Helicity in Classical Electrodynamics and its Topological Quantization

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Any divergenceless vector field defined in a 3-dimensional manifold defines an integral quantity called the helicity that measures the way in which any pair of integral lines curl to one another. In the case of Classical Electrodynamics in vacuum, the natural helicity invariant, called the electromagnetic helicity, has an important particle meaning: the difference between the numbers of right- and left-handed photons. In a topological model of Classical Electrodynamics, the helicity is topologically quantized, in a relation that connects the wave and particle aspects of the fields.

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1. Introduction

The helicity of a divergenceless vector field was first used by Woltjer in 1958 [1] in an astrophysical context. Moreau [2] showed the helicity conservation in certain flows in fluid dynamics, and Moffatt, in a seminal paper [3], coined the term helicity and began the study of its topological meaning.

Let $\mathbf{X}(\mathbf{r})$, $\mathbf{r} \in V$, be a real vector field defined in a parallelizable 3-dimensinal manifold V, i.e. $\mathbf{X} : V \to R^3$. Let $\mathbf{X}$ be a divergenceless vector field, i.e. $\nabla \cdot \mathbf{X} = 0$. In this case, another vector field exists in V, at least locally, called a vector potential $\mathbf{Y}(\mathbf{r})$, such that $\mathbf{X} = \nabla \times \mathbf{Y}$. (The question of the local or global definition of $\mathbf{Y}$ is related to the cohomology of the manifold V.) We define the helicity of the divergenceless vector field $\mathbf{X}(\mathbf{r})$ in V as the integral quantity

$$h(\mathbf{X}, V) = \int_V \mathbf{X} \cdot \mathbf{Y} d^3 r. \tag{1}$$

We will write $h(\mathbf{X})$ or simply h if there is not risk of confussion. There are two physical contexts in which the helicity (1) has been specially useful: in fluid dynamics, where $\mathbf{Y}$ is the flow velocity $\mathbf{v}(\mathbf{r}, t)$, $\mathbf{X}$ is the vorticity $\mathbf{w} = \nabla \times \mathbf{v}$, and $h(\mathbf{w}, V)$ is called vortex helicity, and in electromagnetism, specially in plasma physics, where it is common to talk about the magnetic helicity,

$$h = \int_V \mathbf{A} \cdot \mathbf{B} d^3 r. \tag{2}$$

It can be proved (see [4]) that the helicity (2) is proportional to the linking number of the field lines. Even in the case that these lines are not closed, the notion of linkage takes sense, because a mean value of an asymptotic linking number can be defined, and this value

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coincides with the helicity. An important consequence is that the helicity of an unlinked magnetic field is zero.

The helicity has interesting features in the special case of Maxwell's theory in vacuum, because in this situation the electric field is dual to the magnetic field. This duality allows us to define a natural invariant for Maxwell's theory: the electromagnetic helicity, that is the sum of electric and magnetic helicities. This quantity has an important particle meaning, that complements its topological meaning. In fact, we will see that the electromagnetic helicity is the classical quantity formally equivalent to the quantum helicity operator, defined as the operator that, by acting on a photonic state, is the difference between the numbers of right- and left-handed photons. For a pedagogical review, see [5].

2. The electromagnetic helicity

In standard classical electrodynamics, the Maxwell equation $d\mathcal{F} = 0$, where $\mathcal{F}$ is the Faraday 2-form $\mathcal{F} = 1/2 F_{\mu\nu} dx^\mu \wedge dx^\nu$, becomes a Bianchi identity by using the electromagnetic potential $\mathcal{A}$, defined as $\mathcal{F} = d\mathcal{A}$. The dynamical equation for this field in empty space is $d{*}\mathcal{F} = 0$, where $*$ is the duality operator, ${*}\mathcal{F} = 1/2 F^*_{\mu\nu} dx^\mu \wedge dx^\nu$ and $F^*_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$. But the Minkowski spacetime R^4 has trivial cohomology. This means that the dynamical equation $d{*}\mathcal{F} = 0$ implies that ${*}\mathcal{F}$ is a closed 2-form, so it is also an exact form and we can write ${*}\mathcal{F} = d\mathcal{C}$, where $\mathcal{C}$ is another potential 1-form in the Minkowski space. Now the dynamical equation becomes another Bianchi identity. This simple idea is a consequence of the electromagnetic duality, that is an exact symmetry in vacuum. In tensor components, with $\mathcal{A} = A_\mu dx^\mu$ and $\mathcal{C} = C_\mu dx^\mu$, we have $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $F^*_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu$ or, in vector components,

$$\begin{aligned} \mathbf{B} &= \nabla \times \mathbf{A} = \frac{\partial \mathbf{C}}{\partial t} + \nabla C^0, \\ \mathbf{E} &= \nabla \times \mathbf{C} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla A^0. \end{aligned} \tag{3}$$

Note that the equations (3) are clearly invariants under the gauge transformations $A_\mu \mapsto A_\mu + \partial_\mu \alpha$, $C_\mu \mapsto C_\mu + \partial_\mu \beta$.

The electric and magnetic fields are dual to each other and they have the same properties in Maxwell theory in vacuum. Given the divergenceless vector field $\mathbf{B}$, we have defined the magnetic helicity as

$$h_m = \int_{R^3} \mathbf{A} \cdot \mathbf{B} d^3 r, \tag{4}$$

where $\mathbf{B} = \nabla \times \mathbf{A}$, and we have seen that the magnetic helicity is proportional to the linking number of the magnetic lines. Now, in vacuum, given the divergenceless vector field $\mathbf{E}$, we can also define an electric helicity through

$$h_e = \int_{R^3} \mathbf{C} \cdot \mathbf{E} d^3 r, \tag{5}$$

where $\mathbf{E} = \nabla \times \mathbf{C}$. This quantity will be obviously proportional to the linking number of the electric lines.

Now it is convenient to specify the contour conditions of the fields, that we take in order that the energy and the helicities are finite. So the spatial domain will be a compactification of R^3, that is, a simply-connected domain. By inspection of equations (4) and (5), the condition of finiteness of the helicity means that the electric and magnetic fields must decrease faster than r^{-2} in the surface $r \to \infty$. This implies that the potentials A^μ and C^μ must decrease faster than r^{-1} when $r \to \infty$. We will assume that our fields always satisfy these conditions.

By covariance questions, it is convenient to work with 4-currents of helicity. In [4] the following magnetic helicity current was introduced,

$$\mathcal{H}_m^\mu = A_\nu F^{*\nu\mu}. \tag{6}$$

With this equation as a base, the electric helicity current is

$$\mathcal{H}_e^\mu = C_\nu F^{\mu\nu}. \tag{7}$$

The 4-divergence of both currents is

$$\begin{aligned} \partial_\mu \mathcal{H}_m^\mu &= \frac{-1}{2} F^*_{\mu\nu} F^{\mu\nu} = -2\mathbf{E}\cdot\mathbf{B}, \\ \partial_\mu \mathcal{H}_e^\mu &= \frac{1}{2} F^*_{\mu\nu} F^{\mu\nu} = 2\mathbf{E}\cdot\mathbf{B}. \end{aligned} \tag{8}$$

So the magnetic and the electric helicities are time invariants for singular fields, i.e. fields that satisfy $\mathbf{E}\cdot\mathbf{B} = 0$.

Given any Maxwell field in vacuum, we define the 4-vector density of electromagnetic helicity $\mathcal{H}^\mu$ as the sum of the 4-vectors densities of electric and magnetic helicities (6) and (7),

$$\mathcal{H}^\mu = F^{\mu\nu} C_\nu - F^{*\mu\nu} A_\nu. \tag{9}$$

By construction, and taking into account equations (8), the density of electromagnetic helicity is a conserved current for any Maxwell field in vacuum, i.e. $\partial_\mu \mathcal{H}^\mu = 0$. This implies that the quantity

$$h = \int_{R^3} \mathcal{H}^0 d^3r = \int_{R^3} (\mathbf{A}\cdot\mathbf{B} + \mathbf{C}\cdot\mathbf{E})\, d^3r, \tag{10}$$

is a constant of the motion, $\partial h/\partial t = 0$, called electromagnetic helicity. From now on, we will call (10) helicity, and (9) will be density of helicity, leaving the adjectives for the electric and/or magnetic cases.

The helicity is gauge invariant, so we can work in a particular gauge to inform us about its meaning. The most appropriate one is the Coulomb gauge, in which the duality equations for the potentials (3) are precisely the Maxwell equations. Then, it is obvious that the solutions $\mathbf{A}$ and $\mathbf{C}$ can be written in terms circularly polarized waves, in the same way as in Quantum Electrodynamics [6],

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2\omega}} \left((\mathbf{e}_R a_R + \mathbf{e}_L a_L) e^{-ik\cdot x} + (\mathbf{e}_L \bar{a}_R + \mathbf{e}_R \bar{a}_L) e^{ik\cdot x} \right), \tag{11}$$

where $k^\mu = (\omega, \mathbf{k})$ is null ($k^\mu k_\mu = \omega^2 - \mathbf{k}^2 = 0$) and $k\cdot x = k^\mu x_\mu = \omega t - \mathbf{k}\cdot\mathbf{r}$. The factor $1/\sqrt{2\omega}$ is a normalization factor that allows the measure to be Lorentz invariant. $\bar{z}$ es el complex conjugate of z. The Fourier components a_R and a_L in (11) are functions of the vector $\mathbf{k}$. When the second quantization of the electromagnetic field is done, $a_R(\mathbf{k})$ is interpreted as a destruction operator of photonic states with energy ω, linear momentum $\mathbf{k}$ and spin $\mathbf{k}/\omega$, while the function $\bar{a}_R$ becomes the creation operator a_R^+ of such states.

Analogously, $a_L(\mathbf{k})$ is a destruction operator of photonic states with energy ω, linear momentum $\mathbf{k}$ and spin $-\mathbf{k}/\omega$, and a_L^+ is the correspondent creation operator [6]. For the potential $\mathbf{C}$, we have

$$\mathbf{C}(\mathbf{r},t) = \frac{i}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2\omega}} \left((\mathbf{e}_R a_R - \mathbf{e}_L a_L) e^{-ik\cdot x} - (\mathbf{e}_L \bar{a}_R - \mathbf{e}_R \bar{a}_L) e^{ik\cdot x} \right). \quad (12)$$

Introducing the expressions (11) and (12) we obtain for the helicity the expression

$$h = 2 \int \left(\bar{a}_R(\mathbf{k}) a_R(\mathbf{k}) - \bar{a}_L(\mathbf{k}) a_L(\mathbf{k}) \right) d^3k. \quad (13)$$

This is what we were looking for. In Quantum Electrodynamics, the right hand side of (13) is interpreted (except a factor 2) as the helicity operator, that rests the number of left-handed photons from the number of right-handed photons. We can write the usual expressions

$$N_R = \int \bar{a}_R(\mathbf{k}) a_R(\mathbf{k}) d^3k, N_L = \int \bar{a}_L(\mathbf{k}) a_L(\mathbf{k}) d^3k, \quad (14)$$

and equation (13) is written as

$$h = 2(N_R - N_L). \quad (15)$$

The consequence is that (except a factor 2) the helicity is the classical limit of the difference between the numbers of right-handed and left-handed photons [7, 8, 9]. Note that, in physical units (with $\hbar \neq 1$ and $c \neq 1$), the equation (15) would be

$$h = 2\hbar c(N_R - N_L). \quad (16)$$

We have previously defined the singular fields as those electromagnetic fields that satisfy $\mathbf{E} \cdot \mathbf{B} = 0$. Consider now the case of singular fields in vacuum, with the previously stated contour conditions, that we can sum up by saying that the helicity must be finite. In this case, the Fourier components a_R and a_L should be less singular than $\omega^{-3/2}$ when $\omega \to 0$ and they should decrease faster than ω^{-2} when $\omega \to \infty$. This behaviour allows us to proof the following property [9]: The electric and magnetic helicities of any singular field in vacuum are equal, $h_m - h_e = 0$. The conclussion (16) for singular fields is then

$$h = 2h_m = 2h_e = 2\hbar c(N_R - N_L). \quad (17)$$

3. Topological quantization of the helicity

A topological theory of electromagnetism proposed by one of the authors [10, 11] is based on the idea of electromagnetic knot and turns out to be locally equivalent to Maxwell's standard theory. Electromagnetic knots are electromagnetic fields defined by the condition that their force lines are closed curves and any pair of magnetic lines, or any pair of electric lines, is a link. The linking numbers, respectively n_m and n_e, are two integers that can be interpreted as the Hopf indices of two applications from the sphere S^3 to the sphere S^2 at any instant (taking into account the time dependence, the maps are from $S^3 \times R$ to S^2) [8, 9]. For knots in empty space, that is our case, $n_m = n_e$. Note that these integers give a measure of the curling of any pair of force lines around each other.

In references [8, 9] methods can be found to construct electromagnetic knots. This is done after identifying, via stereographic projection, the physical space R^3 with the sphere

S^3, and the set of complex numbers C with the sphere S^2. This implies to assume that the physical space has only one point at infinity and that there is only one infinite complex number. Any map $S^3 \to S^2$ is thus equivalent to a scalar complex field $\phi(\mathbf{x})$ with only one value at infinity (and vice versa). An electromagnetic knot can be constructed by means of a pair of such complex scalar fields, the level curves of which coincide with the magnetic and the electric lines, respectively. Let two maps $\phi, \theta : S^3 \to S^2$ be given. The pull-backs of the area 2-form ω of S^2 by ϕ and θ, respectively, noted $\mathcal{F} = \phi^*\omega$ and $*\mathcal{F} = \theta^*\omega$, are 2-forms in S^3 with nice properties. In particular their geometrical properties are similar to those of the electromagnetic fields in vacuum. For convenience and because of dimensional reasons, we redefine these 2-forms as $\mathcal{F} = -\sqrt{a}\,\phi^*\omega$ and $*\mathcal{F} = \sqrt{a}\,\theta^*\omega$, the normalizing constant $\sqrt{a}$ being measured in tesla times square meter. The electromagnetic field defined by these 2-forms is called an electromagnetic knot. Note that the scalar fields ϕ, θ have to satisfy the duality equation $- * (\phi^*\omega) = \theta^*\omega$.

A striking consequence follows. As is known, the maps $S^3 \to S^2$ can be classified in classes of homotopy labelled by the so called Hopf index. It is easy to see that, for the electromagnetic knots in vacuum, this means that their magnetic and electric helicities satisfy

$$h_m = h_e = na, \tag{18}$$

where n is the Hopf index of both ϕ and θ, that is the same for the two maps because of the duality conditions. It turns out to be equal to the linking number of any pair of level curves of ϕ, that is of magnetic lines, and to the linking number of any pair of level curves of θ, that is of electric lines [10, 11, 8, 9].

As it was shown in section 2, the magnetic and the electric helicities of any radiation electromagnetic field are equal. Moreover, in the case of the topological model, the helicities of the knots verify (18). Furthermore, the sum of the two helicities $h = h_m + h_e$, was shown to be a constant of the motion for any standard electromagnetic field in empty space. We call it the electromagnetic helicity and it verifies (16). It follows that

$$n = \frac{\hbar c}{a}(N_R - N_L). \tag{19}$$

Consequently, the value of $N_R - N_L$ for a knot is topologically quantized and takes the value $na/\hbar c$. (Note that this is true even if the knots are classical fields.) This suggests a criterion for the value of the normalizing constant. Taking $a = \hbar c$ (in natural units, this is $a = 1$) one has then

$$n = N_R - N_L. \tag{20}$$

Equation (20) relates, in a very simple and appealing way, two meanings of the term helicity, relating to the wave and particle aspects of the field. At left, the wave helicity: the linking number n, characterizing the way in which the force lines — either magnetic or electric — curl around one another. At right, the particle helicity: the difference between the numbers of right-handed and left-handed photons. This is certainly a nice property. It suggests that the electromagnetic knots are worth of consideration. Note that this property gives a new interpretation of the number n. We new that it is a magnetic and electric linking number, and also a Hopf index. We see that it is furthermore the difference of the classical limit of the numbers of right-handed and left-handed photons.

All the electromagnetic knots verify the quantum conditions

$$h_m = h_e = n\hbar c, \quad N_R - N_L = n. \tag{21}$$

Note that the set of the electromagnetic knots contains some with very low energy, for which n is necessarily very small. Even if they can be defined as classical fields, the real system would have quantum behaviour, since the action involved would be of the order of $\hbar$. On the other hand, there are states with n small and even zero, which have however macroscopic energy. They are those for which N_R, N_L are large. When n is large, the photon contents is high and the energy macroscopic. These are the states for which the classical approximation is valid.

This suggest that the set of the electromagnetic knots gives the classical limit of the quantum electromagnetic field with the right normalization.

References

[1] L. Woltjer, *Proc. Nat. Acad. Sci. USA* **44**, 489 (1958).

[2] J. J. Moreau, *C. R. Acad. Sci. Paris* **252**, 2810 (1961).

[3] H. K. Moffatt, *J. Fluid Mech.* **35**, 117 (1969).

[4] A. F. Rañada, *Eur. J. Phys.* **13**, 70 (1992).

[5] J. L. Trueba and A. F. Rañada, *Eur. J. Phys.* **17**, 141 (1996).

[6] B. Hatfield, *Quantum Field Theory of Point Particles and Strings* (Redwood, Addison-Wesley, 1992).

[7] A. F. Rañada, in *Fundamental Problems in Quantum Physics*, ed. M. Ferrero and A. van der Merwe (Dordrecht, Kluwer, 1995).

[8] A. F. Rañada and J. L. Trueba, *Phys. Lett.* **202 A**, 337 (1995).

[9] A. F. Rañada and J. L. Trueba, *Phys. Lett.* **232 A**, 25 (1997).

[10] A. F. Rañada, *Lett. Math. Phys.* **18**, 97 (1989).

[11] A. F. Rañada, *J. Phys.* **A 25**, 1621 (1992).