Prime Integrals in Relativistic Celestial Mechanics

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In relativistic celestial mechanics it is possible to use the Schwarzschild solution, for a fixed spherical central body, in order to describe the orbital motion of a particle around it. This is the astronomic case of a planet's orbital motion around the Sun or of a satellite's orbital motion around the mother planet. This paper reports the study of two prime integrals of the motion.

1. Generality on relativistic problem of motion

The track of a particle moving freely in the space-time with metrics

$$ds^2 = c^2\left(1-\frac{r_g}{r}\right)dt^2 - \frac{1}{1-\frac{r_g}{r}}dr^2 - r^2 dq^2 - r^2 \sin q^2 dj^2,$$

in which $r_g = 2GM/c^2$ is the Schwarzschild radius, r,q and j the polar coordinates, t the time coordinate and also G = gravitational constant, M = mass of central body and c = velocity of light, is determined by the equations of a geodesics

$$\frac{d^2x^a}{ds^2} + \Gamma^a_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} = 0, \tag{1}$$

being the space-time coordinates $x^1=r$, $x^2=q$, $x^3=j$, $x^4=ct$; so a, i, j are varying 1,2,3,4. We start with $a = 2$, putting the only Christoffel Γ^2_{ij} symbols which are not zero, and we obtain

$$\frac{d^2x^2}{ds^2} + \Gamma^2_{12}\frac{dx^1}{ds}\frac{dx^2}{ds} + \Gamma^2_{21}\frac{dx^2}{ds}\frac{dx^1}{ds} + \Gamma^2_{33}\frac{dx^3}{ds}\frac{dx^3}{ds} = 0.$$

The foregoing equation is rewritten so:

$$\frac{d^2q}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{dq}{ds} - \sin q\ \cos q\left(\frac{dj}{ds}\right)^2 = 0, \tag{2}$$

i.e., the differential equation for $q=q(s)$. Because of the spherical symmetry, there is no loss of generality in confining our attention to particles moving in the "equatorial plane" given by $q=p/2$ (the point-mass M is regarded at rest in the origin of the reference frame in polar coordinates). With this constant value for q, eq. (2) is satisfied. Putting a=1 we have

$$\frac{d^2x^1}{ds^2} + \Gamma^1_{11}\frac{dx^1}{ds}\frac{dx^1}{ds} + \Gamma^1_{22}\frac{dx^2}{ds}\frac{dx^2}{ds} + \Gamma^1_{33}\frac{dx^3}{ds}\frac{dx^3}{ds} + \Gamma^1_{44}\frac{dx^4}{ds}\frac{dx^4}{ds} = 0.$$

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Since *q* =constant=*p*/2 and the only surviving Christoffel symbols are

$$\Gamma^1_{11} = -\frac{\frac{r_g}{2}}{r^2\left(1-\frac{r_g}{r}\right)},\ \Gamma^1_{33} = -r\left(1-\frac{r_g}{r}\right),\ \Gamma^1_{44} = \frac{\frac{r_g}{2}}{r^2\left(1-\frac{r_g}{r}\right)},$$

the foregoing equation becomes

$$\frac{d^2r}{ds^2} - \frac{\frac{r_g}{2}}{r^2\left(1-\frac{r_g}{r}\right)}\left(\frac{dr}{ds}\right)^2 - r\left(1-\frac{r_g}{r}\right)\left(\frac{dj}{ds}\right)^2 + \frac{\frac{r_g}{2}}{r^2\left(1-\frac{r_g}{r}\right)}c^2\left(\frac{dt}{ds}\right)^2 = 0. \tag{3}$$

Then for *a*=3

$$\frac{d^2x^3}{ds^2} + \Gamma^3_{23}\frac{dx^2}{ds}\frac{dx^3}{ds} + \Gamma^3_{32}\frac{dx^3}{ds}\frac{dx^2}{ds} + \Gamma^3_{13}\frac{dx^1}{ds}\frac{dx^3}{ds} + \Gamma^3_{31}\frac{dx^3}{ds}\frac{dx^1}{ds} = 0.$$

Since, as before, *q*=constant and the only surviving Christoffel symbol is $\Gamma^3_{13} = \Gamma^3_{31} = 1/r$, we obtain

$$\frac{d^2j}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{dj}{ds} = 0. \tag{4}$$

Finally for *a*=4

$$\frac{d^2x^4}{ds^2} + \Gamma^4_{14}\frac{dx^1}{ds}\frac{dx^4}{ds} + \Gamma^4_{41}\frac{dx^4}{ds}\frac{dx^1}{ds} = 0,$$

the only surviving Christoffel symbol is $\Gamma^4_{14} = \Gamma^4_{41} = \frac{\frac{r_g}{2}}{r^2\left(1-\frac{r_g}{r}\right)}$ and then we have

$$\frac{d^2t}{ds^2} + \frac{r_g}{r^2\left(1-\frac{r_g}{r}\right)}\frac{dr}{ds}\frac{dt}{ds} = 0. \tag{5}$$

Eqs. (4) and (5) are readily integrated. Multiplying by r^2 eq. (4) we obtain

$$r^2\frac{d^2j}{ds^2} + 2r\frac{dr}{ds}\frac{dj}{ds} = 0,$$

or

$$\frac{d}{ds}\left(r^2\frac{dj}{ds}\right) = 0,$$

which is integrated giving

$$r^2\frac{dj}{ds} = h_1 = \text{ integration constant.} \tag{6}$$

It is easy to demonstrate that the expression

$$\frac{dt}{ds}=\frac{k}{1-\frac{r_g}{r}},\ [k=\text{constant}] \tag{7}$$

satisfies eq. (5). Therefore from eqs. (6) and (7) we deduce

$$r^2\frac{dj}{dt}=\frac{h_1}{k}\left(1-\frac{r_g}{r}\right)=L\left(1-\frac{r_g}{r}\right) \qquad [L=\frac{h_1}{k}=\text{constant}]. \tag{8}$$

Eq. (8) is the first prime integral. Let us come back to Schwarzschild metrics rewritten so

$$-1=\frac{1}{\left(1-\frac{r_g}{r}\right)}\left(\frac{dr}{ds}\right)^2+r^2\left(\frac{dj}{ds}\right)^2-\left(1-\frac{r_g}{r}\right)c^2\left(\frac{dt}{ds}\right)^2. \tag{9}$$

Putting, into eq. (9), eqs. (6) and (7) we have

$$-1=\frac{h_1^2}{r^2}-\frac{c^2k^2}{\left(1-\frac{r_g}{r}\right)}+\frac{1}{\left(1-\frac{r_g}{r}\right)}\left(\frac{h_1}{r^2}\frac{dr}{dj}\right)^2,$$

or

$$\frac{r_g}{r}-1=\left(\frac{h_1}{r^2}\frac{dr}{dj}\right)^2+\frac{h_1^2}{r^2}\left(1-\frac{r_g}{r}\right)-c^2k^2,$$

and then

$$\left(\frac{h_1^2}{r^2}\frac{dr}{dj}\right)^2+\frac{h_1^2}{r^2}=\frac{r_g}{r}\frac{h_1^2}{r^2}+c^2k^2-1+\frac{r_g}{r};$$

finally we get

$$\left(\frac{1}{r^2}\frac{dr}{dj}\right)^2+\frac{1}{r^2}=\frac{r_g}{r}\frac{1}{r^2}+\frac{c^2k^2-1}{h_1^2}+\frac{r_g}{rh_1^2}.$$

With substitution $u=\frac{1}{r}$, $\frac{du}{dj}=-\frac{1}{r^2}\frac{dr}{dj}$, we have

$$\left(\frac{du}{dj}\right)^2+u^2=r_gu^3+\frac{c^2k^2-1}{L^2k^2}+\frac{r_g}{L^2k^2}u, \tag{10}$$

since $h_1=Lk$. Differentiating eq. (10) with respect to j

$$2\frac{du}{dj}\frac{d^2u}{dj^2}+2u\frac{du}{dj}=3r_gu^2\frac{du}{dj}+\frac{r_g}{L^2k^2}\frac{du}{dj},$$

and removing the factor $2\frac{du}{dj}$

$$\frac{d^2u}{dj^2}+u=\frac{3r_gu^2}{2}+\frac{r_g}{2L^2k^2}=\frac{3r_gu^2}{2}+\frac{GM}{C_0^2}, \tag{11}$$

with $Lkc=C_0$.

Both eqs. (10) and (11) govern the particle relativistic motion, giving its trajectory in Schwarzschild field. It is very easy to show that the connection with newtonian theory is very close, as of course it must be. Starting from

$$\frac{d\vec{v}}{dt} = -\frac{GM}{r^2}\hat{r} = -\frac{m}{r^2}\hat{r} \qquad [m=GM] \tag{12}$$

of newtonian paradigm ($\hat{r}$ being the unit vector of r direction and $\hat{j}$ the unit vector of transverse direction), the vector velocity is

$$\vec{v} = \dot{r}\hat{r} + r\dot{j}\,\hat{j} \tag{13}$$

Scalar multiplication of eqs. (12) and (13) provides

$$\vec{v}\cdot\frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{1}{2}v^2\right) = -\frac{m}{r^2}\dot{r}$$

which is readily integrated so

$$\frac{1}{2}v^2 = \frac{m}{r} + \text{integration constant.}$$

With a = semi-major of an ellipse, in which point-mass M occupies one of two foci, we have also

$$v^2 = 2m\left(\frac{1}{r} - \frac{1}{2a}\right). \tag{14}$$

Owing to the central force we deduce

$$r^2\dot{j} = r^2\frac{dj}{dt} = C_0 = \sqrt{mp}\ , \tag{15}$$

with p = semi-latus rectum = $a(1-e^2)$ and e = eccentricity of ellipse (Finley-Freundlich, 1958). From eqs. (14) and (15) we get

$$v^2 = \dot{r}^2 + r^2\dot{j}^2 = r^4\dot{j}^2\left[\frac{d}{dj}\left(\frac{1}{r}\right)\right]^2 + r^2\dot{j}^2 = 2m\left(\frac{1}{r} - \frac{1}{2a}\right)$$

and also

$$\left[\frac{d}{dj}\left(\frac{1}{r}\right)\right]^2 + \frac{1}{r^2} = \frac{2m}{C_0^2}\left(\frac{1}{r} - \frac{1}{2a}\right).$$

Putting $u = \frac{1}{r}$ we have

$$\left(\frac{du}{dj}\right)^2 + u^2 = \frac{2m}{C_0^2}u - \frac{m}{aC_0^2}\ . \tag{16}$$

Differentiating eq. (16) with respect to j gives

$$2\frac{du}{dj}\frac{d^2u}{dj^2} + 2u\frac{du}{dj} = \frac{2m}{C_0^2}\frac{du}{dj}\ ,$$

and finally

$$\frac{d^2u}{dj^2}+u=\frac{m}{C_0^2}\ . \tag{17}$$

This equation is the same as eq. (11) if r_g=0, which implies $L \to C_0$.

2. Analytical development of the relativistic model

In item 1 we have shown the prime integral, *i.e.*

$$r^2\dot{j}=L\left(1-\frac{r_g}{r}\right). \tag{8}$$

Now we will provide another prime integral, say, a generalized total energy integral.

In order to show this we start from Eddington's treatise "The mathematical theory of Relativity" (Eddington, 1952). As shown in item 1 we are able to work out the planetary orbits from Einstein's law independently of newtonian paradigm. Nevertheless, in this context, Eddington is looking for a "perturbation theory," *i.e.* he wants to introduce a vectorial perturbing force (for unit mass) $\vec{F}_p=F_{pr}\hat{r}+F_{pj}\hat{j}$ superimposing to the newtonian one $\vec{F}_n=-\frac{m}{r^2}\hat{r}$.

The problem now is to determine F_{pr} and F_{pj} on the basis of GRT (general relativity theory), or better using eqs. (3), (4), (5) of item 1.

After some analytical treatments, Eddington provided the following expressions of radial and transverse components (see Appendix)

$$\left.\begin{aligned} F_{pr}&=\frac{3}{2}\frac{r_g}{r^2}\dot{r}^2-r_g\dot{j}^2-\frac{r_g m}{r^3}\\ F_{pj}&=\frac{r_g}{r}\dot{r}\dot{j}\end{aligned}\right\} \tag{18}$$

It should be noted that, strictly speaking, the net force (per unit mass, or acceleration) on the particle is not a central force.

Substitution of $\dot{j}=\frac{L}{r^2}\left(1-\frac{r_g}{r}\right)$ from eq. (8) into (18) gives

$$\left.\begin{aligned} F_{pr}&=\frac{3}{2}\frac{r_g}{r^2}\dot{r}^2-r_g\frac{L^2}{r^4}\left(1-\frac{r_g}{r}\right)^2-\frac{r_g m}{r^3}\\ F_{pj}&=\frac{r_g}{r^3}\dot{r}L\left(1-\frac{r_g}{r}\right)\end{aligned}\right\} \tag{19}$$

Differential equation (12) becomes

$$\frac{d\vec{v}}{dt}=-\frac{m}{r^2}\hat{r}+\vec{F}_p\ . \tag{20}$$

Again with scalar multiplication and using $\vec{F}_p=F_{pr}\hat{r}+F_{pj}\hat{j}$ we have

$$\vec{v}\cdot\frac{d\vec{v}}{dt}=\frac{d}{dt}\left(\frac{1}{2}v^2\right)=-\frac{m}{r^2}\dot{r}+\dot{r}F_{pr}+r\dot{j}\,F_{pj}\ ,$$

and then using eq. (19)

$$\frac{d}{dt}\left(\frac{1}{2}v^2\right)=-\frac{m}{r^2}\dot{r}+\frac{3}{2}\frac{r_g}{r^2}\dot{r}^3-r_g\frac{L^2}{r^4}\left(1-\frac{r_g}{r}\right)^2\dot{r}-\frac{r_g m}{r^3}\dot{r}+$$
$$+\frac{r_g}{r^4}L^2\left(1-\frac{r_g}{r}\right)^2\dot{r}=-\frac{m}{r^2}\dot{r}+\frac{3}{2}\frac{r_g}{r^2}\dot{r}^3-\frac{r_g m}{r^3}\dot{r}.$$

Integration with respect to time, between t_0 and t, gives

$$\frac{1}{2}\left(v^2-v_0{}^2\right)=\frac{m}{r}-\frac{m}{r_0}+\int_{t_0}^{t}\Gamma dt\ , \tag{21}$$

with $v(t_0)=v_0$, $v(t)=v$ and also $r(t_0)=r_0$, $r(t)=r$. Since

$$\dot{r}=\frac{dr}{dt}=\frac{dr}{dj}\frac{dj}{dt}=\frac{dr}{dj}\dot{j}=\frac{L}{r^2}\left(1-\frac{r_g}{r}\right)\frac{dr}{dj}\ ,$$

we obtain

$$\int_{t_0}^{t}\Gamma dt=\int_{t_0}^{t}\frac{3}{2}\frac{r_g}{r^2}\dot{r}^3 dt-\int_{r_0}^{r}\frac{r_g m}{r^3}dr\ ,$$

and then

$$\int_{t_0}^{t}\Gamma dt=\int_{t_0}^{t}\frac{3}{2}\frac{r_g}{r^2}\frac{L^3}{r^6}\left(1-\frac{r_g}{r}\right)^3\left(\frac{dr}{dj}\right)^3 dt-\int_{r_0}^{r}\frac{r_g m}{r^3}dr\ \ . \tag{22}$$

The first term on the right side of eq. (22) is worked so:

$$\int_{t_0}^{t}\frac{3}{2}\frac{r_g}{r^2}\left(\frac{dr}{dj}\right)^2\frac{L^3}{r^6}\left(1-\frac{3r_g}{r}\right)\left(\frac{dr}{dj}\right)dt=$$
$$\int_{r_0}^{r}\frac{3}{2}\frac{r_g}{r^2}\left(\frac{dr}{dj}\right)^2\frac{L^3}{r^6}\left(1-\frac{3r_g}{r}\right)\frac{r^2}{L\left(1-\frac{r_g}{r}\right)}dr=\int_{r_0}^{r}\frac{3}{2}\frac{r_g L^2}{r^6}\left(\frac{dr}{dj}\right)^2 dr.$$

Finally the eq. (22) becomes

$$\int_{t_0}^{t}\Gamma dt=\int_{r_0}^{r}\frac{3}{2}\frac{r_g L^2}{r^6}\left(\frac{dr}{dj}\right)^2 dr-\int_{r_0}^{r}\frac{r_g m}{r^3}dr=R\ . \tag{23}$$

Putting eq. (23) into (21) we have

$$\frac{1}{2}\left(v^2-v_0{}^2\right)=\frac{m}{r}-\frac{m}{r_0}+R\ . \tag{24}$$

The quantity R, by means of eq. (23), is splitted into two terms

$$R=R_1+R_2=\int_{r_0}^{r}\frac{3}{2}\frac{r_g L^2}{r^6}\left(\frac{dr}{dj}\right)^2 dr-\int_{r_0}^{r}\frac{r_g m}{r^3}dr\ . \tag{25}$$

We note that R_2 is readily calculated, while R_1 requires a little attention in order to gain some insight into integration problem. At first, we see that the integrand contains the factor

$\left(\frac{dr}{dj}\right)^2$. As it is well known the orbit, in the simplest Schwarzschild solution of the problem of motion, is the so-called semielliptic having the equation (Berry, 1989)

$$r = \frac{p}{1 + e\cos\left(1 - \frac{3r_g}{2r}j\right)} \tag{26}$$

and corresponding to a special plane curve whose name is polygasteroid (Loria, 1930).

The motion of the particle exhibits a trajectory which fill up everywhere densely the ring region bounded by two circles, respectively, with the maximum and the minimum r as the radii. As second step, we see from eq. (26) that $\frac{dr}{dj}$ (following GRT paradigm) differs from $\left(\frac{dr}{dj}\right)_{newt}$ (following classical celestial mechanics) on account of $O(r_g)$ terms.

Since $O(r_g^{\,n})$ terms, with $n \geq 2$, are negligible, we can substitute $\left(\frac{dr}{dj}\right)^2_{newt}$ into R_1, in order to perform calculation.

By means of simple calculations we deduce

$$\left(\frac{dr}{dj}\right)^2_{newt} = \frac{r^4 m^2}{L^4}\left[e^2 - \left(\frac{p}{r} - 1\right)^2\right]$$

and putting it into the expression for R_1 we will have

$$\begin{aligned} R_1 &= \int_{r_0}^{r} \frac{3}{2}\frac{r_g m^2}{r^2 L^2}\left[e^2 - \left(\frac{p}{r} - 1\right)^2\right] dr = \\ &\frac{3}{2}\frac{r_g m^2}{L^2}\left(1 - e^2\right)\left(\frac{1}{r} - \frac{1}{r_0}\right) + \frac{r_g m^2 p^2}{2L^2}\left(\frac{1}{r^3} - \frac{1}{r_0^{\,3}}\right) + \frac{3}{2}\frac{r_g m^2 p}{L^2}\left(\frac{1}{r_0^{\,2}} - \frac{1}{r^2}\right) \end{aligned} \tag{27}$$

whereas R_2 is provided by the formula

$$R_2 = -\frac{r_g m}{2}\left(\frac{1}{r_0^{\,2}} - \frac{1}{r^2}\right).$$

Remembering eq. (15) and $p = a(1 - e^2)$, finally we obtain

$$\mathrm{R} = R_1 + R_2 = \frac{3}{2}\frac{r_g m}{a}\left(\frac{1}{r} - \frac{1}{r_0}\right) + r_g m\left(\frac{1}{r_0^{\,2}} - \frac{1}{r^2}\right) + \frac{1}{2} r_g m a\left(1 - e^2\right)\left(\frac{1}{r^3} - \frac{1}{r_0^{\,3}}\right).$$

As in newtonian celestial mechanics, also in relativistic celestial mechanics are available two prime integrals. In fact during the particle motion remains constant the quantity

$$r^2 \dot{j} - L\left(1 - \frac{r_g}{r}\right) = \text{first prime integral},$$

particularly of value zero, but also the quantity

$$\frac{1}{2}v^2 - \frac{m}{r} - E = \text{second prime integral};$$

this last expression is the total energy of the particles. The correction term E is provided by the following formula

$$E = \frac{3}{2}\frac{r_g m}{ar} - \frac{r_g m}{r^2} + \frac{1}{2}\frac{r_g m}{r^3} a(1-e^2)$$

and it depends on m, r, r_g and also on a, e (size and shape of newtonian ellipse).

3. Concluding Remarks

The conclusions are derived here in the first order approximation of quantity r_g. The terms $O(r_g^n)$, with $n \geq 2$, have been dropped out. Similar approximations are used in textbooks using "parametrized post-newtonian (PPN) formalism" (Brumberg, 1991). Possible application of foregoing results to practical spacecraft astrodynamics (orbital maneuvers problems and so on) seems to us not reasonable. In fact the corrections would be negligible ones. Perhaps these results shall be taken into account in the far future in the case of mission towards neutron stars, black holes and similar massive heavenly bodies.

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Appendix

The aim of this appendix is to obtain eqs. (18) starting from eqs. (3), (4) and (5). It is substantially followed the Eddington's method. Let us write down eqs. (3) and (4).

$$\frac{d^2r}{ds^2} - \frac{\frac{r_g}{2}}{r^2\left(1-\frac{r_g}{r}\right)}\left(\frac{dr}{ds}\right)^2 - r\left(1-\frac{r_g}{r}\right)\left(\frac{dj}{ds}\right)^2 + \frac{\frac{r_g}{2}}{r^2\left(1-\frac{r_g}{r}\right)}c^2\left(\frac{dt}{ds}\right)^2 = 0. \tag{A.1}$$

$$\frac{d^2j}{ds^2} + \frac{2}{r}\frac{dr}{ds}\frac{dj}{ds} = 0. \tag{A.2}$$

It is covenient to transform the operator $\frac{d^2}{ds^2}$ in the following way. Let us write the operator $\frac{d}{ds}$ so

$$\frac{d}{ds} = \frac{d}{dt}\frac{dt}{ds} = \frac{dt}{ds}\frac{d}{dt}$$

and then we have

$$\frac{d^2}{ds^2} = \frac{dt}{ds}\frac{d}{dt}\left[\frac{dt}{ds}\frac{d}{dt}\right] = \left(\frac{dt}{ds}\right)^2\frac{d^2}{dt^2} + \frac{dt}{ds}\frac{d}{dt}\left(\frac{dt}{ds}\right)\frac{d}{dt}. \tag{A.3}$$

In eq. (A.3) the right hand second term is rewritten so

$$\frac{dt}{ds}\frac{d}{dr}\left(\frac{dt}{ds}\right)\frac{dr}{dt}\frac{d}{dt}. \tag{A.4}$$

Now we calculate the $\frac{d}{dr}\left(\frac{dt}{ds}\right)$ term using eq. (7):

$$\frac{d}{dr}\left(\frac{dt}{ds}\right)=\frac{d}{dr}\left(\frac{k}{1-\frac{r_g}{r}}\right)=-\frac{k\left(\frac{r_g}{r^2}\right)}{\left(1-\frac{r_g}{r}\right)^2}.$$

Eq. (A.3) becomes

$$\frac{d^2}{ds^2}=\left(\frac{dt}{ds}\right)^2\frac{d^2}{dt^2}+\frac{dt}{ds}\frac{dr}{dt}\frac{d}{dr}\left(\frac{dt}{ds}\right)\frac{d}{dt}=\left(\frac{dt}{ds}\right)^2\frac{d^2}{dt^2}-\frac{k\left(\frac{r_g}{r^2}\right)}{\left(1-\frac{r_g}{r}\right)^2}\frac{dr}{dt}\frac{dt}{ds}\frac{d}{dt}.$$

Again owing eq. (7), *i.e.* $\frac{dt}{ds}=\frac{k}{1-\frac{r_g}{r}}$, the foregoing equation becomes

$$\frac{d^2}{ds^2}==\left(\frac{dt}{ds}\right)^2\left\{\frac{d^2}{dt^2}-\frac{\left(\frac{r_g}{r^2}\right)}{\left(1-\frac{r_g}{r}\right)}\frac{dr}{dt}\frac{d}{dt}\right\}. \tag{A.5}$$

Let us come back to (A.1); applying eq. (A.5) we deduce

$$\left(\frac{dt}{ds}\right)^2\frac{d^2r}{dt^2}-\left(\frac{dt}{ds}\right)^2\frac{\left(\frac{r_g}{r^2}\right)}{\left(1-\frac{r_g}{r}\right)}\left(\frac{dr}{dt}\right)^2-\left(\frac{dt}{ds}\right)^2\frac{\left(\frac{r_g}{2}\right)}{r^2\left(1-\frac{r_g}{r}\right)}\left(\frac{dr}{dt}\right)^2-$$

$$-\left(\frac{dt}{ds}\right)^2\left(\frac{dj}{dt}\right)^2r\left(1-\frac{r_g}{r}\right)+\frac{\left(\frac{r_g}{2}\right)}{r^2\left(1-\frac{r_g}{r}\right)}c^2\left(\frac{dt}{ds}\right)^2=0,$$

and also

$$\frac{d^2r}{dt^2}-\frac{3}{2}\frac{r_g}{r^2}\frac{1}{\left(1-\frac{r_g}{r}\right)}\left(\frac{dr}{dt}\right)^2-r\left(1-\frac{r_g}{r}\right)\left(\frac{dj}{dt}\right)^2+\frac{m}{r^2}\frac{1}{\left(1-\frac{r_g}{r}\right)}=0\ . \tag{A.6}$$

The same procedure applied to (A.2) gives

$$\left(\frac{dt}{ds}\right)^2\left\{\frac{d^2j}{dt^2}-\frac{\left(\frac{r_g}{r^2}\right)}{\left(1-\frac{r_g}{r}\right)}\frac{dr}{dt}\frac{dj}{dt}\right\}+\frac{2}{r}\frac{dr}{dt}\frac{dj}{dt}\left(\frac{dt}{ds}\right)^2=0$$

and then

$$\frac{d^2j}{dt^2}-\frac{r_g}{r^2}\frac{dr}{dt}\frac{dj}{dt}+\frac{2}{r}\frac{dr}{dt}\frac{dj}{dt}=0.$$

Multiplying the foregoing equation by r

$$r\frac{d^2 j}{dt^2}+2\frac{dr}{dt}\frac{dj}{dt}=\frac{r_g}{r}\frac{dr}{dt}\frac{dj}{dt}=\frac{r_g}{r}\dot{r}\dot{j}\ = \text{transverse component of acceleration.}$$

From (A.6) we have

$$\frac{d^2 r}{dt^2}-r\left(\frac{dj}{dt}\right)^2=-\frac{m}{r^2}+\frac{3}{2}\frac{r_g}{r^2}\dot{r}^2-r_g\dot{j}^{\,2}-\frac{r_g m}{r^3}=\text{radial component of acceleration.}$$

We conclude that

$$F_{pr}=\frac{3}{2}\frac{r_g}{r^2}\dot{r}^2-r_g\dot{j}^{\,2}-\frac{r_g m}{r^3}$$

$$F_{pj}=\frac{r_g}{r}\dot{r}\dot{j}\,.$$