Optical Approach to Gravitational Redshift

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This paper explores an optical interpretation of gravitational redshift, and shows how the deflection of light and the radar echo delay can be calculated therefrom. As a result of these considerations, this work establishes new relationships between general relativity and geometrical optics.

1. Introduction

The general theory of relativity is now accepted as the most satisfactory theory of gravitation. This acceptance rests partly on its conceptual and structural elegance, and partly on its agreement with experimental observation. But the theory has been essentially mathematical in character, being concerned with the consequences of a geometrization of the space-time manifold. Thus the application of the theory involves the use of special mathematical methods which, although relevant to optics in many cases, may easily be considered separately from it. In attempting to deduce their optical nature, one may pass from the mathematical language to the physical language, and see how they can be reconciled with each other.

Four classic tests are usually cited as experimental verifications of the general theory of relativity: the gravitational redshift of spectral lines, the deflection of light by the Sun, the precession of the perihelion of the orbit of the planet Mercury, and the time delay of radar echoes passing close to the Sun. Three of these tests examine the influence of the gravitational potential on the propagation of light. Only the planetary orbit precession involves the motion of a particle of finite mass in the gravitational field of the Sun. Because these are optical phenomena, one may raise a question as to whether the three classic tests can also be correctly inferred from the point of view of optics.

It is actually possible to predict these tests in a valid manner on the basis of optics. This paper explores an optical version of gravitational redshift, and shows how the deflection of light and the radar echo delay can be calculated therefrom. The interpretation proposed in this paper attributes the redshift effect to optical phenomenon related to the velocity of propagation of light in a non-uniform medium. This gives a phenomenological derivation of the gravitational redshift, showing how the results of more realistic calculations can be obtained from the point of view of optics. The deflection of light, the radar echo delay, and the plasma effect of solar corona on them are then discussed. It will be of particular interest to note that the deflection of light and the radar echo delay can also be correctly derived from the equation of rays of geometrical optics without using the geodesic equations or the field equations of general relativity. As a consequence of these considerations, this paper will establish new relationships between general relativity and geometrical optics, apparently unrelated areas of physics, with regard to the influence of the gravitational potential on the propagation of light.

2. Redshift of Spectral Lines

In 1911, Einstein[1] predicted the change in the frequency of spectral lines with gravitational potential, generally referred to as the gravitational redshift. The effect was calculated from the time dilatation in a gravitational potential, which follows from the principle of equivalence. The argument, as Einstein explains, was as follows:

> *Let the two points S and S' be placed at rest, a distance* h *apart along the lines of force in a uniform gravitational field of acceleration* g*. In accordance with the principle of equivalence, we are able, in place of the system K in this gravitational field, to set the gravitational-free system K' which is accelerated with* –g*. Consider the process of propagation of radiation from S to S' from a system* K_o*, which is to be free from acceleration. At the moment when the radiation of frequency f is emitted from S to S', let the velocity of K' relative to* K_o *be zero. The radiation will arrive at S' when the time* h/c *has elapsed to a first approximation. But at this moment the velocity of S' relative to* K_o *is* v = gh/c*. Therefore, by Doppler's principle, the radiation arriving at S' does not possess the frequency f but a greater frequency f' which is related to f to a first approximation by the equation*
>
> $$f' = f\left(1 + \frac{v}{c}\right) = f\left(1 + \frac{gh}{c^2}\right). \tag{1}$$
>
> *By the equivalence of K and K', we may replace gh by the gravitational potential if the same process takes place in the system K.*

Astronomical observations, though somewhat ambiguous, have tended to confirm this effect. Since it does not seem possible to predict this interesting effect without using the general theory of relativity, the gravitational redshift is now recognized as resulting from the principle of equivalence. However, contrary to the current recognition, it would always seem possible to find a natural expression for the gravitational redshift from the optical point of view. If one seeks to introduce an optical nature of the gravitational redshift, special attention should be given to the velocity $v = gh/c$, not to the principle of equivalence. At least phenomenologically, the effect would appear to be due to this change of velocity which the radiation experiences during propagation along the lines of force of the gravitational field. Looking for a Newtonian mechanical interpretation, one finds without difficulty that it is equal to the velocity difference due to the medium or fluid which the radiation experiences during the propagation between the places at different gravitational potential.

Let ρ be the density and ϕ be the gravitational potential with $g = -\nabla\phi$. A uniform pressure throughout a fluid mass produces no effect on the motion. The time rate of change of the fluid momentum is equal to and opposite to the pressure gradient force in the medium. If we calculate the velocity difference due to the medium or fluid with differing gravitational potential according to

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -g, \text{ or } \nabla\phi, \tag{2}$$

we obtain the same velocity as that in Eq.(1). This leads to a simple physical interpretation: the redshift effect is attributed to the relative velocity change due to the medium or fluid by which light is affected during propagation in the medium. Such an interpretation, in contrast with its relativistic explanation, can be fitted into the customary point of view of optics, in that it ascribes the effect to an optical phenomenon related to the velocity of propagation of light in a non-uniform medium. Consequently, it leads us to consider the redshift effect as being purely optical in origin. In fact, it is difficult to distinguish a physical difference in form and content between the present interpretation and Einstein's argument here quoted. There is thus no objection in principle to interpreting the redshift result from the present point of view. Although Einstein further explained it as due to the time dilatation between the clocks in different gravitational potentials, what he had found was phenomenologically no more than the fact that the velocity of light is altered, linearly to a first approximation, by the medium as a result of the pressure gradient force.

A difference of interpretation already existed at the first time of observation.[2] Jewell in 1897 and particularly Fabry and Boisson in 1909 found displacements of solar spectral lines toward the red end of the spectrum, and ascribed them to an effect of pressure in the absorbing layer. However, Einstein's theory of general relativity[3] in 1916 established in most physicist's minds the interpretation of the redshift effect as the time dilatation in a gravitational potential, and this rather unusual interpretation has survived until the present. Like most relativistic explanations, the current interpretation is presented in the context of the four-vector space-time approach. The present approach reopens the question of interpretation and reminds us of the effect of pressure on the redshift of solar spectral lines. This means that, apart from the gravitational potential, any change in mechanical pressure, density and temperature of the medium can also give rise to an effect of the same kind on the redshift of spectral lines. To be reconciled with general relativity, however, vacuum in optics should be understood as a vacuum without even gravity.

In order to complete the present description, it is necessary to consider the hydrodynamic equation. The hydrodynamic equation is

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho v) = \nabla(\rho\phi) - \nabla P + F_v + \frac{1}{c}\mathbf{J}\times\mathbf{B}\,. \tag{3}$$

In addition to the gravitational force and pressure terms, we have included viscous and magnetic forces. As the Sun consists of a conducting medium with a magnetic field, it is necessary to include the magnetic force term in the hydrodynamic equation, leading to the magnetohydrodynamic equation. In the limit of very large conductivity, it is convenient to relate the current density **J** to the magnetic induction **B** via Ampere's law. If we use the vector identity and neglect viscous effects, the hydrodynamic equation takes the form

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho v) = \nabla\left(\rho\phi - P - \frac{B^2}{8\pi}\right) + \frac{1}{4\pi}(\mathbf{B}\cdot\nabla)\mathbf{B}\,. \tag{4}$$

Since the time required for light to propagate a path dr is dr/c to a first approximation, the integration of (4) gives

$$v(r) = \frac{1}{c}\left(\phi - \frac{P}{\rho} - \frac{B^2}{8\pi\rho}\right) + \frac{1}{4\pi\rho c}\int(\mathbf{B}\cdot\nabla)\mathbf{B}\,\mathrm{d}r \tag{5}$$

for the relative change of velocity in the medium which light experiences during propagation along the path. The velocity of light at the point of observation thereby becomes

$$c'(r) = c\left[1 + \frac{v(r)}{c}\right] = c\left[1 + \frac{1}{c^2}\left(\phi - \frac{P}{\rho} - \frac{B^2}{8\pi\rho}\right) + \frac{1}{4\pi\rho c^2}\int(\mathbf{B}\cdot\nabla)\mathbf{B}\,\mathrm{d}r\right] \tag{6}$$

as compared with its velocity c at the moment of emission. By Doppler's principle, it can be written in terms of frequency as

$$f'(r) = f\left[1 + \frac{1}{c^2}\left(\phi - \frac{P}{\rho} - \frac{B^2}{8\pi\rho}\right) + \frac{1}{4\pi\rho c^2}\int(\mathbf{B}\cdot\nabla)\mathbf{B}\,\mathrm{d}r\right]. \tag{7}$$

According to the present approach, the redshift effect is attributed not to the time dilatation between clocks in different gravitational potentials but to the velocity of propagation of light as affected by the medium as a result of pressure gradient including gravitational potential. This optical interpretation is consistent with the fact that such a redshift is absent in the spectrum lines of an atom influenced by the Coulomb potential of the atom.

An optical approach to the redshift effect may shed additional light on the formulation and particularly on its relation to property of the medium of propagation. From the point of view of electromagnetic waves, the light wave in Eq. (6) can be thought of as a wave in a medium with an index of refraction given by $c'(r) = c/n(r)$. Thus

$$\frac{1}{n(r)} = 1 + \frac{1}{c^2}\left(\phi - \frac{P}{\rho} - \frac{B^2}{8\pi\rho}\right) + \frac{1}{4\pi\rho c^2}\int(\mathbf{B}\cdot\nabla)\mathbf{B}\,\mathrm{d}r\,. \tag{8}$$

This consideration illustrates how the present picture of redshift offers a natural connection with the framework of optics. To reconcile optics and general relativity, as previously remarked, a vacuum must be understood to exclude fields of any kind. It has a consequence which is of fundamental importance for describing the deflection of light and the radar echo delay from the point of view of optics.

Terrestrial measurements are usually made with respect to a coordinate system fixed in the Earth, which rotates uniformly with a constant angular velocity ω relative to the inertial system. To an observer in the rotating system, it therefore appears as if the medium is moving under the influence of an effective acceleration of gravity[4]

$$g_{eff} = g - 2(\omega \times v) - \omega \times (\omega \times r)\,. \tag{9}$$

The apparent gravitational force acting on the medium is the sum of the actual gravitational force, the Coriolis force and the centrifugal force. After this consideration, we must replace ϕ by an effective potential ϕ_{eff} with g_{eff} in the case of a rotating system.

The redshift effect was qualitatively in agreement with astronomical observations both in the case of the Sun and in the case of white dwarf star like Sirius B where the effect is about thirty times larger. However, the quantitative agreement was not very good. While the frequency shift in Eq. (1) is independent of the point of observation on the solar disk, observations[5] have shown that the wavelength of spectral lines increases as the point of observation moves toward the limb. Furthermore, the solar lines observed at the limb are definitely asymmetric, having pronounced red flanks. There seems to be a systematic change in profile as one approaches the limb. In atomic spectra[6], the broadening of a spectral line due to pressure has shown that the spectral line observed is spread out more on the long wavelength side than it is on the short. With increasing pressure, the mean collision time increases and the time between collisions decreases with the result that, as the line is shifted to the red, it is broadened asymmetrically. From this point of view, the asymmetry observed in limb lines seems to be of pressure character. In fact, Blamont and Roddier[7] found a complete interpretation of their experimental value at the limb when they added to the gravitational redshift the pressure redshift of the Lindholm effect. Assuming this to be so, their interpretation, as well as asymmetric profile, has reminded us of the effect of pressure on the redshift of solar spectral lines, furnishing support for the present approach.

In contrast to astronomical observations, terrestrial experiments using Mossbauer effect are able to test the gravitational redshift to an excellent accuracy. In the experiments,[8] γ-rays in a nuclear resonance passed through an evacuated tube or a tube filled with helium along the lines of force of the gravitational field, and yielded results that were in agreement with the predicted shift in Eq. (1)

after subtraction of the effect due to the temperature difference between the source and absorber. A measurement of redshift in a rapidly rotating system[9] was shown to fit the dependence of g_{eff} on ω in Eq. (9).

3. Light Bending near the Sun

The general theory of relativity states that light rays propagated across a gravitational field undergo deflection. The theoretical value for the deflection of light rays that just graze the Sun's surface is 1.75″. The deflection angle is classically measured by comparing the apparent positions of stars that happen to lie near the solar disk during an eclipse, when their light comes close to the Sun and yet may be detected, with their positions at night six months earlier, when these stars lie on opposite sides of the Earth from the Sun, so their light does not pass close to the Sun on its way to us. In eclipse of 1919, about a dozen stars in all were observed, and yielded values 1.98 ± 0.12″ and 1.61 ± 0.31", in substantial agreement with Einstein's prediction. It is perhaps this dramatic result more than any other success that brought general relativity to the attention of the general public.

In radio astronomy, it is possible to measure the deflection of radio signals by the Sun with potentially far greater accuracy than is possible in optical astronomy. Each October, the quasi-stellar source 3C279 is occulted by the Sun, and radio astronomy groups have taken this opportunity to measure the change in relative position of two discrete radio sources, 3C273 and 3C279, during the period before and after occultation. The observed deflections are a separable combination of the general relativistic effect and of refraction in coronal electron plasma. For example, at radio frequency of 2388 MHz,[10] the maximum possible values of refractivities are 10^{-6} and 5×10^{-7} at $4R_S$ for the general relativistic effect and the coronal electron plasma, respectively, where R_S is the radius of the Sun. On the other hand, the coronal electron plasma can be virtually ignored at frequencies of 8000-10000 MHz.[11] Hence, at frequencies of 2000-4000 MHz, it is necessary to analyze the data in terms of a model, in which part of deflection arises from general relativity, and the rest is produced by the corona.

The Schwarzschild metric, appropriate for the region exterior to a spherically symmetric distribution of mass M, is given in the standard form as

$$c^2\,\mathrm{d}\tau^2 = \left(1-\frac{2GM}{c^2 r}\right)c^2\,\mathrm{d}t^2 - \left(1-\frac{2GM}{c^2 r}\right)^{-1}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\varphi^2). \tag{10}$$

In what follows, we use for the components of the metric tensor the expressions $g_{00}(r) = 1/g_{rr}(r) = 1 - 2GM/c^2r$. Assuming that the whole motion takes place in the plane $\varphi = 0$, we obtain as the equations of motion three differential equations. For light rays propagating along the geodesic lines, we replace the parameter τ by a parameter s describing trajectory. In particular, Weinberg[12] chooses to normalize s so that

$$g_{00}\frac{c\mathrm{d}t}{\mathrm{d}s} = 1 \quad \text{in} \quad g_{00}\frac{c\mathrm{d}t}{\mathrm{d}s} = \text{constant}. \tag{11}$$

On the normalization condition he has combined the three differential equations into one differential equation, which is of the same structure as (10). According to Weinberg, the change in θ as r decreases from infinity to its minimum value r_o is given by

$$\Delta\theta = \int_{\infty}^{r_o}\left[\frac{g_{00}(r_o)r^2}{g_{00}(r)r_o^{\,2}} - 1\right]^{-1/2}\frac{g_{rr}^{1/2}(r)\,\mathrm{d}r}{r}. \tag{12}$$

This integral can be evaluated by expanding in the small parameters GM/c^2r and GM/c^2r_o to first order, giving $4GM/c^2r_o$ for a light ray deflected by the Sun.

Having reviewed the deflection of light by the Sun, we now turn our attention to optics. The gaseous layers surrounding the Sun, through which light propagates, are media of spherically symmetric varying refractive index. The deflection of light can thus be thought as a result of the refraction of light in such gaseous layers from the point of view of optics. If this thought is reasonable, a similar conclusion will also be reached from this point of view. When we check the propagation of light rays to this end, we realize that the equation of rays of geometrical optics has previously had the form of Eq. (12).

In a homogeneous medium, the refractive index is a constant and the light rays have the form of straight lines. Let us consider rays in a medium which has spherical symmetry, *i.e.* where the refractive index depends only on the distance r from a fixed point O: $n = n(r)$. This case is approximately realized by the Earth's atmosphere, when the curvature of the Earth is taken into account. The light rays are then plane curves, situated in a plane through the origin, and along each ray satisfy[13]

$$n(r)r\sin\psi = \text{constant} \tag{13}$$

where ψ is the angle between the position vector **r** and the tangent at the point r on the ray. Since $\psi = \pi/2$ at the point r_o of closest approach of the ray to the origin, Eq. (13) may also be written as $n(r_o)r_o$ = constant. This relation is sometimes called the formula of Bouguer in geometrical optics. If (r,θ) are the polar coordinates of a plane curve, then the angle ψ between the radius vector to a point r on the curve and the tangent at r is given by

$$\sin\psi = \frac{r\mathrm{d}\theta}{(\mathrm{d}r^2 + r^2\mathrm{d}\theta^2)^{1/2}}. \tag{14}$$

From (13), (14), and Bouguer's formula, the equation of rays in a medium with spherical symmetry has been written in the form

$$\Delta\theta = \int\left[\frac{n^2(r)\,r^2}{n^2(r_o)r_o^2} - 1\right]^{-1/2}\frac{\mathrm{d}r}{r}. \tag{15}$$

At first sight, we can see a striking resemblance between this well-known equation and the geodesic equation (12). The equation for rays is lacking a term arising from the difference in path length. For lack of the term, the equation of rays corresponds to the case which is obtained when the curvature of the physical space in a region of strong gravitational potential is neglected. According to general relativity, the deflection of light is due partly to the varying velocity of light and partly to the non-Euclidean character of the spatial geometry. Since these are known to contribute equally to the deflection,[14] it can therefore be stated that the equation of rays will give a deflection of only half of the correct value.

This result is to be expected on optical grounds, because the non-Euclidean character of the spatial geometry has been neglected in optics. In order to compensate for the change in light path due to gravitational potential, one can use the notion of optical path. The optical path represents the distance light travels in a vacuum in the same time it travels a distance in the medium. If a light ray travels in a medium with spherical symmetry, the optical path is given by integral over $n(r)\mathrm{d}r$. This means that the radial interval of integration must be corrected by multiplication with $n(r)$ to take into account the difference in path length due to gravitational potential. Upon integration over $n(r)\mathrm{d}r$ instead of the original integration over $\mathrm{d}r$, it would yield a result in which the difference in path length is taken into consideration. Using the optical path to correct the change in light path, the equation of rays is modified to

$$\Delta\theta = \int \left[\frac{n^2(r)\, r^2}{n^2(r_o)\, r_o^2} - 1 \right]^{-1/2} \frac{n(r)\mathrm{d}r}{r}. \tag{16}$$

The modified equation of rays is then in complete agreement in structure with Eq. (12). From the proposition which has just been proved, one may picture what it is to be a curved space in a region of strong gravitational potential. As viewed from the present approach, the curvature of the physical space in the gravitational field of the Sun can best be understood in terms of the medium with spherical symmetry in which the path of rays is to be curved.

A comparison of (16) with (12) identifies $n^2(r)$ with $g_{rr}(r)$. In the equation of rays of geometrical optics, $n^2(r)$ plays exactly the same role $g_{rr}(r)$ has played in the geodesic equation of general relativity. This suggests introducing an optical metric tensor $n^2(r)$ consisting of the gravitational potential plus the mechanical pressure and the magnetic pressure. Taking only the leading gravitational potential into account, both of these equations give the same result for the deflection of light by the Sun. Thus the optical metric here proposed complements the Schwarzschild metric to this extent. Its line element will then be reconciled with the eikonal equation. The most important is that the existence of the deflection of light and its value can also be derived in the explicit form on the basis of geometrical optics. The emphasis should be on the fact that the deflection of light can be interpreted as a result of the refraction of light in the gaseous layers surrounding the Sun.

If we further include a frequency-dependent dielectric constant defined by Maxwell's equation $\varepsilon(\omega) = n^2(\omega)$, we can obtain the explicit and integrated form of the optical metric and exhibit completely its frequency dependence. Using Eq. (8) and the dielectric constant with $\varepsilon(0) = n^2(0)$, we find far above the highest resonant frequency

$$\begin{aligned} n^2(r,\omega) &= \varepsilon(\omega) n^2(r) \\ &= \left(1 - \frac{4\pi e^2 N}{m\omega^2}\right)\left[1 + \frac{1}{c^2}\left(\phi - \frac{P}{\rho} - \frac{B^2}{8\pi\rho}\right) + \frac{1}{4\pi\rho c^2}\int (\mathbf{B}\cdot\nabla)\mathbf{B}\, dr\right]^{-2}, \end{aligned} \tag{17}$$

where m and e are the mass and charge on the electron, and N is the total number of electrons per unit volume. Using this optical metric in Eq. (16) will provide a theoretical curve in the explicit and integrated form for observation that any beam of radiation is deflected during its passage near the Sun as a result of the general relativistic effect and of refraction in the coronal electron plasma. Since the characteristics of the propagation obviously depend on the index of refraction $n(\omega)$, it seems very natural to expect the frequency dependence of the deflection so discussed. In fact, Muhleman, Ekers, and Fomalont[10] analyzed their experimental data by using geometrical-optics techniques in a spherically symmetric refracting medium of index $n(r,\omega) = 1 + 2GM/c^2r - 2\pi e^2 N(r)/m\omega^2$, where $N(r)$ is the electron-density profile in the corona and interplanetary medium. Their interpretation is qualitatively in agreement with the present approach.

4. Radar Echo Delay

In 1964, Shapiro[15] proposed a fourth test of general relativity. The test involves measuring the time delays between transmission of radar signals from Earth to either Mercury or Venus and detection of the echoes. Because, according to the general theory of relativity, the speed of propagation of light depends on the strength of gravitational potential along its path, the time delays are maximum when the inner planets are at superior conjunction and the radar signals just graze the solar limb. The maximum round-trip excess time delays are estimated to be about 200 μsec. Such a change, equivalent to 60 km in distance, could be measured over the required path length with modern radar equipment by Shapiro and his collaborators.[16] The most reliable of the measured data agree, on the average, with this excess delay predictions of general relativity to well within the experimental uncertainty of ±20%.

According to Weinberg, the time required for light to go from r_o to r' is given by

$$\Delta t = \int_{r_o}^{r'} \left[1 - \frac{g_{00}(r) r_o^2}{g_{00}(r_o) r^2} \right]^{-1/2} \left(\frac{g_{rr}(r)}{g_{00}(r)} \right)^{1/2} \frac{\mathrm{d}r}{c}. \tag{18}$$

The integral can be evaluated by expanding in the small parameters GM/c^2r and GM/c^2r_o to first order, giving 240 μsec for the maximum round-trip excess time delay.

We are now in a position to derive an optical form of expression for the excess time delay. Its explicit form will follow from an equation which specifies the path of rays. An accurate expression we seek can then be obtained by converting the equation of rays into an equation for the path of rays.

A procedure starts from Eq. (14). Substitution of (14) into (13) gives

$$\frac{n(r) r^2 \mathrm{d}\theta}{(\mathrm{d}r^2 + r^2 \mathrm{d}\theta^2)^{1/2}} = \text{constant}. \tag{19}$$

Since the path of rays is $\mathrm{d}s = (\mathrm{d}r^2 + r^2\mathrm{d}\theta^2)^{1/2}$ in the polar coordinates of plane curve, this may also be written as

$$\frac{n(r) r^2 \mathrm{d}\theta}{\mathrm{d}s} = \text{constant}. \tag{20}$$

Solving for ds, we have

$$\mathrm{d}s = \frac{n(r) r^2 \mathrm{d}\theta}{n(r_o) r_o}. \tag{21}$$

In the above equation, Bouguer's formula $n(r_o)r_o = \text{constant}$ has been used. By making use of the integral in (15), the variable of integration can be changed from dθ to dr, thereby obtaining the result:

$$\Delta s = \int \left[1 - \frac{n^2(r_o) r_o^2}{n^2(r) r^2} \right]^{-1/2} \mathrm{d}r. \tag{22}$$

Hence, by dividing ds by $c'(r) = c/n(r)$, the time of propagation of rays is found to be

$$\Delta t = \int \left[1 - \frac{n^2(r_o) r_o^2}{n^2(r) r^2} \right]^{-1/2} \frac{n(r)\mathrm{d}r}{c}, \tag{23}$$

where $c'(r)$ is the speed of propagation of light in a region of gravitational potential. Although the details are altered by the new form of expression, the optical characteristics of (23) remain the same as in (15). Thus, for the correct calculation of excess time delay, we must consider in addition the difference in the path of rays due to gravitational potential.

As discussed in the equation of rays for the deflection, this requires integrating the resulting equation along the optical path. However, it draws a clear distinction between geometrical optics and general relativity, because Eq.(23) has already manifested the form of the integral over the optical path. To be reconciled with general relativity, in addition to the varying velocity of light with gravitational potential, the difference in path length must also be taken into consideration. If we make correction in the radial component of the path of rays, that is, in the component of the path along the lines of force of the gravitational field, the integral in (23) becomes

$$\Delta t = \int \left[1 - \frac{n^2(r_o)r_o^2}{n^2(r)r^2} \right]^{-1/2} \frac{n^2(r)\mathrm{d}r}{c}. \tag{24}$$

The modified integral and that in (18) are again in complete agreement in structure. There is indeed no difference, to first order in GM/c^2r and GM/c^2r_o, between (18) and (24). This once again identifies $n^2(r)$ with $g_{rr}(r)$ in their roles, leading us to consider $n^2(r)$ as an optical metric tensor and to reconcile its eikonal equation with the line element. It is then apparent that the equation of rays of geometrical optics also predicts the radar echo delay in exactly the same form as given by the geodesic equation of general relativity.

5. Plasma Effect of Corona

Having established new relationships between general relativity and geometrical optics, we can now discuss the classic tests of general relativity from the point of view of optics. In particular, the present approach affords a straightforward way to calculate the plasma effect of the solar corona on the deflection of light and the radar echo delay. Simultaneous equivalently accurate measurements at various frequencies will allow the plasma effect to be deduced, since the plasma effect is frequency dependent and the general relativistic effect is not. The plasma effect of the solar corona is evaluated in this section.

As a first important example of the plasma effect of the solar corona, we consider the deflection of light as a combination of the general relativistic effect and of refraction in the coronal plasma. The expected angular deflection can be accurately computed using the frequency-dependent refractive index expressed in (17) in the equation of rays (16):

$$\Delta\theta = \int \left[\frac{n^2(r,\omega)\,r^2}{n^2(r_o,\omega)r_o^2} - 1 \right]^{-1/2} \frac{n(r,\omega)\mathrm{d}r}{r}. \tag{25}$$

In order to evaluate this integral, we use in the integrand expansions in the small parameters. It is both easier and more instructive to evaluate the integral after the expansions. The calculations can be carried to first order in the small parameters with high accuracy. We now carry out the integration of (25).

The argument of the square root in (25) can be expanded to first order in the small parameters as

$$\begin{aligned} \frac{n^2(r,\omega)r^2}{n^2(r_o,\omega)r_o^2} - 1 &\cong \frac{r^2}{r_o^2}\left[1 + \frac{2GM}{c^2}\left(\frac{1}{r} - \frac{1}{r_o}\right) - \frac{4\pi e^2}{m\omega^2}\big(N(r) - N(r_o)\big) \right] - 1 \\ &\cong \left(\frac{r^2}{r_o^2} - 1\right)\left[1 - \frac{2GMr}{c^2 r_o(r + r_o)} + \frac{4\pi e^2 r^2}{m\omega^2(r_o^2 - r^2)}\big(N(r) - N(r_o)\big) \right], \end{aligned} \tag{26}$$

so (25) gives

$$\Delta\theta \cong \int \left(\frac{r^2}{r_o^2} - 1\right)^{-\frac{1}{2}} \frac{\mathrm{d}r}{r}\left[1 + \frac{GM}{c^2 r} + \frac{GMr}{c^2 r_o(r + r_o)} - \frac{2\pi e^2 N(r)}{m\omega^2} - \frac{2\pi e^2 r^2\big(N(r) - N(r_o)\big)}{m\omega^2(r_o^2 - r^2)} \right]. \tag{27}$$

Consequently, the deflections from the individual effects are combined linearly. Refraction effect in the solar corona is now represented by

$$\delta\theta_c = \int \left(\frac{r^2}{r_o^2} - 1\right)^{-\frac{1}{2}} \frac{\mathrm{d}r}{r}\left[\frac{2\pi e^2 N(r)}{m\omega^2} + \frac{2\pi e^2 r^2\big(N(r) - N(r_o)\big)}{m\omega^2(r_o^2 - r^2)} \right]. \tag{28}$$

This must be an addition to the general relativity deflection.

In the Allen-Baumbach model,[17] the electron-density profile in the corona is assumed to have the form $N(r) = 1.55 \times 10^8 (R_S/r)^6$ electron/cm^3. Using more recent results on the corona, Erickson[18] found that $N(r) = 5 \times 10^5 (R_S/r)^2$ electron/cm^3 represents the data reasonably well from $4R_S$ to $20R_S$. Refraction effect is significant where $r < 3R_S$, at which the $(R_S/r)^6$ term dominates. Hence we use the electron distribution of the Allen-Baumbach model, resulting in:

$$` \delta\theta_c = \frac{6.24\times10^{15}}{f^2}\int\left(\frac{r^2}{r_o^2}-1\right)^{-\frac{1}{2}}\frac{\mathrm{d}r}{r}\left[\left(\frac{R_S}{r}\right)^6+\frac{r^2}{r_o^2-r^2}\left\{\left(\frac{R_S}{r}\right)^6-\left(\frac{R_S}{r_o}\right)^6\right\}\right]. \tag{29}$$

The integration for $\delta\theta_c$ is straightforward, and gives

$$\delta\theta_c = \frac{6.24\times10^{15}}{f^2}\left(\frac{R_S}{r_o}\right)^6\left[\frac{105}{48}\theta+\frac{57}{48}\cos\theta\sin\theta+\frac{11}{24}\cos^3\theta\sin\theta+\frac{1}{6}\cos^5\theta\sin\theta\right], \tag{30}$$

where $\cos\theta = r_o/r$.

The total change in θ as r decreases from infinity to its minimum value r_o and then increases again to infinity is just twice its change from ∞ to r_o, that is, $2\Delta\theta$. Hence the deflection of the path of rays from a straight line is given by $\delta\theta = 2\Delta\theta - \pi$, which is calculated positively if concave toward the Sun and negatively if convex. Putting in the numerical factors, the total deflection is

$$\delta\theta \cong 1.75''\left(\frac{R_S}{r_o}\right)-\frac{6.24\times10^{15}}{f^2}\left(\frac{105\pi}{48}\right)\left(\frac{R_S}{r_o}\right)^6. \tag{31}$$

Equation (31) describes interesting behavior of the radiation bending near the Sun. The first term represents the general relativistic effect by which the path of rays is bent toward the Sun. The second term represents the coronal refraction by which the path of rays is bent away from the Sun to the contrary. This is not surprising when we see the difference in sign between these terms. Actually, experimental values at radio frequencies of the general relativity deflection were determined by fitting, by the method of least squares, the measured data to curve of a model bearing difference of sign between these effects. At optical frequencies, coronal refraction is extremely small, so it can be neglected. However, at radio frequencies, it plays an important part in the deflection, as shown in Figure 1, which shows the deflection angle as a function of frequency for the distances in solar radii of the ray's point of closest approach to the Sun's center.

The question might be raised as to whether varying velocity of light in the coronal plasma also gives rise to a change in path length therein. If we assume that varying velocity of light in the coronal plasma does not give rise to a change in path length of rays therein, the radial interval of integration must still be corrected by multiplication with $n(r)$ even in the coronal plasma, not with $n(r,\omega)$ as used in (25). We must then drop the fourth term in the integrand of the integral in (27), that is, the first term in (28). Coronal refraction thus obtained will be exactly the same as what one finds by evaluating the original equation of rays (15) on purely optical grounds. Note that there is a complete agreement in the form of expression for the plasma effect between Eq. (15) and Eq. (25) with such an assumption. In fact, the evaluation of coronal refraction from the equation of rays (15) was carried out to first order by Bracewell, Eshelman, and Hollweg.[19] Their calculation gives $82(R_S/r_o)^6$ sec for the angular deviation of a ray of frequency 9.6 GHz in the corona assuming the Allen-Baumbach model. When Erickson's coronal model is instead assumed, the angular deviation is given by $0.14(R_S/r_o)^2$ sec. Seielstad, Sramek, and Weiler[11] used in data analysis these values as parameters describing refraction effects in the solar corona, when they measure the deflection of 9.602 GHz radiation from 3C279 in the solar gravitational field using an interferometer at the Owens Valley

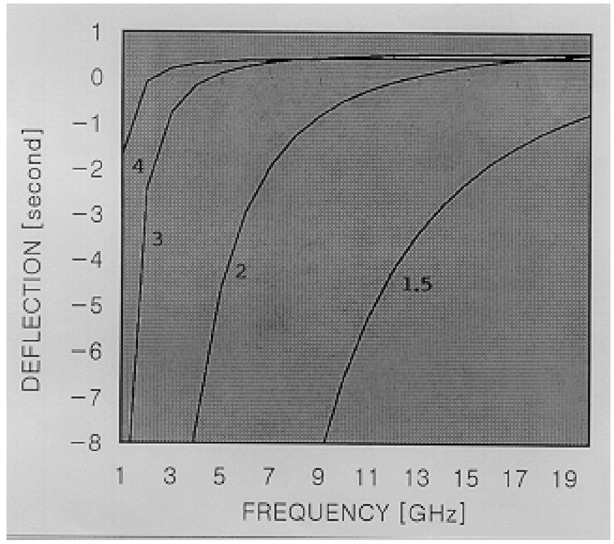

Figure 1. Deflection angle as a function of frequency for r_o/R_S = 1.5, 2, 3, 4.

Radio Observatory. The results of their calculation are indeed in exact agreement with what we would obtain from each model if we excluded the first term from the integrand of the integral in (28) on that assumption. However, on the assumption that varying velocity of light in the coronal plasma also gives rise to the change thereby in path length of rays, Eq. (31) gives coronal refraction of $96(R_S/r_o)^6$ sec for 9.6 GHz frequency. If we used Erickson's coronal model, we would obtain coronal refraction of $0.21(R_S/r_o)^2$ sec. To say it in terms of the metric tensor here proposed, the difference of calculation can be viewed in a way that is very helpful in providing physical insight into the term arising from the difference in path length. In Eq. (31), we have used the metric tensor of the components

$$g_{00} = \frac{1}{\varepsilon(\omega)n^2(r)} \quad \text{and} \quad g_{rr} = \varepsilon(\omega)n^2(r). \tag{32}$$

As viewed from the present approach, their calculation corresponds to the case which is obtained when the components of the metric tensor are

$$g_{00} = \frac{1}{\varepsilon(\omega)n^2(r)} \quad \text{but} \quad g_{rr} = n^2(r) \quad \text{or} \quad 1. \tag{33}$$

The reason for this difference is readily understood by referring to the equations of rays (16) and (15) from which angular deviations were respectively calculated.

As a second example of the plasma effect, let us calculate the plasma effect of the solar corona on the radar echo delay. The time required for light to go from r_o to r' is given by Eq. (24). To evaluate the dependence of the radar echo delay on the frequency being propagated, we must consider Eq. (24) with the frequency-dependent refractive index in (17):

$$t(r_o, r') = \int_{r_o}^{r'} \left[1 - \frac{n^2(r_o,\omega)r_o^2}{n^2(r,\omega)r^2}\right]^{-1/2} \frac{n^2(r,\omega)\mathrm{d}r}{c} \ . \tag{34}$$

In order to evaluate this integral, we once again use in the integrand the expansions in the small parameters to first order. Proceeding in exactly the same way as for (25), Eq. (34) gives

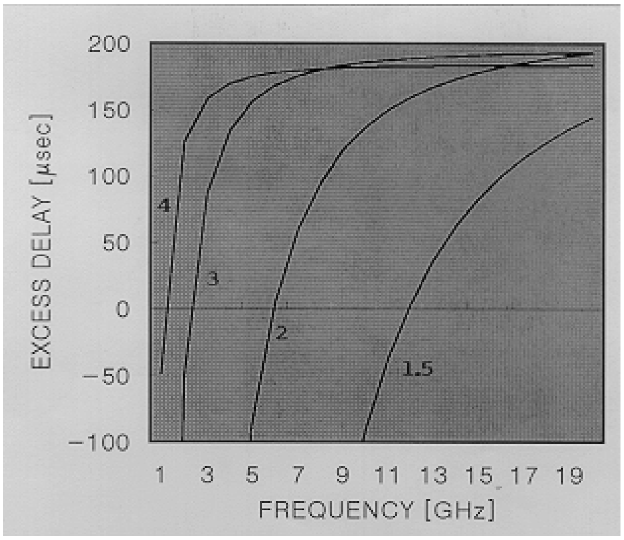

Figure 2. Excess time delay as a function of frequency for r_o/R_S = 1.5, 2, 3, 4.

$$t(r_o, r') \cong \int_{r_o}^{r'} \left(1 - \frac{r_o^2}{r^2}\right)^{-\frac{1}{2}} \frac{\mathrm{d}r}{c} \left[1 + \frac{2GM}{c^2 r} + \frac{GMr_o}{c^2 r(r + r_o)} - \frac{4\pi e^2 N(r)}{m\omega^2} - \frac{2\pi e^2 r_o^2 \left(N(r_o) - N(r)\right)}{m\omega^2 (r^2 - r_o^2)}\right] . \quad (35)$$

The time required for radar signals to travel to Mercury and be reflected back to Earth is $2[t(r_E,r_o) + t(r_o,r_M)]$, where r_E and r_M are astronomical radii of the orbits of the Earth and the Mercury around the Sun. The round-trip excess time delay is then given by $\delta t = 2[t(r_E,r_o) + t(r_o,r_M) - T(r_E,r_o) - T(r_o,r_M)]$, where $T(r_E,r_o)$ and $T(r_o,r_M)$ are the times required for radar signals to travel the paths in straight lines at speed c. The distance r_o of closest approach of the radar wave to the center of the Sun is much smaller than the distances r_E and r_M of the Earth and Mercury from the Sun.

Assuming the electron distribution of the Allen-Baumbach model, the integral yields

$$\delta t \cong \frac{4GM}{c^3}\left[1 + \ln\left(\frac{4r_E r_M}{r_o^2}\right)\right] - \frac{6.24 \times 10^{15}}{f^2}\left(\frac{21\pi r_o}{4c}\right)\left(\frac{R_S}{r_o}\right)^6 . \quad (36)$$

If, instead, we use Erickson's coronal model, we then have

$$\delta t \cong \frac{4GM}{c^3}\left[1 + \ln\left(\frac{4r_E r_M}{r_o^2}\right)\right] - \frac{2.01 \times 10^{13}}{f^2}\left(\frac{6\pi r_o}{c}\right)\left(\frac{R_S}{r_o}\right)^2 . \quad (37)$$

As in the case of the deflection, because of the sign difference, the plasma effect of the solar corona on the time delay is opposite to what is usually expected from the general relativistic effect. For either equation, the positive terms on the right describe a general relativistic delay in the time it takes a radar signal to travel to Mercury and back. In contrast, the negative terms describe a coronal plasma contraction, that is, a radar time contraction. Figure 2 shows frequency dependence of round-trip excess time delay for the distances of closest approach of the radar wave to the center of the Sun.

We now compare Shapiro's calculation with the results obtained from the preceding equations. At the beginning of measurement, Shapiro estimated $\delta t \cong 1.4 \times 10^{-4} - 3.7 \times 10^{-4}$ sec for observations of Mercury near superior conjunction with $r_o \approx 4R_S$ at 430 MHz frequency of the Arecibo Ionospheric Observatory. This difference in time delays between the general relativistic effect and the coronal plasma effect was nowhere large enough and positive for a really reliable result to be obtained solely from Arecibo data. He was thus tried to reduce the plasma effect by a factor of almost 400 by using measurements made at 8350 MHz frequency of Haystack radar of Lincoln Laboratory. For observation of Mercury with $r_o = 4R_S$ at 430 MHz radiation, Eq. (36) gives $\delta t \cong 1.8 \times 10^{-4} - 12.6 \times 10^{-4}$ sec, and Eq. (37), in which the same model as used by Shapiro has been assumed, gives $\delta t \cong 1.8 \times 10^{4} - 11.9 \times 10^{4}$ sec. The values obtained for the plasma effect are about three times larger than Shapiro's estimate. This is because we make a correction in the radial interval of the path and integrate the resulting equation along the optical path bending near the Sun, unlike Shapiro's calculation along the optical path in straight line without correction. If we assume that the difference in path length is due solely to differing gravitational potential, the radial interval of integration must still be corrected by multiplication with $n(r)$ even in the coronal plasma, not with $n(r,\omega)$ as used in (34). The excess time delays are then given by $1.8 \times 10^4 - 8.4 \times 10^4$ sec and $1.8 \times 10^4 - 7.9 \times 10^4$ sec, respectively. The values so obtained for the plasma effect are just what we should expect if the excess time delays were calculated from the equation of rays without any correction on purely optical grounds.

6. Conclusion

The discussion in this paper shows that the redshift effect is attributed not to the gravitational potential alone but to the pressure gradient including the gravitational potential in the medium through which light propagates. Asymmetry observed in limb lines furnishes physical support for the effect of pressure on the redshift of solar spectral lines. It has shown that the equation of rays of geometrical optics can also predict the correct values for the deflection of light and the radar echo delay by the Sun. Furthermore, agreement in structure between the equations of rays of geometrical optics and the geodesic equations of general relativity suggests introducing an optical metric of $n^2(r)$ consisting of the gravitational potential, the mechanical pressure, the magnetic pressure, and viscous effects in the medium of propagation. These results are obtained without direct calculations only by comparing the equations of rays with the geodesic equations formulated by Weinberg. In the light of this fact, Weinberg's formalism has opened a door to establish new relationships between general relativity and geometrical optics. Indeed, it was the formal similarity between (12) and (15) that enabled the present approach to be proposed.

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Appendix: Review of Planetary Motion

According to the general theory of relativity, the elliptical orbit of a planet rotates in its own plane in the same direction as the planet moves. For the motion of Mercury in which the effect can be detected most easily, the theory predicts an advance of the perihelion of angle 43.03" per century. This value is in excellent agreement with 42.56 ± 0.94" that is left after subtraction of all other known effects from the total observed motion.[20] Einstein's announcement of the general theory of relativity in its definitive form was immediately hailed by some astronomers as explaining a previously unaccountable discrepancy between the observed and theoretical motions of this planet, although some astronomers were, however, intuitively opposed to relativity. It must be by far the most important experimental verification of general relativity, both by means of the formal clarity brought to the theory by a space-time geometrization and by virtue of its high accuracy.

In this appendix, we wish to review the equation of the orbit in terms of the energy and angular momentum. The planet's path is not an ellipse but an exceedingly complicated space-curve due to the disturbing effects of all of the other planets. The equation of the orbit may therefore be meaningless, but we can still talk in terms of the system energy and the system angular momentum. For many applications, the equation of motion containing the energy and angular momentum is the natural one. In order to discuss the comparison with Newton's theory or the transition to quantum theory, it is important that the description of the motion be in terms of its energy and angular momentum.

Let us see what can be learned from Einstein's theory, using only the equations of motion in the Schwarzschild metric field, before deriving the orbit equation. We begin by pointing out that the path length of a particle in a static isotropic gravitational field is $\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$. The squares of velocity are then given by

$$c^2 \to g_{00}c^2,\ v^2 \to g_{rr}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + r^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2. \quad \text{(A1)}$$

In consequence of these relations the Schwarzschild metric in (10) can be written

$$\mathrm{d}\tau = g_{00}^{1/2}\mathrm{d}t\left(1-\frac{v^2}{c^2}\right)^{1/2}, \quad \text{(A2)}$$

to a first approximation. This form of equation reduces to the familiar equation leading to the Lorentz time dilation in the limit as g_{00} approaches to unity. In this sense one may say the relation in (A2) as the Schwarzschild time dilation.

The equations of motion in the Schwarzschild field yield two constants of motion. One of them is given by

$$g_{00}\frac{\mathrm{d}t}{\mathrm{d}\tau} = \text{constant}, \quad \text{(A3)}$$

which corresponds to the energy of the system. The other constant is obtained from $r^2(\mathrm{d}\theta/\mathrm{d}\tau) = \text{constant}$, and is absorbed immediately into the definition of the angular momentum. It would seem at first sight that the constant in (A3) is of no importance, for clearly any constant of integration can be added to the right of (A3) without affecting the validity of the equation. However, it is evident that the constant has an important physical significance, for it can bring the formulation of the resulting relativistic mechanics in terms of the energy of a particle as in the case of Hamiltonian in Newtonian mechanics. The relativistic equations of motion must be such that in the nonrelativistic limit they go over into the customary forms given by Newton's theory. Thus the task of identifying the constant is greatly facilitated by seeking the form which it would have in the nonrelativistic limit. In the nonrelativistic limit, Eq. (A3) can be expanded as

$$g_{00}\frac{\mathrm{d}t}{\mathrm{d}\tau} \cong \frac{1}{mc^2}\left(mc^2 + \frac{1}{2}mv^2 - \frac{GMm}{r}\right), \quad \text{(A4)}$$

where m is the mass of a particle. By comparison with Newton's theory, we can identify the constant with

$$g_{00}\frac{\mathrm{d}t}{\mathrm{d}\tau} = \frac{1}{mc^2}\left(mc^2 + E\right). \quad \text{(A5)}$$

Consequently it yields the expression

$$E = mc^2 g_{00}\frac{\mathrm{d}t}{\mathrm{d}\tau} - mc^2 \quad \text{(A6)}$$

for the energy of a particle of mass m and velocity $\mathbf{v}$ in the static isotropic gravitational field. From Eq.(A1), the momentum is defined by

$$\mathbf{p}_r = mg_{rr}^{1/2}\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}\tau} \text{ and } \mathbf{p}_\theta = mr\frac{\mathrm{d}\theta}{\mathrm{d}\tau} = \frac{l}{r}. \quad \text{(A7)}$$

Equations (A6) and (A7) are the necessary relativistic generalizations for the energy and momentum of a particle, consistent with the conservation laws and the postulates of general relativity.

As in the special theory of relativity, it is natural to attempt to identify the four equations of energy and momentum conservation as relations among the energy-momentum four-vectors. We observe that the momentum in (A7) is proportional to the spatial components of the four-velocity defined as $\sqrt{g_{\alpha\alpha}}\mathrm{d}x^\alpha/\mathrm{d}\tau$ in (A1). The time component of this four-velocity is $\sqrt{g_{00}}c\mathrm{d}t/\mathrm{d}\tau$. Comparison with (A6) shows that the energy of a particle differs from its time component by the rest energy mc^2. We are thus led to

$$E = mc^2 g_{00}\frac{\mathrm{d}t}{\mathrm{d}\tau} \quad \text{(A8)}$$

as the covariant form of the total energy, for then the momentum $\mathbf{p}$ and $E/c\sqrt{g_{00}}$ form an energy-momentum four-vector. Formally the connection between the energy E and the momentum is expressed in the Schwarzschild metric in (10) or in the statement that the magnitude of the momentum four-vector is constant:

$$\frac{E^2}{c^2 g_{00}} - (p_r^2 + p_\theta^2) = m^2c^2. \quad \text{(A9)}$$

Equation (A9) is an extension of the relation $T^2/c^2 - p^2 = m^2c^2$ in special relativity that meets the requirements of general relativity.

Note that the gravitational potential lends itself to incorporation in the metric of space-time geometrization, so the potential energy is absorbed automatically into the path length of a particle and its motion therein. In general relativity, therefore, such energies as kinetic energy and potential energy become meaningless any more; only one energy of a particle is needed to solve the equations of motion in their most general forms.

Having formulated the relativistic expression for energy, we can now review the relativity effect in the equation of the orbit of planet in terms of the energy and angular momentum of the system. For comparison with Newton's theory, it is preferable to define the energy E as in (A6), which would bring E in line with the nonrelativistic value. The Schwarzschild metric in (10) can now be expressed in terms of two constants of motion E and l as

$$\frac{(mc^2+E)^2}{c^2 g_{00}} - m^2 g_{rr}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 - \frac{l^2}{r^2} = m^2c^2. \quad \text{(A10)}$$

This equation can also be obtained from a combination of the differential equations resulting from the geodesic equations. But most often we are more interested in the shape of orbits, that is, in r as a function of θ, than in their time history. The angular momentum relation can then be used directly to convert (A10) into the differential equation for the orbit; this gives

$$\frac{(mc^2+E)^2}{c^2 l^2 g_{00}} - \frac{g_{rr}}{r^4}\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 - \frac{1}{r^2} = \frac{m^2c^2}{l^2}. \quad \text{(A11)}$$

The solution may thus be determined by a quadrature:

$$\Delta\theta = \int\left[\frac{(mc^2+E)^2}{c^2 l^2 g_{00}} - \frac{m^2c^2}{l^2} - \frac{1}{r^2}\right]^{-\frac{1}{2}} \frac{g_{rr}^{1/2}\mathrm{d}r}{r^2}. \quad \text{(A12)}$$

At perihelia and aphelia, r reaches its minimum and maximum values r_+ and r_- , and at both points $\mathrm{d}r/\mathrm{d}\theta$ vanishes, so (A11) gives

$$\frac{(mc^2+E)^2}{c^2 l^2 g_{00}(r_\pm)} - \frac{1}{r_\pm^2} = \frac{m^2c^2}{l^2}, \quad \text{(A13)}$$

where $g_{00}(r_\pm) = 1 - 2GM/c^2 r_\pm$. From these two equations we can derive values for the two constants of the motion:

$$\left(1+\frac{E}{mc^2}\right)^2 = \frac{r_+^2 - r_-^2}{r_+^2 g_{00}^{-1}(r_+) - r_-^2 g_{00}^{-1}(r_-)},$$
$$\frac{m^2c^2}{l^2} = \frac{r_+^{-2} g_{00}(r_+) - r_-^{-2} g_{00}(r_-)}{g_{00}(r_-) - g_{00}(r_+)}. \quad \text{(A14)}$$

The expressions for the energy and angular momentum appear here in somewhat different forms involving the metric tensors $g_{00}(r_\pm)$, but their equivalence in the limit as $g_{00} \to 1$ with the respective nonrelativistic Newtonian relations are shown by expanding the equations to a first approximation as

$$E \cong -\frac{GMm}{r_+ + r_-}, \quad l^2 \cong \frac{2GMm^2}{r_+^{-1} + r_-^{-1}}. \quad \text{(A15)}$$

Using the exact values of the constants given by (A14) in (A12) yield the formula for $\Delta\theta$ as

$$\Delta\theta = \int \frac{g_{rr}^{1/2}(r)\mathrm{d}r}{r^2} \times \quad \text{(A16)}$$
$$\left[\frac{r_+^{-2}\left(g_{00}^{-1}(r) - g_{00}^{-1}(r_-)\right) - r_-^{-2}\left(g_{00}^{-1}(r) - g_{00}^{-1}(r_+)\right)}{g_{00}^{-1}(r_+) - g_{00}^{-1}(r_-)} - \frac{1}{r^2}\right]^{-\frac{1}{2}}$$

We can make the argument of the first square root in the integrand a quadratic function of $1/r$ which vanishes at $r = r_\pm$, so

$$\Delta\theta \cong \int\left[C\left(\frac{1}{r_-} - \frac{1}{r}\right)\left(\frac{1}{r} - \frac{1}{r_+}\right)\right]^{-\frac{1}{2}}\left(1 + \frac{GM}{c^2 r}\right)\frac{\mathrm{d}r}{r^2}, \quad \text{(A17)}$$

where $C \cong 1 - (2GM/c^2)(1/r_+ + 1/r_-)$. The constant C could be determined by letting $r \to \infty$.

We can obtain the same result more simply if we use the expansion in GM/c^2r in the formal solution of the equation of motion given by (A12). The process of arriving at the orbit equation is particularly simple here. The angle swept out by the position vector is then given by Eq. (A12) as

$$\Delta\theta \cong \int\left[\begin{array}{l}\frac{2mE}{l^2}\left(1+\frac{E}{2mc^2}\right) + \frac{2GMm^2}{l^2 r}\left(1+\frac{2E}{mc^2}\right) \\ -\frac{1}{r^2}\left(1 - \frac{4G^2M^2m^2}{c^2l^2}\right)\end{array}\right]^{-\frac{1}{2}}$$
$$\times\left(1+\frac{GM}{c^2 r}\right)\frac{\mathrm{d}r}{r^2}. \quad \text{(A18)}$$

As it stands, this indefinite integral is of the standard form. The integrand differs from the corresponding nonrelativistic expression in that the second term in each parenthesis represents the relativistic correction. In a certain sense, Eq. (A18) is the general-relativity analogue of Sommerfeld's treatment of the hydrogen atom in special relativity.

On carrying out the integration, the equation of the orbit is found to be

$$\frac{1}{r} \cong A\left[1 + \varepsilon\cos\left(\begin{array}{l}(\theta - \theta_o)\left(1 - \frac{3G^2M^2m^2}{c^2l^2}\right) \\ -\frac{GM}{c^2r^2}\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)\end{array}\right)\right], \quad \text{(A19)}$$

where

$$A \cong \frac{GMm^2}{l^2}\left(1 + \frac{2E}{mc^2} + \frac{4G^2M^2m^2}{c^2l^2}\right),$$
$$\varepsilon \cong \left[1 + \frac{2El^2}{G^2M^2m^3}\left(1 - \frac{7E}{2mc^2} - \frac{4G^2M^2m^2}{c^2l^2}\right)\right]^{\frac{1}{2}}, \quad \text{(A20)}$$

and θ_0 is a constant of integration. In addition to the motion of the perihelion of a planet, the relativity effect produces the term $(GM/c^2r^2)(\mathrm{d}r/\mathrm{d}\theta)$ in the angle swept out by the radius vector of the planet. This term is not a new result but merely a result of rewriting the square root in the integrand which the integration of (A17) or (A18) actually yields, using (A12) to a first approximation. It is evident therefore that the relativity effect in planetary motion obeying (A17) or (A18), is to cause not only the precession of the perihelion of the orbit of a planet but also the change in the angular displacement of the planet due to its radial velocity. The additional change appearing in the angular displacement of the planet, which does not appear in a circular orbit, seems to represent an effect due to the finite velocity of propagation of the solar gravitational field.

In order to calculate $\Delta\theta$ to first order in GM/c^2r, we need g_{00} to second order in GM/c^2r. To say it in another way, the current discussion of the planetary orbit precession serves as a touchstone for the possible forms of metric tensor by requiring the degree of agreement to second order. Equations (A17) and (A18) are obtained by setting $g_{00}(r) = 1 - 2GM/c^2r$. But the optical approach discussed in sections 3 and 4 gives a metric tensor that can be expressed as $g_{00}(r) = (1 - GM/c^2r)^2$, when only the gravitational potential is taken into consideration. Unless the optical approach is incorrect, we can gain further insight into the metric tensor by assuming that the usual rules for the motion of particles and light rays in a given metric field $g_{\mu\nu}$ still apply, but that the metric for the motion of particles may be different from that for light rays. While the metric for the motion of particles is determined by comparing the geodesic equations with Newton's equation of motion of a particle, indeed, the metric for light rays is determined in this paper by comparing the geodesic equations with the hydrodynamic equation describing the state of motion of the medium of propagation. In the optical approach, it has shown that the metric tensor g_{rr} is identified with the square of a refractive index of the medium with spherical symmetry. However, it is true that such an interpretation of the metric tensor g_{rr} cannot be applied identically to the equation of motion of material particles.

In a previous paper[21], the author has pointed that, phenomenologically, a particle velocity itself would appear dilated to the observer, keeping its time intact. This means that the physical reality of the fourth coordinate is not *ct'* but *c't* in an arbitrary coordinate system (*ct*, **x**)*'*. The time intervals in the Schwarzschild metric should therefore be understood in terms of d(*ct*) and d(*c't*), or briefly *c* and *c'*, contrary to the current recognition. It is then at once evident that Eq. (A2) is just what we should expect if we extend the redshift effect in Eq. (6) to the case of a moving source of light. To be consistent with physics literatures and textbooks, nevertheless, the current notation has been adopted in this subsequent paper.

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Note added in proof. The reviewer's criticism of the redshift result is so instructive that the author would like to introduce his criticism to the readership.

[C1] The redshift equation, Eq. 1 should hold for any acceleration. If derivable from a potential one could postulate that gh/c be replaced by the right member of Eq. 5 and then define $n(r)$ by Eq. 8. This would avoid the potential confusion of fluid drag effects, as in Fizeau's experiments, and also divorce the proposed effect from fluid flow *per se*. As the effects all stem from relative accelerations of reference frames, a clear description of what constitutes proper frames within and outside a "fluid" is needed.

[C2] I do not think that it is always possible to attribute the redshift to a relative velocity between an inertial frame and a frame moving with a fluid. It may be possible that a relative acceleration between frames causes a local redshift and a cumulative time delay. I think that an appeal to the principle of equivalence is still necessary in order to include all causes of acceleration as causes of redshift.

[C3] Without doing all of the manipulations required for proof, I suspect that one could retrieve the redshift implied by Eq. 5 from the field equations of general relativity. The terms beyond gravitational potential should be contained within the stress-energy tensor of the right member of the field equations. This is no criticism of Eq. 5; in fact it is good to have an intuitive way of arriving at the redshift result.

[C4] The author describes Einstein's interpretation of redshift as a manifestation of time dilation in a gravitational potential as "rather unusual." I believe that this is unwarranted. At this point in the development of the author's optical analogy he has tied everything to changes caused by properties of the "medium" of propagation. Since there are effects clearly due to a gravitational field in the absence of any other "medium", and other interpretations are lacking at this point, I believe that it is incorrect to call Einstein's interpretation "rather unusual."

Author's reply to [C4]: It reveals a difference of standpoint looking at the relativity theory between the current paradigm (reviewer) and the opponent (author). What the experiment has actually demonstrated is a change in wavelength or frequency with gravitational potential, not a change in rate of clock with it. One can see from the original paper that Einstein presented the interpretation of the redshift effect as a result of fitting the predicted speed of light $c' = c(1 + gh/c^2)$ even deliberately into the postulate of the constancy of the speed of light. I think that Einstein put the cart before the horse in its interpretation, apart from the controversy of whether the postulate is consistent or inconsistent.