A Compensating Term for the 'Side Force Component' Term in the Barometric Equation

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The derivation of the barometric equation concerns central forces in three dimensions. It has recently been shown that the sum of the incremental volume's side force components in the direction of the center of gravity (the $\left[(-2p/z)(\Delta x)(\Delta y)(\Delta z)\right]$ force) must be included in the Cartesian derivation. That results in the side force component term $(-2p/z)$ being added to the differential equation, or $dp/dz = -\left[\left(NmMG/z^2\right) + (2p/z)\right]$. Meteorological data does not, though, conform unambiguously to the corrected barometric equation. This implies that an approximately compensating term might exist. Such a term results from consideration of the central force due to atmospheric particle random motion perpendicular to the radial from the center of gravity in a central force field (the $\left[+\left(2(f)p/3z\right)(\Delta x)(\Delta y)(\Delta z)\right]$ force, where "*f*" represents the degrees of freedom). The more-complete barometric equation is: $dp/dz = -\left(NmMG/z^2\right) + \left[2(f-3)p/3z\right]$.

Keywords: atmosphere; atmospheric particle motion; barometric; barometric equation; degrees of freedom; fluid; hydrostatic; Newton; side force component

1. Introduction

Newton's[1] derivation of the barometric equation contains an important inconsistency[2] in his treatment of central forces. He started by erecting constant cross-section, flat-bottomed cylinders that extended radially away from the primary's surface. These cylinders were of very small, rectangular, cross-sectional area, and were in contact with each other at the surface so that none of the surface remained uncovered. This divided the fluid into two parts, one part was within the cylinders and the other part was without. Newton claimed that the part of the fluid not within the cylinders was held up archwise and contributed nothing to the pressure within the constant cross-section cylinders. He ignored his own third law, that for every action there is an equal and opposite reaction. If the sides of the imaginary cylinders hold up the 'without' fluid archwise, the 'without' fluid must apply a pressure to the cylinder walls. The pressure against any incremental area of a cylinder's wall produces a force that has a component in the direction of the center of gravity. Newton failed to include those force components in his derivation.

When this inconsistency, which led Newton[1] to an incomplete form of the barometric equation, is corrected we obtain[2]

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$$\frac{dp}{dz} = -\left(\frac{NmMG}{z^2}\right) + \left[\frac{2(f-3)p}{3z}\right], \tag{1}$$

where p is the atmospheric pressure, z is the vertical height above the center of gravity, N is the number of molecules per unit volume of atmosphere, m is the mass of a molecule, M is the mass of the primary (planet or other body), G is the gravitational constant, the atmosphere consists of only one type of molecule, and the positive direction is away from the center of gravity.

Despite the absolute value of the additional term being of the same order of magnitude or greater than the centrifugal force at Earth's surface, measurements of the earth's atmosphere do not reveal the effect of this additional term. This could be due to instrumentation not yet sufficiently accurate and precise to detect the difference, or it could be that the additional negative term is approximately balanced by a comparable positive term not yet considered. This latter possibility suggests there might be yet another important physical condition not used by Newton. That possible condition must yield a central force which is directed radially away from the center of gravity.

The purpose of this paper is to investigate the concept that random motion of atmospheric molecules provides the approximately compensating term.

2. Theory

The obvious candidate for the approximately compensating term is the acceleration experienced, with respect to the center of gravity, by a particle with a component of its velocity perpendicular $(v_\perp)$ to a radial (r) connecting the particle and the center of gravity. That effect yields an $\left(mv_\perp^2\right)/r$ force magnitude. In the Cartesian coordinate system, the magnitude of this outward force would be $\left(mv_\perp^2\right)/z$.

The magnitude of the outward force due to translational motion of the particles in the incremental volume is

$$\mathrm{F}_{5\mathrm{C}} = \frac{\left(Nmv_{\perp A}^2\right)}{z}(\Delta x)(\Delta y)(\Delta z), \tag{2}$$

where $\left(Nmv_{\perp A}^2\right)$ is the average of the squares of the speeds, perpendicular to the radial from the center of gravity, of all of the particles in the incremental volume.

To make equation (2) compatible for combination with the magnitudes of the other forces used to derive equation (1), I express it in terms of pressure (p). $\left(Nmv_{\perp A}^2\right)/z$ can be changed to a more useful form by using the ratio of $\left(v_{\perp A}^2\right)/\left(v_A^2\right)$, where $\left(v_A^2\right)$ is the average of the squares of the speeds of all of the particles in the incremental volume.

Start by finding $\left(v_A^2\right)$.

$$\left(v_A^2\right) = \int_0^\infty \int_0^p \frac{(vU)^2(2p)(\sin\Omega)(d\Omega)(dv)}{E} \tag{3}$$

where

$$E = \int_0^\infty \int_0^p (U)^2(2p)(\sin\Omega)(d\Omega)(dv), \tag{4}$$

where U is the probability of particles within the delta volume having a speed that lies within dv at v, and Ω is the polar angle between the particles' direction of motion and the radial from the primary's center of gravity to the delta volume.

The variables in equation (3) are separable[3] so the equation can be rewritten as:

$$\left(v_A^2\right)=\left(2p\right)\int_0^{\infty}\left(vU\right)^2\left(dv\right)\int_0^{p}\left(\sin\Omega\right)\left(d\Omega\right)/E \tag{5}$$

Next I find $\left(v_{\perp A}^2\right)$. The relation between $\left(v_{\perp A}\right)$ and $\left(v_A\right)$ is

$$\left(v_{\perp A}\right)=\left(v_A\right)\left(\sin\Omega\right) \tag{6}$$

Therefore:

$$\left(v_{\perp A}^2\right)=\left(2p\right)\int_0^{\infty}\left(vU\right)^2\left(dv\right)\int_0^{p}\left(\sin\Omega\right)^3\left(d\Omega\right)/E \tag{7}$$

The relationship of $\left(v_{\perp A}^2\right)$ to $\left(v_A^2\right)$ is thus

$$\frac{\left(v_{\perp A}^2\right)}{\left(v_A^2\right)}=\int_0^{p}\left(\sin\Omega\right)^3/\left(d\Omega\right)\int_0^{p}\left(\sin\Omega\right)\left(d\Omega\right)=\frac{2}{3} \tag{8}$$

Equation (8) enables $\left(Nmv_{\perp A}^2\right)$ in equation (2) to be expressed as

$$\left(Nmv_{\perp A}^2\right)=\frac{2}{3}\left(Nmv_A^2\right) \tag{9}$$

The translational kinetic energy of the average gas particle is $1/2\left(mv_A^2\right)$, which equals $3/2\left(kT\right)$ in the kinetic theory of gasses[4,5]. So,

$$\left(mv_A^2\right)=\left(3kT\right) \tag{10}$$

where k is the Boltzmann constant, and T is the absolute temperature of the particles. Thus,

$$\left(Nmv_{\perp A}^2\right)=\frac{2}{3}\left(Nmv_A^2\right)=\frac{2}{3}\left(N\right)\left(3kT\right)=2\left(NkT\right). \tag{11}$$

In turn, $\left(NkT\right)$ is the gas pressure[4,5] $\left(p\right)$, so

$$\frac{\left(Nmv_{\perp A}^2\right)}{z}\left(\Delta x\right)\left(\Delta y\right)\left(\Delta z\right)=\left(\frac{2p}{z}\right)\left(\Delta x\right)\left(\Delta y\right)\left(\Delta z\right). \tag{12}$$

The speed $\left(v_{\perp A}\right)$ is not limited, though, to only the translational motion of the atmospheric molecules. There are two other $\left(v_{\perp A}^2\right)$ motions that must be considered. One of those is the kinetic motion of atomic vibration in the case of molecules consisting of two or more atoms[6,7], such as N^2 or CO^2. The other is molecular rotation of such molecules. The kinetic energy of motion is $kT/2$ for each mode. For atmospheric molecules, the collection of $\left(v_{\perp A}^2\right)$ effects contribute $\left[2fp/3z\right]\left(\Delta x\right)\left(\Delta y\right)\left(\Delta z\right)$ to the force on the delta volume, where f represents the degrees of freedom.

This means that, for atmospheres, F_{5C} becomes

$$F_{5C} = \left[\frac{2fp}{3z} \right] (\Delta x)(\Delta y)(\Delta z) \tag{13}$$

Newton was the scientist who first derived that $\left(mv_{\perp}^2\right)/r$ force. He also knew that the 'static' atmosphere consists of molecules that are moving rapidly with respect to each other. His not incorporating that force in his derivation is another inconsistency.

Granted, Newton could not have achieved equation (13) because the Kinetic Theory of Gasses was not sufficiently developed. He could, though, have entered the result in a form equivalent to equation (2). He did not.

2.1 The More-complete Barometric Equation

The addition of F_{5C} to the derivation of equation (1) leads to a replacement for equation (1). The new Cartesian coordinate system resultant equation is:

$$\frac{dp}{dz} = -\left(\frac{NmMG}{z^2} \right) + \left[\frac{2(f-3)p}{3z} \right]. \tag{14}$$

When $f = 3$, this resulting equation (14) is Newton's barometric equation for a static atmosphere in a non-rotating, non-accelerating coordinate system.

Obviously there are two types of atmospheres to be considered. One is a "single-atom molecules" atmosphere where $f = 3$, the second is a "multiple-atom molecules" atmosphere where $f \rangle 3$. Real atmospheres range from a low near " $f = 3$ " to a high of " $f = 9$ " or higher. The multiplicative constant " $2(f-3)/3$ " therefore ranges from a minimum value of "0" to a value of "4" or higher depending upon three parameters: the density of the gasses, the types of the gasses, and the temperature of the atmosphere. Earth's atmosphere is primarily nitrogen and oxygen[8], so the multiplicative constant in our case[7] is approximately 1.33 . This means that for Earth,

$$\frac{dp}{dr} = -\left(\frac{NmMG}{r^2} \right) + \left[\frac{1.33p}{r} \right]. \tag{15}$$

Newton "lucked-out". His second and third inconsistencies cancel each other as long as you constrain use of his barometric equation to " $f = 3$ " atmospheres with the atmospheric gasses at appropriate temperatures. Earth's atmosphere is[7,8] approximately " $f = 5$ "; thus, as long as instrumentation is not extremely accurate, his equation erroneously appears to be correct.

Other forces could be introduced—such as the van der Waals' force at the surfaces of the delta volume. Those forces, though, are not germane to the purposes of this paper.

For physical correctness the term $-2p/z$ must be added to the equation that Newton derived. That has already been done[2], but it leads to a wrong result for the atmosphere's pressure unless the integral form of the $\left(mv_{\perp A}^2\right)/z$ force term is also added. The two additions cancel for an " $f = 3$ " atmosphere. They do not cancel for Earth's atmosphere, nor do they cancel for other planets with " $f \rangle 3$ " atmospheres. The correction is obviously significant for several reasons, including theoretical correctness and anticipation of increased drag on Earth-orbiting objects with perigees inside the sensible atmosphere.

For Earth, the correction is also important because a barometric equation is used as part of the equations needed for some attempts at forecasting weather in a chaotic atmosphere. Each subsequent second of time, the result from the barometric equation for the preceding second forms part of the initial condition for evolving weather. Even small errors have enormous consequences. Therefore, replacement of Newton's erroneous barometric equation by a more correct barometric equation is quite important.

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