Mechanical Interpretation of the Klein-Gordon Equation

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The substratum for physics can be seen microscopically as an ideal fluid traversed in all directions by straight vortex filaments. Small disturbances of an isolated filament are considered. The Klein-Gordon equation without mass corresponds to elastic stretching of the filament. The wave function has the meaning of the curve's position vector. The mass part of the Klein-Gordon equation describes the rotation of the helical curve about the screw axis due to the hydrodynamic self-induction of the bent vortex filament.

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Vortex filament

We consider the motion of an isolated vortex filament in an ideal fluid. The vortex filament appears as a microscopic constituent of the substratum for physics historically referred to as the vortex sponge. The latter is usually seen as an ideal fluid traversed randomly in all directions by straight vortex filaments. We are interested in small perturbations of the filament. There are two kinds of perturbation. In stretching deformation the filament behaves as an elastic string. Because of hydrodynamic self-induction the bent vortex filament evolves, changing the form and position in the space.

Let the filament be directed along the x axis. Then a small perturbation of the filament can be specified by considering the position vector $\mathbf{r}$ as a function of x

$$\mathbf{r}(x,t) = x\mathbf{i}_1 + y(x,t)\mathbf{i}_2 + z(x,t)\mathbf{i}_3 \tag{1}$$

where dependence on the time t is also included.

Elasticity

In small stretching deformation a vortex filament behaves as an elastic string. The motion of an elastic string is governed by the d'Alembert equations

$$\begin{gathered}\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}\\ \frac{\partial^2 z}{\partial t^2} = c^2 \frac{\partial^2 z}{\partial x^2}\end{gathered} \tag{2}$$

Normally, this describes a plane wave propagating along the string with the speed c:

$$af\left(kx - w\ t\right),\ w = ck$$

Taking the phase shift for two sinusoidal transverse waves as $p\ /2$, a circularly polarized wave can be constructed:

$$\begin{gathered}y = a\cos\left(kx - w\ t\right)\\ z = a\sin\left(kx - w\ t\right)\end{gathered}$$

This wave has the shape of a helix with the wave number k taking the meaning of the curve's torsion t . The longitudinal motion of the helix appears as rotation about the x axis with the angular velocity w. The circular wave can be conveniently expressed in the complex valued form

$$j\,(x,t) = a \exp\left[i\left(kx - w\ t\right)\right] \tag{3}$$

It obeys the Klein-Gordon equation without mass

$$\frac{\partial^2 j}{\partial t^2} = c^2 \frac{\partial^2 j}{\partial x^2}\,. \tag{4}$$

Self-induction

The fluid element of the bent vortex filament moves in space due to the hydrodynamic interaction of the adjacent elements. In the local induction approximation the motion of the filament is described by the equation

$$\frac{\partial \mathbf{r}}{\partial t} = n\,\frac{\partial \mathbf{r}}{\partial l} \times \frac{\partial^2 \mathbf{r}}{\partial l^2} \tag{5}$$

where n = const and l is the length measured along the filament. For small perturbations $l \approx x$. Using this and (1) in (5) gives[1], neglecting quadratic terms

$$\frac{\partial \mathbf{r}}{\partial t} = n\,\mathbf{i}_1 \times \left(\frac{\partial^2 y}{\partial x^2}\mathbf{i}_2 + \frac{\partial^2 z}{\partial x^2}\mathbf{i}_3\right)$$

The latter can be conveniently rewritten in complex valued form

$$\frac{\partial j}{\partial t} = in\,\frac{\partial^2 j}{\partial x^2} \tag{6}$$

where

$$j = y(x,t) + iz(x,t)$$

The Schroedinger equation (6) is satisfied by the helical curve

$$j(x,t) = a \exp\left[i\left(tx - nt^{2}t\right)\right] \tag{7}$$

where $t = \mathrm{const}$ is the torsion of the helix and the curvature is given by $k = at^{2}$. The helix rotates around the x axis with the angular velocity nt^{2}.

Klein-Gordon equation

The motion of a stretched vortex filament combines both self-induction and elasticity. For an isolated filament the solution must have the form of a helix (7), or (3), although a correction to the frequency must be made.

Notice that the left-hand part of the dynamic equation (4) has the meaning of the acceleration and the right-hand part, of the force. Differentiating (7) twice with respect to the time t we find the acceleration in the self-induction circular motion

$$-n^{2}t^{4}j$$

It is just this quantity which corrects the elastic equation (4):

$$\frac{\partial^{2} j}{\partial t^{2}} = c^{2}\frac{\partial^{2} j}{\partial x^{2}} - n^{2}t^{4}j \tag{8}$$

Next, we will express the coefficient $n^{2}t^{4}$ from (8) in physical terms. For this case the relations found earlier in [1] will be taken into account. First, a soliton on a vortex filament moves along the x axis with the velocity [2]

$$u = 2nt$$

Alternatively, the latter is the group velocity for the small amplitude self-induction wave on a vortex filament [1]. In order to use this in (8) we choose for u the maximal value c. As was shown in [1], the smaller the amplitude a of the soliton, the greater its translational velocity u. This implies the concept of minimal amplitude given by

$$a_0 = 2n / c$$

and the corresponding elementary helix. The notions of the soliton's mass, momentum and the energy have also been introduced. For the mass this leads to

$$m_0 = Va_0$$

where V is the linear density of the fluid along the filament. By comparing the mechanical model of the wave-particle with the standard description [1] the Planck constant has been found to be:

$$\hbar = 2nVa_0$$

Now, using the above relations in (8) we come to the standard form of the Klein-Gordon equation

$$\frac{\partial^2 j}{\partial t^2} = c^2 \frac{\partial^2 j}{\partial x^2} - \frac{m_0^2 c^4}{\hbar^2} j \tag{9}$$

It was stated before [1] that the mass particle requires an extra segment of the vortex filament added to its linear configuration. Here we deal solely with the stretching of a filament. Hence, no real mass can be associated with the disturbance thus formed. Parameter m_0 in (9) corresponds to the controversial fictitious mass of the photon.

Maxwell's equations

The equation for the electromagnetic wave is a trivial consequence of (2). In the elastic model [3] the magnetic vector potential corresponds to the displacement field of the quasisolid aether, *i.e.*,

$$A_1 \propto y \ , \ A_2 \propto z$$

In the turbulent aether it is modeled [4] by the perturbation of the average fluid velocity:

$$\mathbf{A} \propto d\langle \mathbf{u} \rangle$$

or rather by the average velocity, since the background value vanishes in the averaging. The perturbation of the fluid velocity field due to the torsional wave on the vortex filament can be found by substituting (1) with (3) in the Biot-Savart law for hydrodynamic induction:

$$d\mathbf{u} = \frac{\Gamma}{4p}\int_l \frac{d\mathbf{r}\times\mathbf{s}}{s^3} - \frac{\Gamma}{4p}\int_x \frac{d\mathbf{x}\times\mathbf{s}}{s^3} \approx \frac{\Gamma}{4p}\int_x \frac{d\vec{j}\times\mathbf{s}}{s^3} \tag{10}$$

where Γ is the circulation of fluid velocity around the filament, $\mathbf{s}$ is the radius vector from a point on the filament to the point in the fluid, h is the least distance from the latter to the filament. Here the complex wave function (3) has been treated as a vector quantity. For $ka << 1$, (10) can be evaluated as

$$d\mathbf{u} \propto \frac{\Gamma}{4p}\frac{\partial j}{\partial x}\int_x \frac{d\mathbf{x}\times\mathbf{s}}{s^3} = i\frac{\Gamma k}{4ph} j$$

where the velocity has been treated as a complex value.

References

[1] V.P.Dmitriyev, "Mechanical analogy for the wave-particle: helix on a vortex filament," *Apeiron* 8, No. 2 (2001).

[2] H.Hasimoto, "A soliton on a vortex filament," *J. Fluid Mech*. 51, part 3 (1972) 477-485.

[3] V.P.Dmitriyev, "The elastic model of physical vacuum," *Mechanics of Solids* 26, No. 6 (1992) 60-71.

[4] V.P.Dmitriyev, Towards an exact mechanical analogy of particles and fields, *Nuovo Cimento* 111A, No. 5 (1998) 501-511; physics/9904029.