The Inequivalence of Haldane Statistics and the Ambiguous Statistics of Medvedev

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Two forms of modified B.E. and F.D. statistics are discussed and show to be inequivalent to first order in a perturbation parameter used to express deviations from both Bose-Einstein and Fermi-Drac statistics

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1. Introduction

One of the great achievements of relativistic quantum field theory is the spin statistics connection that is derived from a spin 1/2 theory and intergral spin theory assuming relativistic invariance and commutation of field operators over spacelike separations.[1,2,3] The spin statistics connection also depends on the fact that the state vector of a quantum field system can be expanded in terms of free states representing individual noninteracting particles. It has been pointed out that of all of the tenants of nonrelativistic quantum theory, the exclusion principle represents the most mysterious and ununderstandable aspect of the theory.[4,5] Geroch et.al.[6] have pointed out that the exclusion principle might be a principle that cannot be derived from the properties of space-time but but is the result of topological properties of spin space. Whatever the generic foundations of the exclusion principle are, we must admit that many key features of the structure of matter depend on it.[7] In this regard both atomic structure and nuclear structure depend on the antisymmetric state for an ensemble of fermions. Also the hadron spectrum is dependent on the fact that the spin flavor color function for quarks is an antisymmetric state. With regard to bosons, probably the most accurate test of Bose statistics is in the spectral distribution of photons in the CMB which confirms Bose statistics to a higher degree of accuracy. There are however concrete reasons to study nonconventional statistics based on studies of the the (2+1) dimensional quantatized Hall effect[8] and studies of (2+1) dimensional anayons.[9] Over the years there have been various constructions intended to generalize Fermi Bose statistics , the first is due to Gentile[10] who considered a statistics with up to k particles in a single quantum state. A more recent attemp is in this study of "quons" which involves the deformation of the commutation relations of creation and annihilation operators to read $a_k a_l^+ - a_l^+ a_k = d_{kl}$ [11]. When q = 1 we have bosons, for q = -1 we have fermions. For values of q between 1 and -1 all representations of the symmetric group are possible. For a n quon state there are $n!$ linearly independent states while for q = ± 1 there is only one state. One of the weaknesses of the quon theory is that observables do not commute over spacelike separations which may render the theory inconsistent.

As a consequence of studying a modification of statistics in (2+1) dimensions Haldane[12] and Wu[13] developed a modification of Bose Fermi statistics that interpolated between Bose and Fermi statistics when the

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parameter in the theory goes from α= 0 to α = 1. We have applied Haldane statistics to both anomalous photons[14] and anomalous electrons[15] and have calculated corrections to the black body spectrum and the properties of a free electron gas (specific heat) that are induced by Haldane statistics. In another generalization of Fermi Bose statistics Medvedev[16] has introduced a novel form of statistics wherein each particle has a probability of being a fermion (P_f). and a boson (P_b). In a separate note we have applied this form of "ambiguous statistics" to the black body spectrum.[17] Though the limits on the violation of Bose Fermi statistics are very stringent[18] it is not out of the question that violations might be found under extreme conditions of temperature and pressure and for condensed astrophysical objects containing exotic matter. The purpose of the present note is to show that Haldane statistics is inequivalent to the ambiguous statistics of Medvedev to first order in a perturbation away from Bose and Fermi statistics. This is important for it demonstrates that the distributions upon which the statistics are founded admit different notions of probability and combinatorics which pertain to the particles that they describe.

2. Haldane Statistics and the Ambiguous Statistics of Medvedev

To emphasize the difference between the statistical approaches mentioned above we first consider a generalized expression for the number of ways of realizing a system of N particles within the context called Haldane statistics (Ref. 12, 13), the expression for w is (number of ways of realizing system)

$$W = \prod_i \frac{\left(g_i + \left(N_i - 1\right)\left(1 - a\right)\right)!}{N_i!\left(g_i - a N_i - \left(1 - a\right)\right)!} \qquad (2.1)$$

$$\left(\Sigma N_i = N\right)$$

(g_i = size of cell containing N_i particles), () ≤ α ≤ i). Here α = 1 for Fermi-Drac statistics, α = 0 for Bose-Einstein statistics. Taking the natural log of Eq. (2.1), varying with respect to N_i, and using Sterling's approximation $\left(\ln_e N! = N_i \ln_e N_i - N_i\right)$ we have

$$\sum \begin{pmatrix} (1-a)\ln_e\left(g_i + \left(N_i - 1\right)\left(1-a\right)\right) + \left(1-a\right) - \left(1-a\right) \\ -\ln_e N_i + 1 - 1 + a\ \ln_e\left(\left(g_i - a N_i - \left(1-a\right)\right) + a - a\right) \end{pmatrix} \qquad (2.2)$$

In Eq. (2.2) we set $N_i - 1 = N_i$ and neglect (1 - α) in comparison to $g_i - a N_i$. Eq. (2.2) gives

$$\sum_i \ln_e \frac{\left(g_i + N_i\left(1-a\right)\right)^{1-a}\left(g_i - a N_i\right)^a}{N_i} dN_i = 0 \qquad (2.3)$$

We now impose the conditions

$$\sum dN_i = 0 \qquad (2.4)$$

$$\sum e_i dN_i = 0 \qquad (2.5)$$

Multiplying Eq. (2.4) by $\frac{m}{t}$ and Eq. (2.5) by $\frac{-1}{t}$ and adding to Eq. (2.3) we have

$$\sum\left(\ln_e \frac{\left(g_i + N_i\left(1-a\right)\right)^{1-a}\left(g_i - a N_i\right)^a}{N_i} + \frac{m}{t} - \frac{e_i}{t}\right) dN_i =$$

or

$$\frac{\left(g_i+(1-\alpha)N_i\right)^{1-\alpha}\left(g_i-\alpha N_i\right)^{\alpha}}{N_i}=e^{\frac{\varepsilon_i-\mu}{\tau}} \tag{2.6}$$

Here $\tau = kT$, μ = chemical potential. In Eq. (2.6) we set $\alpha = 1 - \varepsilon$ (small deviations from F.D. statistics). Eq. (2.6) becomes upon taking natural logs

$$\varepsilon \ln_e\left(\left(g_i\right)\left(1+\frac{\varepsilon N_i}{g_i}\right)\right)+(1-\varepsilon)\ln_e\left(\left(g_i-N_i\right)\left(1+\frac{\varepsilon N_i}{g-N_i}\right)\right)-\ln_e N_i=\frac{\varepsilon_i-\mu}{\tau} \tag{2.7}$$

Eq. (2.7) becomes to order ε after exponentiating,

$$\frac{g_i-N_i}{N_i}=\left(\frac{g_i-N_i}{g_i}\right)^{\varepsilon}\left(1-\frac{\varepsilon N_i}{g_i-N_i}\right)X_i \qquad \left(where \quad X_i=e^{\frac{\varepsilon_i-\mu}{\tau}}\right) \tag{2.8}$$

Eq.. (2.8) can further be written as

$$\frac{g_i-N_i}{N_i}=\left(1+\varepsilon\ln_e\left(1-\frac{N_i}{g_i}\right)\right)\left(1-\frac{\varepsilon N_i}{g_i-N_i}\right)X_i \tag{2.9}$$

We now expand N_i as $N_i = N_{i0} + \varepsilon N_{i1}$ and Eq. (2.9) becomes

$$\frac{g_i}{N_{i0}}\left(1-\frac{\varepsilon N_{i1}}{N_{i0}}\right)-1=X_i+\varepsilon\ln_e\left(1-\frac{N_{i0}}{g_i}\right)-\frac{\varepsilon N_{i0}}{g_i-N_{i0}}X_i$$

To zeroth order

$$\frac{g_i}{N_{i0}}-1=X_i \tag{2.10}$$

And to first order

$$\frac{g_i N_{i1}}{N_{i0}{}^2}=\left(\frac{N_{i0}}{g_i-N_{i0}}\right)X_i-X_i\ln_e\left(1-\frac{N_{i0}}{g_i}\right) \tag{2.11}$$

Solving Eq. (2.11) for N_{i1} and using Eq. (2.10) we find

$$N_{i1}=g_i\left(\frac{1}{\left(1+X_r\right)^2}+\frac{X_i}{\left(1+X_i\right)^2}\ln_e\frac{\left(1+e^{\frac{\varepsilon_i-\mu}{\tau}}\right)}{e^{\frac{\varepsilon_i-\mu}{\tau}}}\right)$$

Thus for a small deviation from F.D. statistics finally

$$N_i=\frac{g_i}{1+e^{\frac{\varepsilon_i-\mu}{\tau}}}+g_i\varepsilon\left[\frac{1}{\left(1+e^{\frac{\varepsilon_i-\mu}{\tau}}\right)^2}+\frac{e^{\frac{\varepsilon_i-\mu}{\tau}}}{\left(1+e^{\frac{\varepsilon_i-\mu}{\tau}}\right)^2}\ln_e\left(\frac{1+e^{\frac{\varepsilon_i-\mu}{\tau}}}{e^{\frac{\varepsilon_i-\mu}{\tau}}}\right)\right] \tag{2.12}$$

If in Eq. (2.6) we set $\alpha = \varepsilon$ (small deviation from B.E. statistics) a similar argument gives

$$N_i = \frac{g_i}{\left(e^{\frac{e_i - m}{t}} - 1\right)} - g_i e \left[\frac{1}{\left(e^{\frac{e_i - m}{t}} - 1\right)^2} + \frac{e^{\frac{e_i - m}{t}}}{\left(e^{\frac{e_i - m}{t}} - 1\right)^2} \ln_e \left(\frac{e^{\frac{e_i - m}{t}}}{e^{\frac{e_i - m}{t}} - 1} \right) \right] \tag{2.13}$$

Thus Eq. (2.12) and Eq. (2.13) represent a generalized formula for the occupation of the various states in a statistics that deviate slightly from F.D. and B.E. statistics.

We now consider the amibiguous statistics of Medevdev (Ref. 16). We consider each particle to have a probability of being represented as a Boson (P_b) or a Fermion (P_f), the expression for the number of ways for realizing the system is

$$W = \prod W_j, \; W_j = \sum_{k-0}^{N_j} \frac{\left(g_j + k - 1\right)!}{\left(g_j - 1\right)! k!} \frac{g_j ! P_b^k P_f^{\left(N_j - k\right)}}{\left(g_j - N_j + k\right)! \left(N_j - k\right)!} \binom{N_j}{k} \tag{2.14}$$

In Eq. (2.14) it is the sorting process of N_j particles to groups of k bosons and to $N_j - k$ fermions that puts them into a statistical classification (here $\binom{N_j}{k}$ = number of ways of choosing k bosons from N_j particles). When the log of Eq. (2.14) is varied with respect to N_j and the constraints (Eq. (2.4) and Eq. (2.5)) are used with Lagrange multipliers $\frac{m}{t}, \frac{-1}{t}$ we find (Ref. 16)

$$\frac{N_j}{g_j} = \frac{\left(P_f + P_b\right) e^{\frac{e_j - m}{t}}}{\left(e^{\frac{e_j - m}{t}} + P_f\right)\left(e^{\frac{e_j - m}{t}} - P_b\right)} \left[1 + \sqrt{1 - \frac{\left(P_f - P_b\right)^2 \left(e^{\frac{e_j - m}{t}} + P_f\right)\left(e^{\frac{e_j - m}{t}} - P_b\right)}{\left(P_f + P_b\right)^2 e^{\frac{e_j - m}{t}} \left(e^{\frac{e_j - m}{t}} + P_f - P_b\right)}} \right] \tag{2.15}$$

In the Eq. (2.15) we first consider the case $P_f = \varepsilon$, $P_b = 1 - \varepsilon$ (slight deviation from Bose statistics) Eq. (2.15) becomes with $X = e^{\frac{e_j - m}{t}}$

$$\frac{N_j}{g_j} = \frac{X}{X(X-1)\left(1 + \frac{e}{X}\right)\left(1 + \frac{e}{X-1}\right)} \left[1 + \sqrt{1 - (1 - 4e)\left(1 + \frac{e}{X}\right)\left(1 + \frac{e}{X-1}\right)\left(1 - \frac{2e}{X-1}\right)} \right]$$

or

$$\frac{N_j}{g_j} = \frac{X}{X(X-1)} \left(1 - \frac{e}{X}\right)\left(1 - \frac{e}{X-1}\right) \left[1 + 2e^{\frac{1}{2}} \sqrt{1 + \frac{1}{4(X-1)} - \frac{1}{4X}} \right]$$

or finally

$$\frac{N_j}{g_j} = \frac{1}{X-1} - \frac{e}{X(X-1)} - \frac{e}{(X-1)^2} + \frac{2e^{\frac{1}{2}}}{X-1} \sqrt{1 + \frac{1}{4(X-1)} - \frac{1}{4X}} \tag{2.16}$$

In Eq. (2.16) we have neglected in terms of order ε within the square root for they will lead to terms of order $e^{\frac{3}{2}}$ upon expansion of the square root by the binomial expansion. We see that Eq. (2.16) can never be made equivalent to Eq. (2.13) for any redefinition of the parameter. Thus ambiguous statistics for slight deviation from the Bose statistics $P_b = 1 - \varepsilon$, $P_f = \varepsilon$ is not equivalent to the statistics of haldane for small α which represents a small deviation from Bose Statistics. If we now consider $P_f = 1 - \varepsilon$, $P_b = \varepsilon$ (slight deviation from Fermi statistics) Eq. (2.15) becomes

$$\frac{N_j}{g_j} = \frac{X}{X(X+1)\left(1-\frac{e}{X}\right)\left(1-\frac{e}{X-1}\right)}\left[1+\sqrt{1-\frac{(1-4e)(X)(X+1)\left(1-\frac{e}{X}\right)\left(1-\frac{e}{X+1}\right)}{X(X+1)\left(1-\frac{2e}{X+1}\right)}}\right] \quad (2.17)$$

Expanding Eq. (2.17) to order ε we find

$$\frac{N_j}{g_j} = \frac{1}{X+1}\left(1+\frac{e}{X}+\frac{e}{X+1}\right)+\frac{2e^{\frac{1}{2}}}{X+1}\sqrt{1+\frac{1}{4X}-\frac{1}{4(X+1)}} \quad (2.18)$$

We note that Eq. (218) can never be made equivalent to Eq. (2.12) (Haldane statistics for slight deviation from F.D. statistics) for any redefinition of the parameter ε. Thus ambiguous statistics for slight deviation from Fermi statistics is not equivalent to Haldane statistics for slight deviation from Fermi statistics. The fundamental reason that these two statistical formulations disagree in their predictions stems from the basic constraints for w in Eq. (2.1) and Eq. (2.14). Eq. (2.1) admits to fractional factorials which destroy our sense of an integral number of particles in a quantum state, Eq. (2.14) does not destroy this property but admits to a combinatoric formula with the product of probabilities for bosons and fermions as $P_b^k P_f^{N_j-k}$ for each cell of phase space g_j. One of the unsolved problems is to relate each statistics (Haldane and ambiguous statistics) to a quantum algebra. To date this has not been achieved, if such a relation was found it would shed light on the notions of locality and relativistic invariance with regard to identical particles.

3. Conclusion

As mentioned in the introduction in Ref. 14 and Ref. 15, we have applied Haldane statistics to photons and electrons when these particles admit to a statistics differing slightly from B.E. and F.D. statistics. In these two studies we actually set $\left(\frac{g+N}{g_r}\right)^e$ and $\left(\frac{g_i-N_i}{g_r}\right)^e$ equal to 1 which leads to a slightly different formula as that derived in Eq. (2.13) and Eq. (2.12). Eq. (2.12) and Eq. (2.13) contain all corrections to order ε. To test the above two statistical schemes, Eq. (2.12) and Eq. (2.13) along with Eq. (2.16) and Eq. (2.18) would have to be applied to the black body spectrum for bosons (Eq. (2.13) and Eq. (2.16)) and to a free electron gas for fermions Eq. (2.12) and Eq. (2.18) and compared with experiment to see if either statistical scheme is in accord with the experimental deviations from B.E. and F.D. statistics. Also precise measurements of both white dwarf[19] and neutrons star[20] limiting masses might provide us with a laboratory to look for deviations from Fermi statistics. A last place and probably the best place to look for deviations from F.D. and B.E. statistics is in the deep inelastic scattering of e^- off of nucleons[21]. Here the structure function depends on the statistics of sea-quarks and gluons which in turn determines the cross section for scattering. Since data on Q.C.D. processes can only be interpreted to an accuracy of 10% the study of these processes might provide us with an excellent window through which to set limits on the parameters describing deviations from B.E. and F.D. statistics.

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