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Spreading of Wave Packets, Uncertainly Relations and the de Broglie Frequency

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The spreading of quantum mechanical wave packets is studied in two cases. Firstly we look at the time behaviour of the packet width of a free particle confined in the observable Universe. Secondly, by imposing the conservation of the time average of the packet width of a particle driven by a harmonic oscillator potential, we find a zero-point energy which frequency is the de Broglie frequency.

The quantum mechanical wave-packet spreading is a subject of current interest as can be verified in some recently published papers [1] and [2]. As pointed out by Grobe and Fedorov [1] the ionization of atoms can be suppressed in superstrong fields. This phenomenon has been called stabilization and is characterized by decreasing probability with increasing laser intensity. The wave-packet spreading plays a key role in the final degree of stabilization. On the other hand, Dodonov and Mizrahi [2] have addressed to the "Strict lower bound for the spatial spreading of a relativistic particle

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"where they provide a strict inequality for the minimal possible extension of a wave packet corresponding to the physical state of a relativistic particle.

In this letter we intend to study the spreading of wave-packets in two particular situations. In the first case we explore the consequences of the finiteness of the Universe in the spreading of a free particle wave-packet. In the second one, we want to study the wave-packet of a particle described by a one-dimensional harmonic oscillator. As we will see this can lead to interesting consequences related to the interpretation of the de Broglie frequency of a particle. For a one-dimensional wave-packet let us define [3].

$$(\Delta q)^2 = \langle q^2 \rangle - \langle q \rangle^2, \tag{1}$$

$$(\Delta p)^2 = \langle p^2 \rangle - \langle p \rangle^2 \tag{2}$$

Where $(\Delta q)^2$ and $(\Delta p)^2$ in the above relations are, respectively the variances of the quantities q and p, representing the position and the momentum of a particle. An interesting interpretation of wave-packet spreading can be found in Gasiorowicz [4].

As pointed out by Messiah [3], the spreading law for a free wave-packet turns out to be quite simple if the wave packet is taken to be minimum at the initial time, namely:

$$\Delta q_0 \, \Delta p_0 = \frac{1}{2}\hbar . \tag{3}$$

Besides this, the Heisenberg equation of motion gives in this case:

$$\left(\Delta q(t)\right)^2 = (\Delta q_0)^2 + \frac{(\Delta p_0)^2}{m^2} t^2 \tag{4}$$

Now, looking at relation (4) we verify that a lower bound for the spreading in the particle localization corresponds to a lower bound on the initial (t=0) variance of the particle momentum (Δp_o). Can we associate this (Δp_o) minimum with the finiteness of the Universe? In the following we are going to look for this possibility.

In his paper: "Is the Universe a Vacuum Fluctuation?" Tryon [5] states that the positive mass energy of a particle could be cancelled by an equal amount of negative gravitational energy, due to the interaction of this particle with the rest of the Universe. In this way the classical mechanical energy of a particle is equal to zero, so that the particles are free. However, from the point of view of the quantum mechanics, we may permit fluctuations in the energy of this particle. Before pursuing further in the calculations we would like to stress that if we consider the averaged behavior of a certain particle of the Universe (the electron, for instance) we don't have freedom to prepare any kind of wave packets with arbitrary initial conditions. So let us consider the uncertainty in time to be equal to the Hubble time H_0^{-1}. Now, we write the minimum time-energy uncertainty relation, namely:

$$\Delta E \, \Delta t = \frac{1}{2}\hbar \,. \tag{5}$$

Putting $\Delta t = H_0^{-1}$ in (5), we obtain:

$$\Delta E = \frac{1}{2}\hbar \, H_0 \equiv E_1 \tag{6}$$

We observe that the Hubble constant is related to the radius of the Universe through the equation

$$H_0 = \frac{c}{R_0} \tag{7}$$

Therefore the lower bound on the kinetic energy of a particle reflects the fact that the Universe has a finite radius. We assume that in cosmological grounds each particle has zero total (mechanical) energy, but with fluctuations about this zero given by

$$\Delta E \equiv \Delta E_1.$$

Now we suppose that the minimum uncertainty in the kinetic energy of a particle confined in the Universe corresponds to this lowest energy level E_1 and we write the equality:

$$\frac{(\Delta p_0)^2}{2m} = E_1 = \frac{1}{2}\hbar H_0 \tag{8}$$

By using the relation $\Delta p_0 = m\Delta u_0$, and after solving for Δu_0 we get:

$$(\Delta u_0)_{\mathrm{min}} = \sqrt{\frac{\hbar H_0}{m}} \tag{9}$$

The minimum uncertainty in velocity given by (9) can be interpreted as a lower bound on a particle velocity. For the electron we have $(\Delta u_0)_{min} = 4.10^{-12}\ m/s$.

From (8), we also have:

$$(\Delta p_0) = \sqrt{m\hbar H_0}\ . \tag{10}$$

Putting (10) into (3) (the minimum uncertainty relation) we obtain

$$(\Delta q_0) = \frac{1}{2}\sqrt{\frac{\hbar}{mH_0}} = \frac{1}{2}\sqrt{l_{rc}R_0}\ , \tag{11}$$

where $R_0 = \dfrac{c}{H_0}$ is the radius of the Universe and $l_{rc} = \dfrac{\hbar}{mc}$ is the reduced Compton wavelength of the particle. Then we see that, for the minimum uncertainty in the momentum, the particle has an uncertainty in position which is the geometric average between l_{rc} (

a characteristic length of the particle) and R_0 (the radius of the observable Universe). Despite this maximum initial uncertainty in the position of the particle being very large (it is of the order of 10^6m, for the electron) it is, 10^{20} times smaller than the radius of the Universe. This maximum initial variance of a physical coordinate of a particle coupled to the Universe must be compared with its minimum [2] which is given by $\frac{1}{2} l_{rc}$.

Eq. (11) also deserves the following comment. Thinking in terms of a hypothetical Universe where R is a variable quantity, we can write:

$$\left(\Delta q_0\right) = \frac{1}{2}\sqrt{l_{rc}R} \ . \tag{12}$$

In this way the initial variance of a particle position will be able to vary from its minimum to its maximum value, with R running from l_{rc} to R_0 .

Now, if we consider that the initial time corresponds to the present time, let us see what happens with the spreading of the maximum initial uncertainty wave-packet if we wait a time equal to the Hubble time $\left(t = H_0^{-1}\right)$. Using (4) we get:

$$\left(\Delta q\left(t_H\right)\right) = \sqrt{\frac{5}{4}}\sqrt{\frac{\hbar}{mH_0}} = \sqrt{5}\left(\Delta q_0\right) \tag{13}$$

In obtaining (13), we also used (10) and (11).

Therefore we verify that if we wait a time equal to the Hubble time, the initial maximum variance of a physical coordinate of a particle is not substantially modified.

We will now look to the second case when we study the spreading of a wave-packet of a particle described by a one-dimensional harmonic oscillator.

As pointed out by Messiah [3]: In order that the motion of a wave packet may be likened to the motion of a classical particle, it is first of all necessary that its position and momentum follow the laws of classical mechanics. Also according to Messiah the two most interesting cases are those of the harmonic oscillator and the free particle, cases for which the motion of the center of the packet is rigorously identical to that of a classical particle. Let us turn now our attention to the harmonic oscillator case. In a pedagogical paper [6] the Heisenberg representation was used as a means to study the spreading of wave-packets in some simple examples. For the harmonic oscillator, whose hamiltonian is given by:

$$H = \frac{p^2}{2m} + mw^2 \frac{q^2}{2}, \tag{14}$$

the evolution in time of the width in the position distribution is given by [6]:

$$\left(\Delta q(t)\right)^2 = \left(\Delta q_0\right)^2 \cos^2(wt) + \frac{\left(\Delta p_0\right)^2}{(mw)^2} \sin^2(wt) + \left(\frac{1}{2}\langle qp + pq\rangle_0 - \langle q\rangle_0 \langle p \right. \tag{15}$$

Now let us make the requirement that the variance in the position $\left(\Delta q(t)\right)^2$ averaged in time will be conserved. Then we have:

$$\left[\Delta q(t)^2\right]_{time\ \ \text{average}} = \frac{\left(\Delta q_0\right)^2}{2} + \frac{1}{2}\frac{\left(\Delta p_0\right)^2}{(mw)^2} = \left(\Delta q_0\right)^2, \tag{16}$$

where we have averaged $\left(\Delta q(t)\right)$ in a period of time equal to $T = \dfrac{2p}{w}$. Relation (16) implies that:

$$\left(\Delta p_0\right) = mw\left(\Delta q_0\right). \tag{17}$$

Putting (17) into the minimum uncertainty relation (3), we obtain:

$$mw\left(\Delta q_0\right)^2 = \frac{1}{2}\hbar \tag{18}$$

Mutiplying both sides of (18) by *w*, we get:

$$mw^2\left(\Delta q_0\right)^2 = \frac{1}{2}\hbar w. \tag{19}$$

On the order hand, for a classical harmonic oscillator of amplitude A, we can write:

$$q(t)_{class} = A\cos(wt). \tag{20}$$

The above relation leads to:

$$\left[\Delta q(t)^2 class\right]_{timeaverage} = \frac{A^2}{2}. \tag{21}$$

Making the requirement that the classical variance to be identified with the quantum variance $\left(\Delta q_0\right)$, we obtain

$$mw^2\left(\Delta q_0\right)^2 = \frac{1}{2}mw^2A^2 = \frac{1}{2}\hbar w \tag{22}$$

Second and third terms of equation (22) show a classical harmonic oscillator which mechanical energy is equal to the zero-point energy of the corresponding quantum oscillator.

An interesting consequence of relation (22) is obtained when we make the maximum velocity of the particle undergoing classical harmonic motion to be in magnitude equal to the speed of light c. Putting $wA = c$ in equation (22), we get

$$mc^2 = \hbar w \equiv \hbar w_{dB} \ . \tag{23}$$

and

$$A = \frac{\hbar}{mc} \qquad 24)$$

Therefore we see that (23) reproduces the definition of the de Broglie frequency [7] implying also that the classical amplitude of the oscillator to be equal to the reduced Comptom wavelength. The driving force amplitude of this oscillator is given by:

$$F_1 = mw^2 A = \frac{m^2 c^3}{\hbar} \ . \tag{25}$$

It can also be interpreted as a string constant. Some numerical estimates of it gives order of magnitudes of $10^{-1}N$ for the electron and $10^5 N$ for the nucleons (protons or neutrons). The fact that the force F_1 is proportional to the squared mass of the particle and that is can be defined for electrons, protons, neutrons or any other kind of elementary particles, lead us to think in the only common kind of interaction experimented by these various particles, namely: the gravitational interaction. If we multiply and divide equation (25) by G (the gravitational constant), we obtain:

$$F_1 = \frac{Gm^2}{l_P^2} . \tag{26}$$

where

$$l_P^2 = \frac{G\hbar}{c^3} \tag{27}$$

is the square of the Planck radius.

In conclusion, we would like to take into account the following considerations [8]: It was pointed out by Penrose [9] that the existence of accurate clocks is ultimately due to the fact that each particle of mass m has associated with it, a natural frequency w_{dB} given by the Einstein-Planck's law $\Delta E = mc^2 = \hbar w_{dB}$.

Therefore we can associate this natural frequency with the de Broglie frequency, with the driving force behind this clock being attributed to the internal degree of freedom of the particle described by a harmonic oscillator potential. The same conclusion was reached by one of the present authors [10] starting from other initial assumptions.

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References

[1] R. Grobe and M.F. Fedorov, *Phys. Rev. Lett*. 68 (1992) 2592.

[2] V.V. Dodonov and S.S. Mizrahi, *Phys. Lett. A* 117 (1993) 394.

[3] A. Messiah, *Quantum Mechanics*, Vol. I, John Wiley and Sons, N. York, 1958

[4] S. Gasiorowicz, *Quantum Physics*, John Wiley and Sons, N. York, 1974

[5] E.P. Tryon, *Nature* 246 (1973) 396

[6] J.F. Perez, *Revista Brasileira de Ensino de Física* 17 (1995) 123

[7] L. de Broglie, *Ann. Phys*. 2 (1925) 10, see also J.W. Haslett, *Am. J. Phys*. 40 (1972) 1315

[8] J.S. Anandam, *Quantum Theory of Gravitation*, Ed. A. R. Marlow, *Proc. of Symposium held at Loyola University*, New Orleans, Academic Press, 1980

[9] R. Penrose, In *Battelle Recontres*, Eds. C.M. De Witt and J.A. Wheeler, Benjamin, N. York, 1968.

[10] P.R. Silva, *Phys. Essays* 10 (1997) 628.