Cantorian Fractal Space-Time Fluctuations in Turbulent Fluid Flows and the Kinetic Theory of Gases

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Fluid flows such as gases or liquids exhibit space-time fluctuations on all scales extending down to molecular scales. Such broadband continuum fluctuations characterise all dynamical systems in nature and are identified as selfsimilar fractals in the newly emerging multidisciplinary science of *nonlinear dynamics and chaos.* A cell dynamical system model has been developed by the author to quantify the *fractal* space-time fluctuations of atmospheric flows. The earth's atmosphere consists of a mixture of gases and obeys the gas laws as formulated in the *kinetic theory of gases* developed on probabilistic assumptions in 1859 by the physicist *James Clerk Maxwell*. An alternative theory using the concept of *fractals and chaos* is applied in this paper to derive the fundamental equation of the kinetic theory of ideal gases and the *Maxwell*'s distribution of molecular speeds.

Keywords: fractals, chaos, kinetic theory of gases, gas laws

1. Introduction

The kinetic theory of ideal gases is a study of systems consisting of a great number of molecules, which are considered as bodies having a small size and mass (Kikoin and Kikoin, 1978). Classical statistical methods of investigation (Dennery, 1972; Yavorsky and Detlaf, 1975; Kikoin and Kikoin, 1978; Rosser, 1985; Guenault, 1988; Gupta, 1990; Ruhla, 1992; Dorlas, 1999; Chandrasekhar, 2000) are employed to estimate average values of quantities characterizing aggregate molecular motion such as mean velocity, mean energy, *etc*., which determine the macro-scale characteristics of gases. The mean properties of ideal gases are calculated with the following assumptions. (1) The intra-molecular forces are completely absent instead of being small. (2) The dimensions of molecules are ignored, and considered as material points. (3) The above assumptions imply the molecules are completely free, move rectilinearly and uniformly as if no forces act on them. (4) The ceaseless chaotic movements of individual molecules obey Newton's laws of motion.

The observed nonlinear space-time fluctuations of microscopic objects such as atoms and molecules in an ideal gas are now (since 1980s) identified as *fractals* generic to macro-scale real world dynamical systems in nature such as, fluid flows, stock market price fluctuations, heart beat patterns, *etc*. The apparently chaotic (nonlinear) *fractal* fluctuations however exhibit self-similarity, *i.e.*, long-range space-time correlations. The identification of the physical laws governing *fractal* fluctuations is an intensive field of research in the newly (since 1980s) emerging science of *Nonlinear Dynamics and Chaos* (Gleick, 1987). Mary Selvam (1990) has developed a general systems theory for the simulation and prediction of the observed *fractal* space-time fluctuations in dynamical systems of all size scales ranging from the microscopic scale of atoms and molecules to macro-scale turbulent fluid flows. The model concepts are applied to derive the following classical relationships for an ideal gas: (1) pressure exerted by an ideal gas (2) the Boltzmann distribution for molecular energies (3) the Maxwell distribution of molecular velocities. The derivation of the above relationships according to classical statistical methods is briefly described followed by a detailed discussion of the *fractal* concepts applied to derive the same equations.

The important new contributions of the general systems theory applied to model ideal gases are as follows: (1) *fractal* fluctuations are signatures of *quantum-like chaos* on all scales ranging from subatomic dynamics of quantum systems to real world macro-scale fluid flows (2) quantum mechanical laws are applicable to dynamical systems of all size scales.

The general systems theory concepts used in the derivation of the fundamental equations for the kinetic theory of gases have been applied earlier by the author for the simulation and prediction of both microscopic and macro-scale dynamical systems (Selvam, 1990; Selvam, 1993; Selvam, and Fadnavis, 1998; Selvam and Suvarna Fadnavis, 1999a; Selvam, and Suvarna Fadnavis, 1999b; Selvam, 2001).

In the following, Sections 2, 3 and 4 deal respectively with application of the model concepts to derive the following three classical relationships for an ideal gas: (1) pressure exerted by an ideal gas (2) the Boltzmann distribution for molecular energies (3) the Maxwell distribution of molecular velocities. The derivation of the above relationships according to classical statistical methods is briefly described followed by a detailed discussion of the *fractal* concepts applied to derive the same equations. In conclusion Section 5 discusses the universal characteristics of *fractal* space-time fluctuations, a signatures of quantum-like chaos exhibited by dynamical systems of all size scales ranging from sub-atomic dynamics of quantum systems to macro-scale turbulent fluid flows. The model shows that quantum mechanical laws are applicable to macro-scale real world dynamical systems and also provides physically consistent interpretations for *wave-particle duality* and *non-local connection* exhibited by microscopic-scale quantum systems which so far do not have a satisfactory explanation on the basis of current concepts in quantum mechanics.

2. Pressure exerted by an ideal gas

2.1 Classical statistical physics

A brief summary of the method for calculating *pressure* based on classical statistical physics concepts is given in the following The molecular collisions exert a force on the walls of the vessel containing the gas and this force is measured by the parameter *pressure*, which is equal to the force per unit area perpendicular to the direction of the force.

The pressure p is calculated as

$$p = \frac{1}{3} nm\overline{v^2}$$

where n is the number of molecules per unit volume, m is the mass of one molecule and $\frac{\overline{v^2}}{3}$ represents the mean square velocity in any one direction x, y or z. The pressure p may be written as

$$p = \frac{2}{3} n \frac{m\overline{v^2}}{2} = \frac{2}{3} nE_k \tag{1}$$

In Equation (1) E_k is equal to the mean kinetic energy $\frac{m\overline{v^2}}{2}$ of one molecule of a gas. Consequently the pressure of a gas equals two-thirds of the mean kinetic energy of the molecules contained in a unit of its volume. This is one of the most important conclusions of the kinetic theory of an ideal gas. Equation (1) establishes a relationship between molecular quantities, *i.e.*, quantities relating to a separate molecule, and the value of the pressure characterizing a gas as a whole – a macroscopic quantity directly measured in experiments. Equation (1) is sometimes called the *fundamental equation* of the kinetic theory of ideal gases.

2.2 General systems theory

One of the most convincing demonstrations that gases really are made up of fast moving molecules is Brownian motion, the observed constant jiggling around of tiny particles, such as fragments of ash in smoke. This motion was first noticed in 1827 by the British botanist, Robert Brown who initially assumed he was looking at living creatures, but then found the same motion in what he knew to be particles of inorganic material. Einstein showed how to use Brownian motion to estimate the size of atoms (Kikoin and Kikoin, 1978; Fowler, 1997; Lee and Kelvin).

Chaotic fluctuations such as those executed by molecules in a gas are now identified as *fractals* generic to space-time fluctuations of dynamical systems in nature (Mandelbrot, 1977; 1983; Gaspard *et al.*, 1998). Identification of the physics of *fractal* fluctuations and quantification is an intensive field of research in the newly emerging (since 1980s) multidisciplinary science of *Nonlinear Dynamics and Chaos* (Gleick, 1987). It has been long supposed that the existence of chaotic behaviour in the microscopic motions of atoms and molecules in fluids or solids is responsible for their equilibrium and nonequilibrium properties. Recently this hypothesis of microscopic chaos has been verified experimentally by Gaspard *et al*. (1998) who found evidence for microscopic chaos in fluid systems, by the observation of Brownian motion of a colloidal particle suspended in water.

Mary Selvam (1990) has developed a general systems theory (Capra, 1996) for the observed space-time *fractal* fluctuations in dynamical systems, which enable quantification of large-scale fluctuations in terms of inherent smaller scale fluctuation characteristics. The irregular *fractal* fluctuations occur on all space-time scales and may be considered to result from the superimposition of a continuum of eddies or waves such as sine waves. An eddy is basically a circular motion characterized by the radius r and r.m.s (root mean square) circulation speed w_*. Larger scale fluctuations result from the integration of enclosed smaller scale fluctuations. The relationship between the r.m.s circulation speeds W and w_* of large and small eddy of respective radii R *and* r is given as (Townsend, 1956; Mary Selvam, 1990)

$$W^2 = \frac{2}{p}\frac{r}{R}w_*^2 \tag{2}$$

The above equation represents the growth of an eddy continuum with formation of a hierarchy of successively larger eddies from enclosed smaller scale eddies. The square of the eddy amplitude, *i.e.*, W^2 represents the eddy energy (kinetic). The large eddy energy is the integrated mean of the enclosed smaller scale eddy energies and therefore the eddy energy spectrum will follow statistical normal distribution according to the *Central Limit Theorem* (Ruhla, 1992). Such a result that the additive amplitudes (W) of eddies, when squared (W^2) represent the statistical normal distribution is exhibited by subatomic dynamics of quantum systems such as the electron or photon (Maddox, 1998; 1993).

By analogy with quantum mechanics the square of the eddy amplitude W^2 represents the kinetic energy E given as (from Equation 2)

$$E = Hn$$

In the above Equation the parameter n (proportional to $1/R$) is the frequency of the large eddy and H is a constant equal to $\frac{2}{p} r w_*^2$ for the growth of large eddies sustained by constant energy input proportional to w_*^2 from fixed primary small scale eddy fluctuations. Energy content of eddies is therefore similar to quantum systems which can possess only discrete quanta or packets of energy content hn where h is a universal constant of nature (Planck's constant) and n is the frequency in cycles per second of the electromagnetic radiation.

The macro-scale eddy continuum represented by Equation (2) obeys quantum-like mechanical laws, a manifestation of quantum-like chaos. The apparent paradox of wave-particle duality exhibited by microscopic scale quantum systems such as an electron or photon is however physically consistent in the context of real world macro-scale dynamical systems as explained in the following. The bi-directional energy flow intrinsic to eddy circulations is associated with bimodal, *i.e.*, formation and dissipation respectively of phenomenological form for manifestation of energy such as the formation of clouds in updrafts and dissipation of clouds in adjacent downdrafts resulting in the observed discrete cellular geometry to cloud structure. The commonplace occurrence of clouds in a row is a manifestation of wave-particle duality in macro-scale atmospheric flows. By analogy, the molecules (atoms) of an ideal gas may be visualised as the manifestation of matter during a half-cycle of an eddy circulation (Mary Selvam, 1990; Selvam and Fadnavis, 1999a). The primary perturbation of r.m.s circulation speed w_* and eddy radius r represents the wave-like structure of a molecule or atom in the ideal gas, the manifestation of matter of molecular mass m occurring during half a cycle of the complete circulation as explained above.

The length scale ratio R/r in the above Equation (2) represents the fractional volume intermittency of occurrence of small-scale (*fractal*) structures (Mary Selvam, 1993) across unit area of large eddy surface as shown in the following.

Considering two large eddy circulations of respective radii R_1 and R_2 (R_2 being greater than R_1) and corresponding r.m.s circulation speeds W_1 and W_2 which grow from the same small-scale primary perturbation of radius r and r.m.s circulation speed w_*, we have from Equation (2)

$$\frac{R_2}{R_1} = \frac{W_1^2}{W_2^2} \tag{3}$$

Introducing the factor $\frac{R_1^3}{R_2^3}$ representing eddy volumes on both sides of the above equation we have

$$\frac{R_2}{R_1}\frac{R_1^3}{R_2^3} = \frac{R_1^2}{R_2^2} = \frac{W_1^{\,2}}{W_2^{\,2}}\frac{R_1^3}{R_2^3} \tag{4}$$

Therefore

$$\frac{R_2}{R_1} = \frac{W_2^2}{W_1^2}\frac{R_2^3}{R_1^3}\frac{R_1}{R_2} \tag{5}$$

Substituting for R_1/R_2 on the right hand side from Equation (3) we have the following relation for fractional volume intermittency of occurrence of small-scale fluctuations given by the fourth moment about the mean for the relative eddy transports as

$$\frac{R_2}{R_1} = \frac{W_2^4}{W_1^4}\frac{R_2^3}{R_1^3} \tag{6}$$

The length scale ratio R_2/R_1 is equal to the transport of fractional volume of small-scale fluctuations in the environment of the large eddy (per unit volume of large eddy), basically by eddy mixing process. Considering large and small eddies of respective radii R and r and r.m.s circulation speeds W and r the corresponding mass transport M of gas across unit area for half cycle of large eddy circulation in terms of molecular mass is equal to $\frac{W}{2}\frac{R}{r}nm$. The molecular mass m corresponds to the small-scale primary eddy perturbation. Multiplying both sides of Equation (2) by $nm/2$ and rewriting

$$\frac{W}{2}\frac{R}{r}nmW = MW = \frac{2}{p}n\frac{mw_*^2}{2} \tag{7}$$

In the above equation the large eddy circulation speed W represents the acceleration since it is computed as an incremental value relative to its earlier stage of eddy growth. The *pressure* p exerted by the gas is given by the product MW equal to the rate of change of momentum due to molecular impact across unit area of the large eddy surface. Equation (7) may now be written as

$$MW = p = \frac{2}{p}n\frac{mw_*^2}{2} \tag{8}$$

The r.m.s eddy circulation speed w_* represents by concept the average molecular speed in any direction and the average kinetic energy of one molecule designated by E_k is equal to $\frac{mw_*^2}{2}$. The above Equation (8) may now be written as

$$p = \frac{2}{p} nE_k \tag{9}$$

Equation (9) is almost the same as Equation (1), the *fundamental equation* of the kinetic theory of ideal gases, namely, $p = \frac{2}{3} nE_k$.

The important differences in the physical concepts underlying the derivation of the *fundamental equation* of the kinetic theory of ideal gases by classical statistical physical methods and the general systems theory for *fractal* space-time fluctuations are as follows: (1) The general systems theory visualises molecules or atoms as manifestation of matter during half a cycle of eddy circulation. Classical physics visualises molecules and atoms as point objects. (2) The r.m.s velocity w_* represents the average velocity for computation of mean molecular kinetic energy in the general systems theory. The mean square velocity of the molecule or atom in any one direction (x, y or z) equal to $\frac{\overline{v^2}}{3}$ is used for computing the molecular kinetic energy in classical physics.

3. Boltzmann distribution for molecular energies in an ideal gas

3.1 Classical physics

For any system large or small in thermal equilibrium at temperature T, the probability P of being in a particular state at energy E is proportional to $e^{-\frac{E}{k_B T}}$ where k_B is the *Boltzmann's constant*. This is called the *Boltzmann distribution* and may be written as

$$P \propto e^{-\frac{E}{k_B T}} \tag{10}$$

3.2 General systems theory

The physical concepts of the general systems theory enables to show that precise ordered mathematical patterns underlie the apparently chaotic space-time fluctuations of dynamical systems. The irregular *fractal* fluctuations of dynamical systems may be visualized to result from the superimposition of an ensemble of eddies, namely an eddy continuum. An eddy continuum by concept consists of a hierarchy of eddies, the larger scale eddies enclosing smaller scale eddies. The larger scale eddies grow by the spatial integration of enclosed smaller scale eddies and the growth trajectory traces an overall logarithmic spiral flow path with the quasiperiodic Penrose tiling pattern for the internal structure (Mary Selvam, 1990; Selvam and Fadnavis, 1998). The ratio of radii (R_2/R_1) or r.m.s. circulation speeds (W_2/W_1) corresponding to the successive growth steps of the large eddy generating the geometry of the quasiperiodic Penrose tiling pattern is equal to the *golden mean* τ ($\cong$ 1.618).

$$\frac{R_2}{R_1} = \frac{R_4}{R_3} = \frac{R_6}{R_5} = t \cong 1.618$$

and also (11)

$$\frac{W_2}{W_1} = \frac{W_4}{W_3} = \frac{W_6}{W_5} = t \cong 1.618$$

The r.m.s circulation speed W of the large eddy follows a logarithmic relationship with respect to the length scale ratio z equal to R/r as given below

$$W = \frac{w_*}{k} \log z \tag{12}$$

In Equation (12) the variable k represents for each step of eddy growth, the fractional volume dilution of large eddy by turbulent eddy fluctuations carried on the large eddy envelope and is given as

$$k = \frac{w_* r}{WR} \tag{13}$$

Incidentally, Equation (12) represents the observed logarithmic spiral air flow structure in the planetary atmospheric boundary layer and the constant k called the *von Karman's constant* is determined from observations to be equal to about 0.38 (Mary Selvam, 1990; Selvam and Fadnavis, 1998).

From Equations (11) and (13) it is seen that, for successive large eddy growth steps generating the quasiperiodic Penrose tiling pattern, the value of k is equal to $1/\tau^2$ (≅ 0.38) where . τ is the *golden mean* (≅ 1.618). Substituting for k in Equation (12) we have

$$W = w_* \frac{WR}{w_* r} \log z = \frac{WR}{r} \log z$$

$$\text{and} \tag{14}$$

$$\frac{r}{R} = \log z$$

Therefore

$$z = \frac{R}{r} = e^{\frac{r}{R}}$$

$$or \tag{15}$$

$$\frac{r}{R} = e^{-\frac{r}{R}}$$

The ratio r/R represents the fractional probability P of occurrence of small-scale fluctuations (r) in the large eddy (R) environment. Considering two large eddies of radii R_1 and R_2 (R_2 greater than R_1) and corresponding r.m.s circulation speeds W_1 and W_2 which grow from the same primary small-scale eddy of radius r and r.m.s circulation speed w_* we have from Equation (2)

$$\frac{R_1}{R_2} = \frac{W_2^2}{W_1^2} \tag{16}$$

From Equations (15) and (16)

$$\frac{R_1}{R_2} = e^{-\frac{R_1}{R_2}} = e^{-\frac{W_2^2}{W_1^2}} \tag{17}$$

The square of r.m.s circulation speed W^2 represents eddy kinetic energy. Following classical physical concepts (Kikoin and Kikoin, 1978) the primary (small-scale) eddy energy may be written in terms of the eddy environment temperature T and the *Boltzmann's constant* k_B as

$$W_1^2 \propto k_B T \tag{18}$$

Representing the larger scale eddy energy as E

$$W_2^2 \propto E \tag{19}$$

The length scale ratio R_1/R_2 therefore represents fractional probability P of occurrence of large eddy energy E in the environment of the primary small-scale eddy energy k_BT (Equation 18). The expression for P is obtained from Equation (17) as

$$P \propto e^{-\frac{E}{k_BT}} \tag{20}$$

The above Equation (20) is the same as the *Boltzmann's equation* (Equation 10).

The derivation of *Boltzmann's equation* from general systems theory concepts visualises the eddy energy distribution as follows: (1) The primary small-scale eddy represents the molecules whose eddy kinetic energy is equal to k_BT as in classical physics. (2) The energy pumping from the primary small-scale eddy generates growth of progressive larger eddies (Mary Selvam, 1990). The r.m.s circulation speeds W of larger eddies are smaller than that of the primary small-scale eddy (Equation 2). (3) The space-time *fractal* fluctuations of molecules (atoms) in an ideal gas may be visualised to result from an eddy continuum with the eddy energy E per unit volume relative to primary molecular kinetic energy (k_BT) decreasing progressively with increase in eddy size.

4. Maxwell-Boltzmann distribution of molecular speeds

4.1 Classical physics

The distribution of molecular speeds was derived by *Maxwell* based on the probabilistic assumptions, namely (i) uniform distribution in space, (ii) mutual independence of the three velocity components and (iii) isotropy as regards the directions of the velocities (Ruhla, 1992). These assumptions were also used in deriving the fundamental gas law at Equation (1) for a perfect gas. *Maxwell*'s distribution of molecular speeds is given by the following equation.

$$r(n) = 4p\left(\frac{m}{2pk_BT}\right)^{\frac{3}{2}}\left[\exp\left(-\frac{mn^2}{2k_BT}\right)\right]n^2 \tag{21}$$

where $r(v)$ is the probability density assigned to the speed v, T is the absolute temperature of the perfect gas, m is the mass of a molecule and k_B is the *Boltzmann's constant*. For a given gas at a fixed temperature T, the probability density $r(v)$ may be written as

$$r(n) \propto \exp(-n^2)n^2 \tag{22}$$

A graph of *Maxwell*'s distribution of molecular speeds is shown in Figure 1.

4.2 General systems theory

The steady state upward transport of small-scale fluctuation of speed w_* and size scale r in the environment of larger scale fluctuation of speed W and size R is given as (Mary Selvam, 1990; Selvam and Fadnavis, 1998)

$$f = \sqrt{\frac{2}{pz}}\log z \tag{23}$$

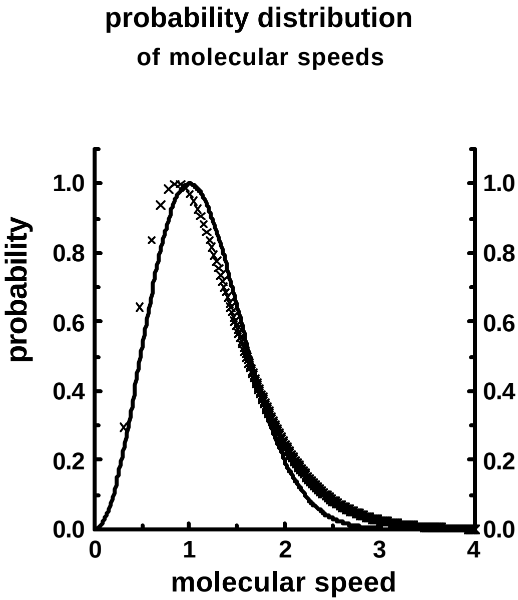
In Equation (23) z is the length scale ratio equal to R/r. Considering three-dimensional fluctuations the fractional contribution (probability density) of smaller length scale r fluctuations in the environment of the larger length scale R fluctuation is given by f^3. The eddy circulation speeds follow the logarithmic law with respect to the length scale ratio z (Equation 12), namely

$$W = \frac{w_*}{k} \log z$$

The eddy circulation speeds are therefore proportional to *log z*, that is

$$W \propto \log z \tag{24}$$

A graph of f^{3} versus $\log z$ will give the probability density distribution for molecular speeds. The cell dynamical system model predicted molecular speed distribution in a perfect gas is shown as crosses in Figure 1. The distributions (Maxwell's and model predicted) are normalized with respect to the maximum speed. There is close agreement between the Maxwell's and model-predicted distributions for molecular speeds in a perfect gas.



continuous line-----> Maxwell's distribution

xxxxxxx----------> f^3 distribution

x and y-axes units are arbitrary

probability is normalised to its maximum value

Figure 1

5. Conclusions

Dynamical systems of all size scales ranging from microscopic scale quantum systems to macro-scale turbulent fluid flows exhibit self-similar *fractal* space-time fluctuations. Self-similarity implies long-range space-time correlations or *non-local* connections such as that observed in quantum systems. A general systems theory developed by the author enables to show quantitatively that the observed *fractal* space-time fluctuations generic to dynamical systems in nature are signatures of *quantum-like chaos*. The model concepts for Cantorian fractal space-time fluctuations is applied to derive the fundamental gas law, namely $p = \frac{2}{3} n E_k$ and also the Maxwell's molecular speed distribution for a perfect gas. The model predictions are in agreement with *Maxwell's kinetic theory of gases* developed in 1859 on probabilistic assumptions.

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