General Spherically Symmetric Solutions of Einstein Vacuum Field Equations With L

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A one-parameter family of inequivalent non-static spacetimes for a point mass in the presence of cosmological constant are investigated.

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I. Intoduction

According to Birkhoff's theorem the Schwarzschild metric for a vacuum spherically symmetric gravitational field with $\Lambda = 0$ (G=c=1),

$$ds^2 = (1 - \frac{2M}{r})dt^2 - (1 - \frac{2M}{r})^{-1} dr^2 - r^2 d\Omega \qquad (1)$$

and with $\Lambda \neq 0$ the Schwarzschild- deSitter metric,

$$ds^2 = (1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2)dt^2 - (1 - \frac{2M}{r} - \frac{\Lambda}{3} r^2)^{-1} dr^2 - r^2 d\Omega \qquad (2)$$

uniquely exhibit the spacetime of a point mass. Eq.(1) has a coordinate singularity at $r = 2M$ and an intrinsic singularity at $r = 0$, while Eq.(2) has two coordinate singularities at $r \approx 2M$ and $r \approx \sqrt{\frac{3}{\Lambda}}$, and an intrinsic singularity at $r = 0$. The intrinsic singularity is irremovable and this is indicated by diverging the Riemann tensor scalar invariant [1]

$$R^a{}_{bcd} R_a{}^{bcd} = \frac{48M^2}{r^6}. \qquad (3)$$

The general enthusiasm for validity of uniqueness of (1) and (2) has been hesitated by providing an infinity of spacetimes that satisfy the postulates for a point mass in several frameworks. One is in the form of general solutions of the spherically symmetric vacuum Einstein field equations that are a one-parameter family physically differing in their limiting lower bound of the surrounding surface area of the source [2]. Other one is in the framework of general Birkhoff's theorem that are a two-parameter family physically differing in their limiting lower bound of the hypersurfaces $\{t = r = \text{const.}\}$[3]. Another one is known as alternative spacetime for point mass that are one-parameter family physically differing in their limiting acceleration of a radially approaching test particle [4,5].

For $\Lambda = 0$, among these works, a common general class of solutions may be expressed, in terms of a dimensionless parameter a, within the interval of $r \geq 0$ by the following line element:

$$ds^2 = (1-\frac{2M}{r+aM})dt^2 - (1-\frac{2M}{r+aM})^{-1}dr^2 - (r+aM)^2 d\Omega \qquad (4)$$

where the source is located at $r = 0$.

We primarily discuss the apparent objection that (4) presents subspaces of the Schwarzschild space in Sec.II. Though it looks like a linear transformation of the radial coordinate, but it is not so and a does affect the curvature of spacetime. By deriving the equation of precession of perihelia and bending of light in a gravitational field given by Eq.(4) and comparing the obtained results with observations, we come to the conclusion that a may take small as well as large values up to 10^3. Even though $a > 2$ may solve both coordinate and intrinsic singularities, still we need to consider $\Lambda > 0$, because recent observations of type Ia supernovae do indicate its existence [6]. On the other hand it has been shown that in the presence of cosmological constant, using a coordinate system that asymptotically leads to a static metric cannot serve as a comoving frame [7]. Then a non-static solution of this system as an alternative for Eq. (2) has been proposed that has the following form [8]:

$$ds^2 = \frac{\sqrt{(1-\frac{2M}{r}-\frac{\Lambda}{3}r^2)^2+\frac{4\Lambda}{3}r^2}+(1-\frac{2M}{r}-\frac{\Lambda}{3}r^2)}{2}dt^2 - e^{2\sqrt{\frac{\Lambda}{3}}t}$$
$$(\frac{\sqrt{(1-\frac{2M}{r}-\frac{\Lambda}{3}r^2)^2+\frac{4\Lambda}{3}r^2}+(1-\frac{2M}{r}-\frac{\Lambda}{3}r^2)}{2})^{-1}dr^2 - r^2 d\Omega \qquad (5)$$

where $r = re^{\sqrt{\frac{\Lambda}{3}}t}$. Evidently this result is free of any singularity for $r > 0$, and is singular at $r = 0$. We show in Sec.III that it is indeed an intrinsic singularity. By making use of the techniques presented in [2,3,7], a general class of non-static solutions will be obtained in Sec.IV that has the functional form of Eq. (5) except that

$r = (r + aM)e^{\sqrt{\frac{\Delta}{3}}t}$. These solutions are smooth and finite everywhere even at $r = 0$. Obviously they should be checked for the completeness requirements before we may call them non-singular.

The geodesic equations for a freely falling material particle in the general case are solved in Sec.V and results in a potential field that is very large but finite near the origin. Finally some concluding remarks will come at the end.

II. Case L = 0, a ¹ 0

Since Eq. (4) transforms to Eq. (1) by simply replacing $r' = r + aM$ with the range of $r' \geq aM$, this may cause a confusion that (4) is a subspace of (1). Usually the proof of completenss for a pseudo-Riemannian sapce is not an easy task. The flaw in this argument will be shown by a Riemannian counter-example. Let us consider a two-space, R^2 of all points with coordinates (r,q) such that its line element is $ds^2 = dr^2 + r^2 dq^2$ where $q = 0$ is identified with $q = 2p$ and $r = 0$ is included. This plane is complete and non-singular. Also consider another two-space, R'^2 of all points with coordinates (r,q) and the line element, $ds^2 = dr^2 + (r + a)^2 dq^2$ where the range of (r,q) is the same as R^2. If we transform $r' = r + a$ this line element will transform to $ds^2 = dr'^2 + r'^2 dq^2$ with $r' \geq a$ that apparently it means $R'^2 \subset R^2$. We will show that indeed this is a false conclusion. Consider a subspace of R^2 and R'^2 by restricting $r < b$. The surface area of the open set $R^2(r < b)$ is pb^2 while the surface area of the open set $R'^2(r < b)$ is $p(b^2 + 2ab)$. This means that for finite b we always have $R^2(r < b) \subset R'^2(r < b)$. If we take the limit $b \to \infty$ this leads to $R^2(r<b)\to R^2$, $R'^2(r<b)\to R'^2$ and $R^2 \subset R'^2$. Since R^2 is complete and nonextendable then there is no way except to conclude that $R^2 = R'^2$. This counter-example shows that how the conclusion that the spaces of (4) are subspaces of the Schwarzschild space may be impulsory. Indeed (1) is a special case of (4) for $a=0$. However it is worth to notice that (1) and (4) both are in the same Schwarzschild coordinates, which manifestly have different forms. In the case of any transformation of (4) by $r' = r + aM$ requiers that (1) be demonstrated in this new coordinate too, which means replacing r by $r' - aM$ in (1). Thus in the new coordinate system also (1) and (4) have different forms.

Next we show how the spaces of (4) for $a>2$ are complete. A manifold endowed with an affine or metric is said to be geodesically complete if all geodesics emanating from any point can be extended to infinite values of the affine parameters in both directions. For a positive definite metric the geodesic completeness and metric completeness are equivalent [9]. Focusing on $a>2$ which are the most likely values of it, in the line element (4), g_{tt} does not change sign in the whole range of $r \geq 0$ so that t always and everywhere is time coordinate. The hypersurfaces $t = const.$ are spacelike with a positive definite line element $ds^2 = (1 - \frac{2M}{r+aM})^{-1} dr^2 + (r+aM)^2 d\Omega$ that is a distance function. Consequently every Cauchy sequence with respect to this distance function converges to a point in the manifold, and this yields metrically completeness.

Now by considering the bending of light we search an upper bound for a. Since 1919 there has been much studies on the gravitational deflection of light by the Sun and gravitational lensing (GL). Under the great vision of Zwicky [10], observation of a QSO showed the first example of the GL phenomena [11], and thereafter it has become the most important tool for probing the universe. GL can give valuable information on important questions, such as masses of galaxies and clusters of galaxies, the existence of massive exotic objects, determination of cosmological parameters and can be also used to test the alternative theories of gravitation [12]. The gravitional deflection of light has now been measured more accurately at radio wavelenghts with using VLBI than at visible wavelenghts with available optical techniques.

Invoking the spherically symmetric nature of the metric in the line element, $ds^2 = Bdt^2 - Adr^2 - Dd\Omega$, we consider the geodesics on the equatorial plane $(q = \frac{p}{2})$, without lose of generality. Following Weinberg [13], we get the equation for the photon trajectories as:

$$f(r) - f_\infty = \int_r^\infty \left[\frac{A(r)}{D(r)}\right]^{\frac{1}{2}} \left[\frac{D(r)}{D(r_0)}\frac{B(r_0)}{B(r)} - 1\right]^{-\frac{1}{2}} dr \; . \qquad (6)$$

The Einstein deflection angle is, $\Delta f = 2\left|f(r_0) - f_\infty\right| - p$. Using Eq. (4) makes the integral in (6) well-defined for $r_0 > (3-a)M$. Since a merely takes positive values, different values of r_0 and a, yield different expansions for B and D, so we get different expressions for deflection angle. Our investigation is on very small and sufficiently large values of a. We gave the details of calculations in [14] and here merely use the results. For $a < 1$, the Einstein deflection angle up to the second order is:

$$\Delta f = 4x + 4x^2\left[\frac{15p}{16} - (1+a)\right] + \ldots \qquad (7)$$

where $x = \frac{M}{r_0}$ and the restriction imposed by the integral singularity is $, 0 < x < \frac{1}{3-a}$, thus the closet approach is $r_0 \approx (3-a)M$. Putting $a = 0$ recovers the well-known Schwarzschild results, which has been extensively examined(see [15] and references therein). However cases with small values of a are qualitatively similar to the case of Schwarzschild but are different quantitatively. Further calculations show that (7) is also valid for $a > 1$, but it may not contain the closest approach. Therefore for the weak field limit $r_0 >> M$ and with all possible values of a, we rewrite the equation as:

$$\Delta f = \Delta f_{fo}\left[1 + \tfrac{M}{r_0}\left(\tfrac{15p}{16} - (1+a)\right)\right] + \ldots \tag{8}$$

where $\Delta f_{fo} = \frac{4M}{r_0}$ is the first order deflection angle. The results of VLBI observations of extragalactic radio sources show radio-wave deflection by the Sun [16] as:

$$\Delta f \approx \Delta f_{fo}(0.9998 \pm 0.0008)\,. \tag{9}$$

Since the order of magnitude of $\frac{M}{r_0}$ for the sun is about 10^{-6}, (8) and (9) give an upper bound $a < 10^3$. For sufficiently large values of a (actually $a > 3$) we may obtain another expression for deflection angle (up to the second order) in the following form

$$\Delta f = 8y + 4y^2\left[\tfrac{15p}{16} - 5(a-1)\right] + \ldots \tag{10}$$

where $y = \frac{r_0}{a(a-2)M}$ and the range of validity is $0 < y < a^{-1}$, thus the closest approach is $r_0 \approx 0$. Eqs. (7) and (10) that contain the closest approach, can be used for testing the general theory of relativity in a strong gravitational field. Although, no test for the theory is known in this region, but there is an open room for such investigations. Several possible observational candidates have been proposed to test the Einstein's theory of relativity in the vicinity of a compact massive object. One of the current topics is the study of point source lensing in the strong gravitational field regions when the deflection angle can be very large [17]. Our calculations confirm that the deflection angle may take any small as well as large values depending on a and it would provide a good tool for the gravitational lensing studies. Consequently bending of light phenomena in the regions of weak and strong field both confine the values of a up to 10^3.

Next we would like to use observational data of the precession of perihelia measurements, in order to find a better bound for a. Following Weinberg [18], for a test particle moving on a timelike geodesic in the plane $q = \frac{p}{2}$, the angle swept is given by:

$$f(r) - f(r_-) =$$

$$\int_{r_-}^{r} dr \left[\frac{D_-(B^{-1}(r) - B_-^{-1}) - D_+(B^{-1}(r) - B_+^{-1})}{D_+ D_-(B_+^{-1} - B_-^{-1})} - \frac{1}{D(r)} \right]^{-\frac{1}{2}} \frac{A^{\frac{1}{2}}(r)}{D(r)} \qquad 11)$$

where $D_\pm = D(r_\pm)$ and $B_\pm = B(r_\pm)$. The orbit precesses in each revolution by an angle $\Delta f = 2\left|f(r_+) - f(r_-)\right| - 2p$. By using the metric components of (4), we have gotten the expression for the precession per revolution (up to the second order), the details are given in Appendix B of [14]. So we may write:

$$\Delta f = \Delta f_{fo}\left[1 + \tfrac{M}{L}\left(\tfrac{19}{6} + \tfrac{e^2}{4} - 2a\,(1 + e^2)\right)\right] + \ldots \qquad 12)$$

where L and e are the semilatus rectum and eccentericity respectively, and $\Delta f_{fo} = \frac{6pM}{L}$ is the well-known first order approximation. Fortunately developments of Long-Baseline Radio Interferometry and analysis of Radar Ranging Data, provided accurate measurements of precession that typically show [19]

$$\Delta f \approx \Delta f_{fo}(1.003 \pm 0.005). \qquad 13)$$

Thus matching the theory with observation, using typical values of $\frac{M}{L}$ and e, we get an upper bound of $a < 10^5$.

We conclude that measurments of these two tests of the general theory of relativity in the weak field limit restrict the allowed values of a to 10^3. Observational data of the GL phenomena would support our presented metric components' role in a strong gravitational field and would also give a more accurate bound for a.

By calculating the Riemann tensor scalar invariant, we receive useful information about the existence of singularities. For the line element (4), it is:

$$R^a{}_{bcd}R_a{}^{bcd}=\frac{48M^2}{D^3}=\frac{48M^2}{(r+aM)^6} \tag{14}$$

As it is evident, the presence of a makes the scalar finite in the whole range of r, meaning that the solutions are free of any intrinsic singularity. Meanwhile there may be a coordinate singularity at $r=(2-a)M$ according to the gotten upper bound.

We would like to mention two points concerning this work. Usually in the literature for discussing this problem coordinate r is defined so that the area of the surfaces $r=const.$, to be $4pr^2$ [20]. This generally is not the case, since before fixing the metric there is no possibility of speaking the distance, so in the same way, there is no possibility of speaking the area. Here we take $r=\sqrt{x^2+y^2+z^2}$, where (x,y,z) are usual Cartesian space coordinates.

The other point worthy enough to be taken with caution concerning r is that, while at first r is taken as a space radial coordinate with the range from zero to infinity and the particle is supposed to be at $r=0$, at the end we come to the conclusion that r is merely a space coordinate in the interval $\left((2-a)M,\infty\right)$. For the rest of the interval $\left(0,(2-a)M\right)$, it is standing as a time coordinate. This contradiction or at least ambiguity raises the question that while the location of the particle is not well-defined, how may we speak of the value of D at this position? This ambiguity particularly needs to have a satisfactory explanation and the criterion of this defination should be justified.

III. Case $\Lambda \neq 0$, $\alpha = 0$

Recently for vaccum spherically symmetric space, non-static solution of Einstein field equations with cosmological constant in the form of the Eq. (5) has been proposed [8]. This result shows a singularity at the origin where the intrinsic nature of it may be checked by calculating the Riemann tensor scalar invariant. This has been calculated in Sec.III of [14] and it is:

$$R^{a}{}_{bcd} R_{a}{}^{bcd} = \frac{48M^{2}}{r^{6} R^{6}(t)} + \frac{8}{3}\Lambda^{2} \tag{15}$$

where $\frac{\dot{R}}{R} = \sqrt{\frac{\Lambda}{3}}$. This evidently exhibits the existence of an intrinsic singularity at the origin. Removing this deficiency leads us to the general form that comes next. As it is expected, (15) with R(t) =1 gives the result of the static case [21].

Let us emphasize some features of Eq. (5) and Eq. (15). Firstly, the existence of a nonzero cosmological constant regardless of its actual value, is sufficient to prevent from occuring of coordinate type singularity at $r \approx 2M$. Recent observations of type Ia supernovae indicating a universal expansion, put forward the possible existence of a small positive cosmological constant [6]. These evidences persuade us that in a cosmological constant dominated universe, we would have no trouble in describing the whole space. Secondly, since there is no singularity for $r > 0$, then there is no ambiguity in defining coordinate r, that mentioed at the end of Sec.II. It is a space coordinate in the whole interval $(0, \infty)$ and we may speak of the value of D at $r = 0$ without any problem. Thirdly a coordinate transformation transforms (5) to (2) [8]. The metric of Eq. (2) has some deficiencies that we would like to discuss briefly. Despite of an intrinsic singularity at the origin similar to the Eq.(5) case, it has a coordinate type singularity at $r \approx \sqrt{\frac{3}{\Lambda}}$ in a Λ-dominated universe. Though the presence of cosmological constant removes the coordinate type singularity from metric of Eq.(5), but in the Schwarzschild-deSitter metric it increases the number of coordinate singularities to two, so the problem of exchanging the meaning of space and time remain. On the other hand when $M = 0$, the assumed FRW background due to homogenity and isotropy of space could not be revisited. More importantly, this metric shows a redshift-magnitude relation that contradicts the observational data [7].

Therefore it is adequate to discard the Schwarzschild-deSitter metric, in favor of our presented metric as a proper frame of reference in the presence of Λ, because it is free of all of the mentioned deficiencies. The metric asymptotically approaches to the non-static deSitter metric that is appropriate for a Λ-dominated universe. Furthermore as we will show next, the presented metric has the suitability eventually to remove the intrinsic singularity.

IV. Case $\Lambda \neq 0$, $\alpha \neq 0$

Since it turns out that there is an intrinsic singularity with the choice $\alpha = 0$, we would like to solve the problem by using general form of the line element. In this section we find out the expressions for metric coefficients asymptotically approching to the FRW universe and getting an analytic metric everywhere. To start we choose the line element in terms of the coordinates (t, r, θ, ϕ) to be:

$$ds^2 = B(r,t)dt^2 - R^2(t)\left[A(r,t)dr^2 + D(r)(d\theta^2 + \sin^2\theta d\phi^2\right] \quad (16)$$

The nonvanishing components of the metric are

$$g_{tt} = -B\,,\ g_{rr} = AR^2\,,\ g_{qq} = DR^2\,,\ g_{ff} = DR^2 \sin^2 q \tag{17}$$

The nonvanishing components of the connection are given in Eq.(45) of [14], and the nonvanishing components of the Ricci tensor are :

$$R_{tt} = \tfrac{B}{AR^2}\left[-\tfrac{B''}{2B} + \tfrac{B'}{4B}\left(\tfrac{A'}{A} + \tfrac{B'}{B}\right) - \tfrac{B'D'}{2BD}\right] + \tfrac{\ddot{A}}{2A} - \tfrac{\dot{A}}{4A}\left(\tfrac{\dot{A}}{A} + \tfrac{\dot{B}}{B}\right)$$

$$+ \tfrac{3\ddot{R}}{R} - \tfrac{\dot{R}}{R}\left(\tfrac{3\dot{B}}{2B} - \tfrac{\dot{A}}{A}\right) \tag{18}$$

$$R_{tr} = -\tfrac{\dot{A}D'}{2AD} - \tfrac{B'\dot{R}}{BR} \tag{19}$$

$$R_{rr} = \tfrac{B''}{2B} + \tfrac{D''}{D} - \tfrac{D'}{2D}\left(\tfrac{D'}{D} + \tfrac{A'}{A}\right) - \tfrac{B'}{4B}\left(\tfrac{A'}{A} + \tfrac{B'}{B}\right)$$

$$- \tfrac{R^2 A}{B}\left[\tfrac{\ddot{R}}{R} + \tfrac{\dot{R}}{R}\left(\tfrac{2\dot{R}}{R} + \tfrac{2\dot{A}}{A} - \tfrac{\dot{B}}{2B}\right) + \tfrac{\ddot{A}}{2A} - \tfrac{\dot{A}}{4A}\left(\tfrac{\dot{A}}{A} + \tfrac{\dot{B}}{B}\right)\right] \tag{20}$$

$$R_{qq} = -1 + \tfrac{D'}{4A}\left(\tfrac{B'}{B} - \tfrac{A'}{A}\right) + \tfrac{D''}{2A} - \tfrac{R^2 D}{B}\left[\tfrac{\ddot{R}}{R} + \tfrac{\dot{R}}{R}\left(\tfrac{2\dot{R}}{R} + \tfrac{\dot{A}}{2A} - \tfrac{\dot{B}}{2B}\right)\right] \tag{21}$$

$$R_{ff} = \sin^2 q R_{qq}$$

where $(')$ and $(\dot{})$ denote derivatives with respect to r and t respectively.

To solve the vacuum field equations $R_{mn} + \Lambda g_{mn} = 0$, we first begin with $R_{tr} = 0$ and introduce a new variable $r = R(t)D^{1/2}(r)$. From this and Eq.(19) we obtain:

$$\tfrac{1}{2}\dot{R}D'D^{-\frac{1}{2}}\left(\tfrac{A^*}{A} + \tfrac{B^*}{B}\right) = 0 \tag{22}$$

where $(^*)$ means differentiation with respect to r. Since $\dot{R}, D' \neq 0$, we have $\tfrac{A^*}{A} + \tfrac{B^*}{B} = 0$. Integrating with respect to r and imposing boundary condition at large distances yields $AB = 1$, in agreement with the FRW background. In the next step, let us consider

$$\frac{R_{tt}}{B}+\frac{R_{rr}}{AR^2}=0\,. \qquad (23)$$

Now by inserting Eq.(18) and Eq.(20) in Eq.(23) we have

$$-\frac{D'^2}{2AD^2R^2}+\frac{D''}{ADR^2}=0 \text{ or } (D'D^{-\frac{1}{2}})'=0\ . \qquad (24)$$

Integrating Eq.(24) with respect to r gives

$$D^{\frac{1}{2}}=r+aM \text{ or } D=(r+aM)^2 \qquad (25)$$

where a is the familiar dimensionless positive constant in the range $0<a<10^3$.

Finally from $R_{qq}+\Lambda g_{qq}=0$, and by inserting Eq. (17) and Eq. (21) in it, we obtain the functional form of $A(r,t)$ as follows:

$$-1+\frac{1}{A}-r\,\frac{A^*}{A^2}-\Lambda r^2(A-1)-\frac{\Lambda}{3}r^3A^*=0$$

or

$$\frac{d}{dr}\left[r(1-\tfrac{1}{A})\right]+\frac{\Lambda}{3}\frac{d}{dr}\left[r^3(A-1)\right]=0\,. \qquad (26)$$

Integration of Eq. (26) with respect to r yields:

$$r(1-\tfrac{1}{A})+\frac{\Lambda}{3}r^2(A-1)=c \qquad 27)$$

where c is integration constant and the post-Newtonian limit gives $c=2M$. Our final result is

$$B=A^{-1}=\tfrac{1}{2}\{\sqrt{(1-\tfrac{2M}{r}-\tfrac{\Lambda}{3}r^2)^2+\tfrac{4\Lambda}{3}r^2}+(1-\tfrac{2M}{r}-\tfrac{\Lambda}{3}r^2)\}\,. \qquad 28)$$

As it is evident from the functional form of A, B and D this metric has no apparent singularity and a straightforward calculation gives the Riemann tensor scalar invariant in the form

$$R^a{}_{bcd}R_a{}^{bcd}=\frac{48M^2}{D^3R^6(t)}+\frac{8}{3}\Lambda^2 \qquad 29)$$

If we set $\Lambda = 0, R(t) = 1$ in (29), (14) will be obtaind, and furthermore with $D = r^2$, (3) is recovered.

It is remarkable that when $a \neq 0$, we may have a well-defined metric in the whole space that asymptotically approaches to the non-static deSitter metric, i.e. the appropriate metric for a Λ-dominated universe.

V. Geodesic Equations

Our next task is to obtain and solve the geodesic equations of a freely falling material particle. We have

$$\frac{d^2x^m}{ds^2} + \Gamma^m_{nl}\frac{dx^n}{ds}\frac{dx^l}{ds} = 0 \qquad 30)$$

Using the nonvanishing components of affine connection, given by Eq (46) in [14] and by putting them in (30) we get

$$\frac{d^2t}{ds^2} + \frac{\dot{B}}{2B}\left(\frac{dt}{ds}\right)^2 + \frac{B'}{B}\frac{dt}{ds}\frac{dr}{ds} + \left(\frac{R\dot{R}A}{B} + \frac{R^2\dot{A}}{2B}\right)\left(\frac{dr}{ds}\right)^2 + \frac{R\dot{R}D}{B}\left(\left(\frac{dq}{ds}\right)^2 + \sin^2 q\left(\frac{df}{ds}\right)^2\right) = 0 \qquad 31)$$

$$\frac{d^2r}{ds^2} + \frac{B'}{2AR^2}\left(\frac{dt}{ds}\right)^2 + 2\left(\frac{\dot{R}}{R} + \frac{\dot{A}}{2A}\right)\frac{dt}{ds}\frac{dr}{ds} + \frac{A'}{2A}\left(\frac{dr}{ds}\right)^2 - \frac{D'}{2A}\left(\left(\frac{dq}{ds}\right)^2 + \sin^2 q\left(\frac{df}{ds}\right)^2\right) = 0 \qquad (32)$$

$$\frac{d^2q}{ds^2} + 2\frac{\dot{R}}{R}\frac{dt}{ds}\frac{dq}{ds} + \frac{D'}{D}\frac{dr}{ds}\frac{dq}{ds} - \sin q\,\cos q\left(\frac{df}{ds}\right)^2 = 0 \qquad (33)$$

$$\frac{d^2f}{ds^2} + 2\frac{\dot{R}}{R}\frac{dt}{ds}\frac{df}{ds} + \frac{D'}{D}\frac{dr}{ds}\frac{df}{ds} + 2\cot q\,\frac{dq}{ds}\frac{df}{ds} = 0 \qquad (34)$$

Since the field is isotropic, we may consider the orbit of our particle to be confined to the equatorial plane, that is $q = \frac{p}{2}$. Then Eq.(33) is satisfied and we may forget about q as a dynamical variable.

By taking $r = Rr$ and $\frac{\dot{R}}{R} = \sqrt{\frac{\Lambda}{3}}$, Eqs. (31), (32), (34) become:

$$\frac{d^2t}{ds^2}+\sqrt{\tfrac{\Lambda}{3}}r(\tfrac{A^*}{2A}+\tfrac{\Lambda}{3}rA^2+\tfrac{\Lambda}{6}r^2AA^*)(\tfrac{dt}{ds})^2-(\tfrac{A^*}{A}+2\tfrac{\Lambda}{3}rA^2+\tfrac{\Lambda}{3}r^2AA^*\tfrac{dt}{ds}\tfrac{dr}{ds}$$
$$+\sqrt{\tfrac{\Lambda}{3}}((A^2+\tfrac{1}{2}rAA^*)(\tfrac{dr}{ds})^2+r^2A(\tfrac{df}{ds})^2)=0 \quad (35)$$

$$\frac{d^2r}{ds^2}-\sqrt{\tfrac{\Lambda}{3}}r\frac{d^2t}{ds^2}-(\tfrac{\Lambda}{3}r+\tfrac{A^*}{2A^3}+\tfrac{\Lambda}{6}r^2\tfrac{A^*}{A})(\tfrac{dt}{ds})^2+\tfrac{A^*}{2A}(\tfrac{dr}{ds})^2-\tfrac{r}{A}(\tfrac{df}{ds})^2=0 \quad (36)$$

$$\frac{d^2f}{ds^2}+\frac{2}{r}\frac{dr}{ds}\frac{df}{ds}=0 \quad 37)$$

Integration of Eq. (37) with respect to s gives $\frac{df}{ds}=Jr^{-2}$ where J is the constant of integration. Using this relation we may rewrite Eq.(31) in the following form:

$$\frac{d}{ds}\left[(\tfrac{1}{A}-\tfrac{\Lambda}{3}r^2A)\tfrac{dt}{ds}\right]+\sqrt{\tfrac{\Lambda}{3}}rA\left[\sqrt{\tfrac{\Lambda}{3}}r\frac{d^2t}{ds^2}+(\tfrac{A^*}{2A^3}+\tfrac{\Lambda}{3}r+\tfrac{\Lambda}{6}r^2\tfrac{A^*}{A})(\tfrac{dt}{ds})^2+\frac{J^2}{Ar^2}\right]$$
$$+\sqrt{\tfrac{\Lambda}{3}}(A+\tfrac{r}{2}A^*)(\tfrac{dr}{ds})^2=0 \quad (38)$$

Substituting Eq. (32) in Eq.(38) yields:

$$\frac{d}{ds}\left[(\tfrac{1}{A}-\tfrac{\Lambda}{3}r^2A)\tfrac{dt}{ds}\right]+\sqrt{\tfrac{\Lambda}{3}}\left[rA\frac{d^2r}{ds^2}+(A+rA^*)(\tfrac{dr}{ds})^2\right]=0$$

or

$$\frac{d}{ds}\left[(\tfrac{1}{A}-\tfrac{\Lambda}{3}r^2A)\tfrac{dt}{ds}+\sqrt{\tfrac{\Lambda}{3}}rA\tfrac{dr}{ds}\right]=0 \quad (39)$$

Eq. (39) may be integrated and in term of integration constant c_1 gets

$$(\tfrac{1}{A}-\tfrac{\Lambda}{3}r^2A)\tfrac{dt}{ds}+\sqrt{\tfrac{\Lambda}{3}}rA\tfrac{dr}{ds}=c_1. \quad (40)$$

Derivation of r with respect to s gives $\frac{dr}{ds}=\sqrt{\frac{\Lambda}{3}}r\frac{dt}{ds}+R\frac{dr}{ds}$. By imposing the boundary condition at infinity for a free fall, $\frac{dr}{ds}=0$ and $\frac{dt}{ds}=1$ we may fix $c_1=1$ in Eq.(40) and it becomes:

$$\frac{dt}{ds}=(1-\sqrt{\tfrac{\Lambda}{3}}rA\tfrac{dr}{ds})(\tfrac{1}{A}-\tfrac{\Lambda}{3}r^2A)^{-1}. \quad (41)$$

The line element in terms of r provide us another equation

$$(\tfrac{1}{A}-\tfrac{\Lambda}{3}r^2A)(\tfrac{dt}{ds})^2+2\sqrt{\tfrac{\Lambda}{3}}rA\tfrac{dt}{ds}\tfrac{dr}{ds}-A(\tfrac{dr}{ds})^2-r^2(\tfrac{df}{ds})^2=1 \quad (42)$$

Inserting (37), (41) in (42) gives

$$(\tfrac{dr}{ds})^2=1-(\tfrac{1}{A}-\tfrac{\Lambda}{3}r^2A)(1+\tfrac{J^2}{r^2}) \quad (43)$$

Using Eq.(27) and differentiating Eq.(43) with respect to s yields:

$$\frac{d^2r}{ds^2}=-\frac{M}{r^2}+\frac{\Lambda}{3}r \qquad (44)$$

The rhs of Eq.(44) may be considered as gradient of a potential field

$$\Phi=-\frac{M}{r}-\frac{\Lambda}{6}r^2. \qquad (45)$$

Although the potential Φ at $r=0$, *i.e.* $r=aMR(t)$ is very large, but it is finite. It seems likely that the potential field of massive stars show this behavior and models of collapses of massive objects help us to find a physical mechanism for fixing a.

It will be a great success if observing extra high energy phenomena in AGN's and cosmic rays. It will provide a lower limit for a.

VI. Remarks

1- The fact that a boundary condition for $r\to 0$ is just as necessary as the one for $r\to\infty$ was first realized by Brillouin [24].

2- A spacetime is said to be spherically symmetric if it admits the group SO(3) as a group of isometries, with the group orbits spacelike two surfaces. A coordinate transformation like $r\to r'=r+aM$ which translates the center of symmetry and thereby breakdowns spherical symmetry does not belong to SO(3).

3- Our choice $D^{\frac{1}{2}}(0) = aM$ may be justified by this fact that the only fundamental lenght available in the theory of gravitation is M .
4- a should be fixed in a more developed theory of gravitation and comparison with more accurate data would be a test for it.
5- In the light of these new considerations the documents in the litrature on the observation of black holes should be revisited. Statements as “Evidence has been progressively mounting and the case is now rather strong for saying that black holes have indeed been observed [25]”, should be taken with more caution. We think the observational data will find a satisfactory explantion and will help us to investigate the nature of a and understanding this concept.
5- It should be emphasized as it was pointed out by the referee, a is a dimensionless fundamental constant of theory and nature like hyperfine structure constant or intractions coupling constants. It is not a particular property of a coordinate system rather it is a measurable quantity of the theory of gravitation. While it could be fixed by comparison with observational data at the same time may be considered as a test for the theory.
6- As it has been mentioned new observations of ultra high energy astrophysical phenomena in the form of gamma rays are very crucial to determine a lower bound for a .
7- The most serious objection that can be raised against the conceptual possibilities presented in this article is from the viewpoint of Hawking and Penrose singularity theorems [26]. A primary assumption for proving these theorems is the existence of closed trapped surfaces. Trapped surfaces (closed or not) are 2-dimensional imbeded spatial surfaces such that any portion of them has, at least initially, a decreasing area along any future evolution direction [27]. In the presented metric and the case of Eq. (5) it can be checked this does not necessarily occur.

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