x/c^2)\cos(i\alpha)[1 + \tan^2(i\alpha)\sin^2(\theta)]^{1/2}}. \tag{43}$$

Substituting $\gamma(v')$ and v'_{rad} in (35)-(36), and then using the fact that $\sin(i\alpha) = i(V/c)\gamma(V)$, $\cos(i\alpha) = \gamma(V)$, and $\tan(i\alpha) = iV/c$, we finally obtain the components in (16)-(17).

Conclusions

We have given geometrical derivations, based on 3D Minkowski diagrams, of the relativistic addition of velocities, and of the electromagnetic interaction between two uniformly moving charged particles. The geometrical approach to special relativity provides us with strong motivation to investigate alternative theories of the electromagnetic interaction, which allow for the variation of the electron's rest mass.

©2003 C. Roy Keys Inc.

Acknowledgments

It is a pleasure to thank John Davies for valuable comments and encouragement.

References

[1] E.M. Purcell, *Electricity and Magnetism*, McGraw-Hill, New York, NY, 1985, p. 238.

[2] C. Galeriu, An Introduction to Minkowski Space, *Mathematical Spectrum*, in print.

[3] C. Galeriu, Time-Symmetric Action-at-a-Distance Electrodynamics and the Structure of Space-Time, *Physics Essays* **13** (2000), 597.

[4] L. Landau and E. Lifshitz, *The Classical Theory of Fields*, Addison-Wesley, Cambridge, MA, 1951, p. 97.

[5] O.D. Jefimenko, Direct calculation of the electric and magnetic fields of an electric point charge moving with constant velocity, *Am. J. Phys.* **62** (1994), 79.

[6] W.G.V. Rosser, *Classical Electromagnetism via Relativity*, Plenum Press, New York, NY, 1968, p. 32.

[7] J.D. Jackson, *Classical Electrodynamics*, Wiley, New York, NY, 1975, p. 580.

[8] J.A. Wheeler and R.P. Feynman, Interaction with the Absorber as the Mechanism of Radiation, *Rev. Mod. Phys.* **17** (1945), 157.

[9] A. K. T. Assis, Changing the Inertial Mass of a Charged Particle, *J. Phys. Soc. Japan* **62** (1993), 1418.

[10] A. K. T. Assis, *Weber 's Electrodynamics*, Kluwer Academic, Dordrecht, 1994.

©2003 C. Roy Keys Inc.

[11] P. Moon, D.E. Spencer, A.S. Mirchandaney, U. Shama, P.J. Mann, The Derivation of a New Gaussian Equation for the Force Between Moving Charges from Fundamental Postulates, *Phys. Essays* **12** (1999), 153.

[12] O. Costa de Beauregard, Effet Mikhailov généralisé: inertie induite par un 4-potentiel irrotationnel, *Ann. Fond. Louis de Broglie* **25** (2000), 129.

[13] P. Graneau and N. Graneau, *Newtonian electrodynamics*, World Scientific, River Edge, NJ, 1996.

[14] V. F. Mikhailov, The Action of an Electrostatic Potential on the Electron Mass, *Ann. Fond. Louis de Broglie* **24** (1999), 161.

[15] V. F. Mikhailov, Influence of an electrostatic potential on the inertial electron mass, *Ann. Fond. Louis de Broglie* **26** (2001), 633.

[16] V. V. Dvoeglazov, Historical Note on Relativistic Theories of Electromagnetism, *Apeiron* **5** (1998), 69.

©2003 C. Roy Keys Inc.