Link between the Non-Abelian Stokes Theorem and the B Cyclic Theorem

Petar K. Anastasovski (1), T. E. Bearden, C. Ciubotariu (2), W. T. Coffey (3), L. B. Crowell (4), G. J. Evans, M. W. Evans (5, 6), R. Flower, S. Jeffers (7), A. Labounsky (8), D. Leporini (9), B. Lehnert (10), M. Meszaros, J. K. Moscicki (11), P. R. Molnar, H. Munera (12), E. Recami (13), D. Roscoe (14) and S. Roy (15)
Institute for Advanced Study, Alpha Foundation, Institute of Physics
11 Rutafa Street, Building H, Budapest, H-1165, Hungary*

It is demonstrated that a non Abelian Stokes Theorem is necessary to describe the $\mathbf{B}^{(3)}$ field of radiation. A simple form of the theorem is built up from the fundamental definition of $\mathbf{B}^{(3)}$ in O(3) gauge field theory, which is a gauge field theory applied to electrodynamics with an O(3) internal gauge symmetry based on a complex basis ((1), (2), (3)). The indices (1) and (2) are complex conjugate pairs based on circular polarization, and the index (3) is aligned with the propagation axis of radiation.

1. Introduction

In gauge field theory the electromagnetic field is considered conventionally to be described by a U(1) symmetry internal gauge group. The U(1) group is abelian and linear. A non-linear commutator of complex vector potentials such as $\mathbf{A} \times \mathbf{A}^*$ is identically zero in the U(1) group [1-5] because it is asserted to be zero in the fundamental definition of the electrodynamic field tensor, the familiar four-curl of potentials. This procedure is however severely self-inconsistent because the object $-i\mathbf{A} \times \mathbf{A}^*$ defines the third Stokes parameter S_3 in free space. It is the archetypical signature of circular polarization, and if S_3 disappears, as in U(1) gauge field theory, so does the field intensity, proportional to the zero order Stokes parameter S_0. For circular polarization [6]:

$$S_o = \pm S_3 \tag{1}$$

and so U(1) gauge field theory is diametrically self-inconsistent because its structure and assumptions mean that circular polarization and field intensity vanish in free space. Since field intensity is proportional to field energy the momentum of the field also vanishes, another reduction to absurdity.

There is no way out of this self-inconsistency in U(1) gauge field theory, because $\mathbf{A} \times \mathbf{A}^*$ is always defined within its basic structure to be zero. If we are to accept gauge field theory as a basis for

* Also at: 1) Faculty of Technology and Metallurgy, Department of Physics, University of Skopje, Republic of Macedonia; 2) Institute for Information Technology, Stuttgart University, Stuttgart, Germany; 3) Department of Microelectronics and Electrical Engineering, Trinity College, Dublin 2, Ireland; 4) Department of Physics and Astronomy, University of New Mexico, Albuquerque, New Mexico; 5) former Edward Davies Chemical Laboratories, University College of Wales, Aberystwyth SY32 1NE, Wales, Great Britain; 6) sometime JRF, Wolfson College, Oxford, Great Britain; 7) Department of Physics and Astronomy, York University, Toronto, Canada; 8) The Boeing Company, Huntington Beach, California, 9) Dipartimento di Fisica, Universita di Pisa, Piazza Toricelli 2, 56100 Pisa, Italy, 10) Alfven Laboratory, Royal Institute of Technology, Stockholm, S-100 44, Sweden, 11) Smoluchowski Institute of Physics, Jagiellonian University, ul Reymonta, Krakow, Poland, 12) Centro Internacional de Fisica, A. A. 251955, Bogota, DC Colombia; 13) Faculty of Engineering, Bergamo State University, 24044 Dalmine, Italy; 14) School of Mathematics, Sheffield University, Great Britain., 15) George Mason University, Virginia, and Indian Statistical Institute, Calcutta, India.

development in field theory a different internal symmetry group is needed, one which allows a commutator such as $\mathbf{A} \times \mathbf{A}^*$ to be non-zero. Recently a considerable amount of development has taken place [7-12] of a gauge field theory with an O(3) internal symmetry applied to electrodynamics both on the classical and on the quantum electrodynamical levels. This means that the fundamental field equations become non-Abelian in nature, with covariant derivatives. The homogeneous field equation for example is a Feynman Jacobi identity [13]. The integral form of these field equations give a Stokes Theorem which is in general a non-Abelian Stokes Theorem, and the purpose of this short paper is to derive a simple example of a non-Abelian Stokes Theorem from the basic equation defining a non-zero $\mathbf{A} \times \mathbf{A}^* = \mathbf{A}^{(1)} \times \mathbf{A}^{(2)}$ in this O(3) symmetry gauge theory. For convenience, we refer to this theory as "O(3) electrodynamics" [7-12].

In O(3) electrodynamics the field quantities are tensors in Minkowski spacetime and vectors in the internal gauge space, ((1), (2), (3)) [7-12]. The fundamental definition of the field tensor [13] in O(3) electrodynamics contains quantities such as:

$$\mathbf{G}^{\mu\nu(1)*} = c\left(\partial^{\mu}\mathbf{A}^{\nu(1)*} - \partial^{\nu}\mathbf{A}^{\mu(1)*} - ig\mathbf{A}^{\mu(2)} \times \mathbf{A}^{\nu(3)}\right) \quad (2)$$

where there occurs a finite conjugate product of complex potentials in addition to the familiar four-curl of U(1) electrodynamics. The Greek indices obey the Minkowski algebra (contravariant-covariant rules) and the superscripts (1), (2) and (3) refer to the basis being used [7-12], one based on the empirical existence of circular polarization. One of the components of such a field tensor is a magnetic field, $\mathbf{B}^{(3)}$, which is oriented along the direction of propagation, and is sometimes referred to as the "Evans Vigier" field. It is defined by:

$$\mathbf{B}^{(3)} = -ig\mathbf{A}^{(1)} \times \mathbf{A}^{(2)} \quad (3)$$

where g is a constant of proportionality [7-12] given by $e/\hbar$, where e is the fundamental charge in coulombs and the Dirac constant. This relation is sometimes known [7-12] as the B Cyclic Theorem. The structure of O(3) electrodynamics is similar to that of Yang Mills theory, but the internal gauge space of O(3) symmetry is defined in the complex basis ((1), (2), (3)) [7-12], not considered by Yang and Mills.

The $\mathbf{B}^{(3)}$ field therefore self-consistently defines circular polarization, because it is proportional to the third Stokes parameter, and and defines magneto-optical effects such as the inverse Faraday effect [14] from the first principles of gauge field theory. In the conventional approach to electrodynamics, based on a U(1) gauge, the inverse Faraday effect is described by the phenomenological introduction of $\mathbf{A} \times \mathbf{A}^*$, which is self-inconsistent because in U(1), $\mathbf{A} \times \mathbf{A}^*$ is zero by definition. The same can be said of the third Stokes parameter. In order to introduce $\mathbf{A} \times \mathbf{A}^*$ self consistently into gauge field theory the field equations become [7-12]:

$$D_{\mu}\tilde{\mathbf{G}}^{\mu\nu} := \mathbf{0} \quad (4)$$

$$D_{\mu}\mathbf{H}^{\mu\nu} = \mathbf{J}^{\nu} \quad (5)$$

where $\mathbf{D}_{\mu}$ are O(3) covariant derivatives, and where $\tilde{\mathbf{G}}$ is the fundamental field tensor. The first equation is the Feynman Jacobi identity [13] corresponding to the homogeneous Maxwell equation in U(1) electrodynamics, and the second equation above is the O(3) equivalent of the inhomogeneous Maxwell equation in U(1) where the symbols $H^{\mu\nu}$ and J^{ν} have the same meaning. Recent work [7-12] has analyzed and considerably developed these field equations. The purpose of the present short paper is to show that the integral form of these differential equations must also be non-Abelian, *i.e.*, non-Abelian Stokes Theorems. A rigorous treatment of the non-Abelian Stokes Theorem requires advanced topology [15] so the treatment in this paper is kept at the simplest possible level. Any attempt [16] to develop or analyze the $\mathbf{B}^{(3)}$ component with a conventional Stokes Theorem is incorrect because $\mathbf{B}^{(3)}$ is defined within a non-Abelian gauge field theory.

The $\mathbf{B}^{(3)}$ field of radiation is therefore part of an O(3) symmetry gauge field theory and is thus a component of a non-Abelian field theory applied to electrodynamics systematically. Application of Stokes' Theorem must therefore be made within the same non-Abelian framework In this short paper a simple illustration of this procedure is made by starting from the basic definition of $\mathbf{B}^{(3)}$ in O(3) gauge theory.

Basic Definition of B$^{(3)}$

In O(3) gauge theory the basic classical definition of $\mathbf{B}^{(3)}$ is:

$$\mathbf{B}^{(3)*} = -i\frac{\kappa}{A^{(0)}}\mathbf{A}^{(1)}\times\mathbf{A}^{(2)} \tag{6}$$

where $\mathbf{A}^{(1)} = \mathbf{A}^{(2)}$ is the conjugate product of vector potentials, κ is the wave-vector, and $A^{(0)}$ is the magnitude of $\mathbf{A}^{(1)} = \mathbf{A}^{(2)*}$. Using the relation [7-12]:

$$B^{(0)} = \kappa A^{(0)} \tag{7}$$

we obtain:

$$\kappa A^{(0)}\mathbf{k} = -i\frac{\kappa}{A^{(0)}}\mathbf{A}^{(1)}\times\mathbf{A}^{(2)} \tag{8}$$

Without loss of generality it is possible to multiply both sides of this equation by the area πR^2 :

$$\kappa A^{(0)}\pi R^2\mathbf{k} = -i\frac{\kappa}{A^{(0)}}\mathbf{A}^{(1)}\times\mathbf{A}^{(2)}\pi R^2 \tag{9}$$

Therefore

$$\pi R\kappa A^{(0)}\mathbf{k}\cdot R\mathbf{k} = \pi\kappa\mathbf{A}^{(0)}\mathbf{R}\cdot\mathbf{R} = \mathbf{B}^{(3)}\cdot Ar\mathbf{k} \tag{10}$$

This expression can be integrated to give the non-Abelian Stokes Theorem in the form:

$$2\pi A^{(0)}\oint\mathbf{R}\cdot\mathrm{d}\mathbf{R} = \iint\mathbf{B}^{(3)}\cdot\mathrm{d}\mathbf{Ar} \tag{11}$$

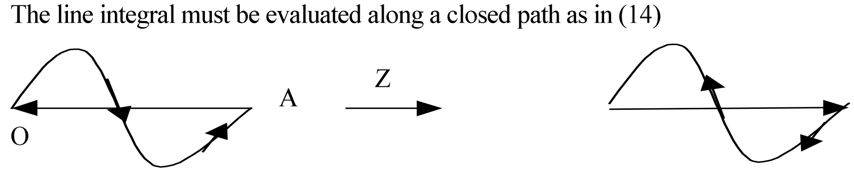
Finally, let

$$R = \frac{1}{\kappa} = \frac{\lambda}{2\pi} \tag{12}$$

and multiply both sides by $g = \kappa/A^{(0)}$ to define the phase:

$$\gamma = 2\pi\oint\mathbf{k}\cdot\mathrm{d}\mathbf{R} = \frac{\kappa}{A^{(0)}}\iint\mathbf{B}^{(3)}\cdot\mathrm{d}\mathbf{Ar} \tag{13}$$

The line integral must be evaluated along a closed path as in (14)



(14

A closed curve is therefore formed from a circle by drawing it out into a helix along the propagation axis (*Z* axis). The circumference of the circle remains $2\pi R$. The line integral starts at the origin O and proceeds around the helix back along the propagation axis from **A** to O. Off the Z axis the line integral is zero. Integration around a circle for example gives

$$\oint \mathrm{d}r = -\int_0^{2\pi} \sin t \mathrm{d}t + \int_0^{2\pi} \cos t \mathrm{d}t = 0 \tag{15}$$

The line integral along the off axial helix is formed from:

$$\Im = -\int_0^{2\pi} \sin\phi \mathrm{d}\phi \mathbf{i} + \int_0^{2\pi} \cos\phi \mathrm{d}\phi \mathbf{j} \tag{16}$$

which is zero for all ϕ.

The line integrals for all **r** (along the *Z* axis) are defined therefore by:

$$\gamma = 2\pi \int_{AO} \bar{\kappa} \cdot \mathrm{d}\mathbf{R} = -2\pi \int_{OA} \bar{\kappa} \cdot \mathrm{d}\mathbf{R} \tag{17}$$

and it is these line integrals that are responsible for interferometry, for example [7-12]. As soon as the concept of line integral is introduced, the area integral follows, and the non-Abelian Stokes Theorem links together the dynamical and topological phases. In terms of the vector potential:

$$\gamma = 2\pi g \oint \mathbf{A}^{(3)} \cdot \mathrm{d}\mathbf{R} = g \iint \mathbf{B}^{(3)} \cdot \mathrm{d}\mathbf{Ar} \tag{18}$$

and as the helical path closes to a circle:

$$R \to 0; \quad \kappa \to \infty \tag{19}$$

and so the line integral becomes inteterminate. The situation is then that of the Sagnac interferometer [7-12].

Discussion

From the basic definition of $\mathbf{B}^{(3)}$ in non-Abelian electrodynamics the Stokes Theorem can be constructed in a simple way and is obviously also non-Abelian in nature. The phase in eqn. (18) for example links two quantities, $\mathbf{A}^{(3)}$ and $\mathbf{B}^{(3)}$, which are not defined in Abelian electrodynamics. Yet $\mathbf{B}^{(3)}$ is an observable through the observable optical conjugate product $\mathbf{A}^{(1)} \times \mathbf{A}^{(2)}$, which also defines the third Stokes parameter. In Abelian electrodynamics, whose gauge group symmetry is U(1), neither $\mathbf{A}^{(3)}$ nor $\mathbf{B}^{(3)}$ are defined. Therefore it is erroneous [16] to apply the Abelian Stokes Theorem to the $\mathbf{B}^{(3)}$ field. This short note has given a simple example of a non-Abelian Stokes Theorem (eqn. (18)) constructed from the basic definition of $\mathbf{B}^{(3)}$ in O(3) gauge theory.

It is clear that the Stokes Theorem (18) has been built up from an initially non-Abelian relation, the definition of $\mathbf{B}^{(3)}$ in eqn. (6). More generally the Stokes Theorem relevant to O(3) electrodynamics is the integral form of eqns. (4) and (5), and in it most general topological form is given by:

$$\int_{\partial M} A = \int_M \mathrm{d}A \tag{20}$$

in advanced topology [13]. If Lie valued forms are used, path ordering of a trace must be used in order to connect the exponential of the Wilson loop to the curvature or field strength $F_{\mu\nu}$. This procedure is quite different from that used in U(1) electrodynamics, where the U(1) Stokes Theorem [1-5] clearly does not apply to $\mathbf{B}^{(3)}$. The latter is not defined in U(1) electrodynamics and is the archetypical signature of O(3) electrodynamics.

Acknowledgments

Many colleagues are thanked for many interesting discussions and the constituent institutes of AIAS for funding.

References

[1] Y. Aharonov and D. Bohm, *Phys. Rev*., **115**, 484 (1959).

[2] T. T. Wu and C. N. Yang, *Phys. Rev*., **12**, 3845 (1975).

[3] J. J. Sakurai, *Ann. Phys*., **11**, 1 (1960).

[4] A. Salam and J. C. Ward, *Il Nuovo Cimento*, **20**, 419 (1961).

[5] H. G. Loos, *J. Math. Phys*., **8**, 2114 (1967).

[6] L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon, Oxford, 1975).

[7] B. Lehnert and S. Roy, *Extended Electromagnetic Theory* (World Scientific, Sinpaore, 1998).

[8] T. W. Barrett and D. M. Grimes, (eds.), *Advanced Electrodynamics* (World Scientific, Singapore, 1995).

[9] M. W. Evans and S. Kielich, *Modern Nonlinear Optics* (Wiley, New York, 1997, paperback edition).

[10] M. W. Evans and A. A. Hasanein, *The Photomagneton in Quantum Field Theory* (World Scientific, Singapore, 1994).

[11]M. W. Evans, J.-.P Vigier, S. Roy and S Jeffers, *The Enigmatic Photon* (Kluwer, Dordrecht, 1994 to 1999) in five volumes.

[12] M. W. Evans and L. B. Crowell, *Classical and Quantum Electrodynamics and the* $\boldsymbol{B}^{(3)}$ *Field* (World Scientific, Singapore, 1999, in prep).

[13] L. H. Ryder, *Quantum Field Theory* (Cambridge Univ. Press, 1987, paperback edition).

[14] B. Talin, V. P. Kaftandjan and L. Klein, *Phys. Rev. A*, **11**, 648 (1975).

[15] R. P. Feynman, "Gauge Theories" in: R. Balian and C. H. Llewellyn Smith, (eds.), *Les Houches 29* (North Holland, Amsterdam, 1977).

[16] E. Comay, *Chem. Phys. Lett*, **261**, 601 (1996); replied to by M.W. Evans and S. Jeffers, *Found. Phys. Lett*., **9**, 587 (1996).