Darboux Transformations and Isospectral Potentials in Quantum Mechanics

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We show that the Darboux transformations allow to construct generalized isospectral potentials for a given standard potential, this in the frame of the Schödinger equation. An application is made for the Hulthén interaction.

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1.- Introduction

In the one-dimensional stationary case, the Schrödinger equation is given by [1]:

$$-\frac{d^2}{dx^2}y + u(x)y = l\,y \tag{1}$$

which is written in natural units taking $\hbar^2/2m = 1$. The values of l represent the energy spectrum allowed for determined boundary conditions and corresponding to the standard potential $u(x)$. In this work, we show the very useful procedure of Darboux [2-4] as to generalize any specific standard potential and to generate in this way new interaction models with the same energy levels. After the procedure, equation (1) is transformed to

$$-\frac{d^2}{dx^2}j + U(x)j = l\,j \tag{2}$$

and we say that the potential $U(x)$ is isospectral to $u(x)$. The Darboux transformation (DT) (explained in Sec. 2) is related to the Sturm-Liouville theory [5,6], and it is easy to see the implicit presence of DT in supersymmetric quantum mechanics [1,4,7-12]. Finally, in Sec. 3 we use DT to construct generalized isospectral potentials for the Hulthén interaction [1,13-16].

2.- Darboux transformations

We suppose that (1) permits the particular solution y_1 for the eigenvalue l_1:

$$-y_1'' + u(x)y_1 = l_1 y_1\,, \tag{3}$$

then we employ y_1 as a ``seed'' to construct the DT [2-4,17]:

$$j(x) = y' - s_1(x)y, \qquad s_1 = \frac{d}{dx}\ln y_1, \tag{4}$$

then (1) adopts the structure (2) with the generalized potential:

$$U(x) = u(x) - 2\frac{d}{dx}s_1 \tag{5}$$

and the same energy spectrum. That is, the Schrödinger equation is covariant with respect to the DT. Selecting other ``seed functions'' we generate many DT and therefore a great family of generalized interactions isospectral to $u(x)$.

The DT is a mathematical technique that applied in quantum mechanics can be interpreted as a supersymmetry [1,4,7-12,18,19]. In [20-26] it is employed the Riccati equation [5,27,28] to construct generalized potentials isospectral to the free particle, harmonic oscillator, hydrogenic, Morse [1,9,11,29-33] and Hulthén [1,13-16,34] interactions. It is possible to obtain all these generalized potentials via an adequate DT, which show the power of the Darboux's procedure. As an example, in the next Section we apply the DT to Hulthén model, as an alternative method to the Riccati equation, in the search of new quantum mechanical potentials for a given spectrum.

3.- Generalized Hulthén potentials.

The Hulthén potential [13] is a useful interaction model that has been used extensively in different areas of Physics, including nuclear [35] and atomic physics [36], due to the fact that it yields to closed analytic solutions for the *s* waves [15,37]. Its expression is given by [15,26]:

$$u(r) = -\frac{V_0}{e^{Ar} - 1} \tag{6}$$

where A and V_0 are positive constants such that $V_0 > A^2$. It is clear that [1,10] the Schrödinger equation for the radial wave function $R(r)$ takes the form (1) with $R = \frac{1}{r} y$ in the case $l = 0$.

According with equation (4) the DT depends on the function y_1 selected, so that now we will show two options verifying (3):

a) y_1 is the usual wave function [15,37] for the ground state associated to (6):

$$y_1(r) = \left(1 - e^{-Ar}\right) e^{-kr}, \qquad l_1 = -k^2 \tag{7}$$

where $k = \frac{1}{2A}(V_0 - A^2) > 0$.

Employing (7), the relations (4) and (5) lead to the generalized potential of Hulthén (Darboux potential):

$$U_m = -\frac{V_0}{e^{Ar} - 1} + \frac{2A^2 e^{Ar}}{\left(e^{Ar} - 1\right)^2} \tag{8}$$

equivalent to (36) of [26], which is isospectral to (6).

b) Another possibility is to utilize:

$$y_1(r) = \frac{1}{b}\left(1 - e^{-Ar}\right) e^{-kr} \left[g + b \int \frac{e^{2kr}}{\left(1 - e^{-Ar}\right)^2} dr \right] \tag{9}$$

satisfying (3) with $l_1 = -k^2$, g and b are arbitrary constants.

Expressions (4), (5) and (9) imply the following generalized Hulthén interaction:

$$U_g = U_m + \frac{2b}{r}\left[\frac{b}{r} - 2\left(k - \frac{A}{e^{Ar} - 1} \right) \right], \tag{10}$$

with $r = b(1 - e^{-Ar}) e^{-kr} y_1$, corresponding to (11) of [26] obtained via Riccati equation.

Finally, we conclude by remarking that the Darboux procedure used to generalize a standard potential is a straightforward method, which is far simpler than equivalent approaches employed to find new families of isospectral known potentials [38].

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