On the Energy-Inertial Mass Relation: II. Kinematic and Geometrical Aspects

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Kinematic and geometrical aspects of the connection between energy and inertial mass are considered. Transformations of coordinate and time are obtained in a two-dimensional flat space. For this case the Euclidean, pseudo-Euclidean and Galilean kinematics are considered. A new interval in a flat four-dimensional anisotropic Finsler space is found. Under certain assumptions the known results follow from the derived relations.

Introduction

In a previous paper (Zaripov 1996) the author used the dynamic approach and the expression for the total energy

$$E = \mathrm{a}\, mc^2 + \mathrm{b} mv^2 + (1-\mathrm{a})m_0c^2 \qquad (1)$$

to obtain the general inertial mass-velocity relation

$$m(v) = \frac{\mathrm{m}}{\left(1-\dfrac{\mathrm{e}v^2}{\mathrm{a}c^2}\right)^{1-1/2\mathrm{e}}},\ (\mathrm{e} = 1-\mathrm{b}) \qquad (2)$$

By varying the parameters a and e, different possible relativistic expressions for the energy and mass were derived. The case with a mathematical singularity was examined where the functions $E = E(v)$ and $m = m(v)$ lost their continuity under certain conditions. A matter of particular interest was the case where the functions were continuous. Several classes of objects were defined for these classes.

The purpose of this work is to continue the investigation and consider the kinematic and geometric aspects of the problem. The kinematic aspects arise from a study of the motion of inertial systems. The geometric aspects emerge from the problem of geometrization of the Lagrange formalism in analytical mechanics.

For the relativistic case of special relativity, we substitute value $\mathrm{a} = \mathrm{e} = 1$ into the Lagrange function:

$$L = \vec{p}\vec{v} - E = -\mathrm{a}\, m_0c^2\left(1-\frac{\mathrm{e}v^2}{\mathrm{a}c^2}\right)^{1/2\mathrm{e}} - (1-\mathrm{a})m_0c^2 \qquad (3)$$

and get $L = -m_0c^2(1-v^2/c^2)^{1/2}$, where $\vec{p} = m\vec{v}$ is the momentum of an object. Geometrization of the variational problem is governed by variation of the following operation

$$\mathrm{d}I = \mathrm{d}\int L dt = -m_0 c\, \mathrm{d}\int ds\,. \qquad (4)$$

The physical meaning of the expression (4) is determined by the value of the proper time of a moving object $d\mathrm{t} = \frac{1}{c}ds$. Free trajectories of object motion are geodesic lines in a four dimensional flat space with the indefinite metric

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2\,. \qquad (5)$$

The kinematic aspects are governed by Lorentz transformations:

$$x' = \frac{x-vt}{\sqrt{1-v^2/c^2}},\ t' = \frac{t-xv/c^2}{\sqrt{1-v^2/c^2}},\ y' = y,\ z' = z, \qquad (6)$$

which retain the length element (5) form-invariant.

In the case of Newtonian dynamics we substitute the value $\mathrm{e} = 1/2$ into (3) and get $L = -m_0c^2 + (m_0v^2)/2$. The problem of geometrizing the Lagrange formalism is described in earlier work (Schouten 1951). A line element of a flat space is $ds^2 = c^2dt^2$ and the kinematic aspects are determined by the Galilean transformations:

$$x' = x - vt,\ t' = t,\ y' = y,\ z' = z\,. \qquad (7)$$

Finally we may consider the motion of an object in a four-dimensional flat space with the definite metric

$$ds^2 = c^2dt^2 + dx^2 + dy^2 + dz^2\,. \qquad (8)$$

The kinematic aspects are governed by the transformations

$$x' = \frac{x-vt}{\sqrt{1+v^2/c^2}},\ t' = \frac{t+xv/c^2}{\sqrt{1+v^2/c^2}},\ y' = y,\ z' = z, \qquad (9)$$

For this relativistic case, by substituting the value $\mathrm{a} = -\mathrm{e} = -1$ in (3), we obtain $L = m_0c^2(1+v^2/c^2)^{1/2} - 2m_0c^2$. A constant term does not enter into the equations of motion in the problem of variation of operation (4).

These three flat geometries for the two-dimensional case (ct, x) and kinematics are desribed in earlier work (Liebscher 1977). The parallel axiom holds true only for these three geometries. In the general case, there are nine plane geometries (Klein 1928).

It follows from the above consideration that relativistic cases with the metrics (5) and (8) correspond to the Riemann approach to geometrization of the Lagrange formalism. This approach, with $ds^2 = g_{ij}dx^idx^j$ $(i = 1,2,3,4)$, is possible only at $\mathrm{e} = 1$ in the expression for the total energy (1).

Let us first consider the Riemann approach. Later, we shall study the general case with $\mathrm{e} \neq 1$ and other approaches to geometrization.

Coordinate and time transformations in a two-dimensional flat space.

$K(x,t)$ and $K(x',t')$. According to the principle of relativity these systems are equivalent. The system $K(x',t')$ moves about the system $K(x,t)$ with the constant velocity v. A transformation of coordinates between the systems can be written in the form

$$x' = \Omega(x - vt)\,,\ x = \Omega(x' + vt') \qquad (10)$$

where $\Omega = \Omega(v) = \Omega(-v)$. Transformations (10) generalize the Galilean transformations (7), for which we have the equality $\Omega = 1$.

We can obtain the transformations of time between the systems from the relation (10)

$$t' = \Omega\left[t + \left(\frac{1-\Omega^2}{\Omega^2 v^2}\right)vx\right],\ t = \Omega\left[t' - \left(\frac{1-\Omega^2}{\Omega^2 v^2}\right)vx'\right]. \qquad (11)$$

Thus, forward and reverse transformations are carried out with the aid of the transition matrix $\hat{A} = \hat{A}(\Omega, v)$ and its inverse $\hat{A}^{-1} = \hat{A}^{-1}(\Omega, -v)$

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \hat{A}\begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} \Omega & -\Omega v \\ \dfrac{1-\Omega^2}{\Omega v} & \Omega \end{pmatrix}\begin{pmatrix} x \\ t \end{pmatrix},\ \det\hat{A} = 1 \qquad (12)$$

$$\begin{pmatrix} x \\ t \end{pmatrix} = \hat{A}^{-1}\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} \Omega & \Omega v \\ -\dfrac{1-\Omega^2}{\Omega v} & \Omega \end{pmatrix}\begin{pmatrix} x' \\ t' \end{pmatrix};\ \begin{cases} \det\hat{A}^{-1} = 1 \\ \hat{A}\hat{A}^{-1} = 1 \end{cases}, \qquad (13)$$

Let us now consider a third inertial system $K(x'',t'')$ which moves about the systems $K(x,t)$ and $K(x',t')$. with the velocities v' and v''. We use the group properties of transformations as the product of the transitional matrices

$$\hat{A}(\Omega'', v'') = \hat{A}(\Omega, v)\hat{A}(\Omega', v')\,, \qquad (14)$$

where $\Omega' = \Omega(v')$, $\Omega'' = \Omega(v'')$. For the Galilean transformations (7) the velocity addition law $v'' = v + v'$ follows from (14). In the case under consideration we have the following relations

$$\frac{1-\Omega^2}{\Omega^2 v^2} = \frac{1-\Omega'^2}{\Omega'^2 v'^2} = \frac{1-\Omega''^2}{\Omega''^2 v''^2} = \frac{\xi}{c^2} = const\,, \qquad (15)$$

$$\Omega\Omega'(v + v') = \Omega'' v''\,, \qquad (16)$$

$$v'' = \frac{v + v'}{1 - \xi\dfrac{vv'}{c^2}}. \qquad (17)$$

In the relation (15), a constant value is written taking into account the dimensions of the velocities of the systems. The velocity value c should be determined from experiments. The value of the parameter ξ is obtained from different agreements concerning synchronization. The problem of defining simultaneity in time was considered by the author in earlier work (Zaripov 1978, 1980, 1984, 1992).

Using the value of the universal constant c we shall consider events in a two-dimensional geometric space. We can then get the transformations of coordinates between the systems $K(x,ct)$ and $K(x',ct')$ from (12) and (13)

$$\begin{pmatrix} x' \\ ct' \end{pmatrix} = \Omega\begin{pmatrix} 1 & -v \\ \xi v & 1 \end{pmatrix}\begin{pmatrix} x \\ ct \end{pmatrix},\ \begin{pmatrix} x \\ ct \end{pmatrix} = \Omega\begin{pmatrix} 1 & v \\ -\xi v & 1 \end{pmatrix}\begin{pmatrix} x' \\ ct' \end{pmatrix}, \qquad (18)$$

$$\Omega = \left(1 + \xi\frac{v^2}{c^2}\right)^{-1/2}.$$

At $\xi = -1$, $\xi = 0$ and $\xi = 1$, the transformations (6), (7) and (9) follow, correspondingly, from (18). In Figure 1 the geometric presentation of the transformations is given. The dashed line corresponds to the isotropic signal with the velocity c. For the transformations (9) the x' axis is perpendicular to the ct' axis. The slopes of different axis are of the same angle ψ.

2. Representation of transformations in terms of binary numbers

Let an event in the inertial system $K(x,ct)$ correspond to the binary number $X = x_0 + ex$ $(x_0 = ict)$. The binary system with operations of addition and multiplication is one of three:

$$\begin{aligned} &1)\ e^2 = -1 \\ &2)\ e^2 = 1 \\ &3)\ e^2 = 0 \end{aligned} \qquad (19)$$

The properties of a binary system, for which x_0 is the real number, are considered in several books (for instance, Kantor and Solodovnikov 1973, Madelung 1957). A system of complex numbers ($e^2 = -1$) has the property of division. Division cannot be performed for all numbers in systems of double ($e^2 = 1$) or dual ($e^2 = 0$) numbers. However, such systems play a large role in mathematics as well.

Transformations of coordinate and time between systems $K(x,t)$ and $K(x',ct')$ are represented by the transformations of binary numbers

$$X' = AX\,,\ X' = x_0' + ex'\,,\ X = x_0 + ex\,,\ A = a + eb \qquad (20)$$

or between their conjugate quantities

$$\overline{X'} = \overline{X}\overline{A}\,,\ \overline{X'} = x_0' - ex'\,,\ \overline{A} = a - eb\,. \qquad (21)$$

Let us consider transformations of binary numbers,

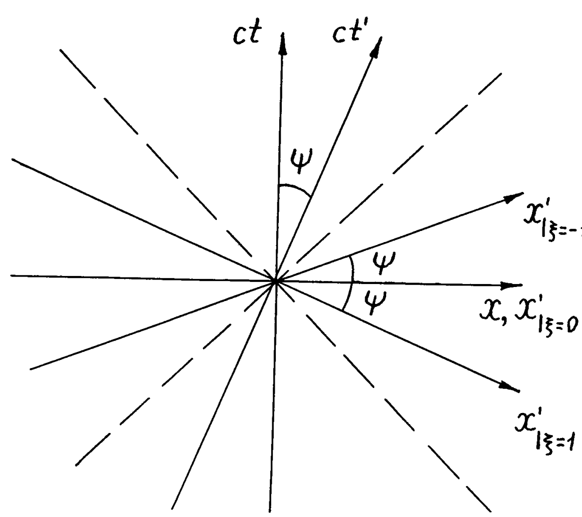

Figure 1

which keep invariant a magnitude of the number

$$X'\overline{X}' = X\overline{X}, \ (ct'^2 + e^2x'^2 = c^2t^2 + e^2x^2). \quad (22)$$

From (20) and (21) we get the equality

$$A\overline{A} = 1, \ (a^2 - e^2b^2 = 1) \quad (23)$$

and the inverse transformations

$$X = A^{-1}X', \ A^{-1} = \overline{A} = a - eb. \quad (24)$$

To determine a and b we can consider the relation

$$\frac{x_0' + ex'}{x_0' - ex'} = \frac{a+eb}{a-eb}\,\frac{x_0 + ex}{x_0 - ex} \quad (25)$$

We will use the equality $x = vt$ at $x' = 0$ (or $x' = -vt'$ at $x = 0$). We then obtain values $iv/c = b/a$ and $a = (1 + e^2v^2/c^2)^{-1/2}$ from (20) and (23). For the number A we have

$$A = A(v) = \frac{1 + e\dfrac{iv}{c}}{\sqrt{1 + e^2v^2/c^2}}. \quad (26)$$

Making use of the group properties of transformations as a product of binary numbers

$$A(v'') = A(v)A(v'). \quad (27)$$

we obtain the law of addition of velocities

$$v'' = \frac{v + v'}{1 - e^2\dfrac{vv'}{c^2}}. \quad (28)$$

At $e^2 = -1$, $e^2 = 1$ and $e^2 = 0$ the transformations (6), (7) and (9) follow, correspondingly, from (20). The cases with $e^2 = -1$ and $e^2 = 0$ were studied earlier by the author (Zaripov 1979).

3. Finsler s approach to geometrization of the Lagrange formalism

Let us consider the case with $e = 0$, for which the inertial mass and total energy

$$m = m_0 e^{-\frac{v^2}{2ac^2}}, \ E = amc^2 + mv^2 + (1 - a)m_0c^2, \quad (29)$$

depend on only one parameter. This case describes ordinary real objects with infinite velocity of motion $0 \le v \le \infty$ (Zaripov 1996). For the Lagrange function we have the expression

$$L = -am_0c^2e^{-\frac{v^2}{2ac^2}} - (1 - a)m_0c^2. \quad (30)$$

For $a = 1$ we obtain the following formulae from (30)

$$L = -m_0c^2e^{-\frac{v^2}{2c^2}}, \ ds^2 = e^{-\frac{v^2}{c^2}}dt^2. \quad (31)$$

We now substitute the value $v^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2$ in (31) and at $(v^2/c^2) \ll 1$ we expand $e^{-\frac{v^2}{c^2}} \approx 1 - v^2/c^2 + \ldots$. We then obtain the formulae of the special relativity with the Riemann metric (5) accurate to $0(c^{-2})$. This is easily seen in the general case. We now generalize the relation (31) and get the interval in a flat four-dimensional space

$$ds^2 = e^{\left[\frac{g_{ij}dx^idx^j}{(n_idx^i)^2} - 1\right]}(n_idx^i)^2, \quad (32)$$

where $g_{ij} = (1,-1,-1,-1)$, n_i is the vector of anisotropy with the norm $|n| = (g^{ij}n_in_j)^{1/2} = 1$. In the case of the formula (31) we have $n = (1,0,0,0)$.

Trajectories of free motion of objects are geodesic lines in a Finsler space with the interval (32). When the inequalities are satisfied

$$g_{ij}dx^idx^j \ll 2(n_idx^i)^2 \quad (33)$$

we can obtain the following expansion

$$ds^2 \approx g_{ij}dx^idx^j\left\{1 - \left[\frac{g_{ij}dx^idx^j}{(n_idx^i)^2} - 1\right]^2 + \ldots\right\} \quad (34)$$

It follows from (34) that in the first approximation we have a pseudo-Euclidean space of the special relativity. Analogous calculations can be carried out for the case $a = -1$. In the first approximation we get Euclidean space with the metric (8).

Conclusions

Thus, the dynamic approach and analysis of the expression for the total energy (1) yield the flat Riemann and anisotropic flat Finsler geometry in the case of geometrization. Experimental investigation of the spectra of primary cosmic-ray protons of superhigh energy require the assumption that the Lorentz transformations (6) are not satisfied at $(1 - v^2/c^2)^{-1/2} \sim 5 \cdot 10^{10}$ (Bogoslovsky 1977). Finsler geometry was suggested with the interval

$$ds^2 = \left[\frac{(n_idx^i)^2}{g_{ij}dx^idx^j}\right]^r (g_{ij}dx^idx^j), \quad (35)$$

where n_i is a vector of anisotropy with the norm $|n| = (g^{ij}n_in_j)^{1/2} = 0$. At $r \to 0$ we have the pseudo-Euclidean space of special relativity. A generalization of (35) to the case of nonstandard clock timing according to Reichenbach-Grünbaum (Reichenbach 1958, Grünbaum 1973) is given in earlier work by the author (Zaripov 1992).

Different kinds of Finsler spaces show anisotropy (see, for example, Asanov 1979, Pimenov 1987, Rund 1959). The following analysis favours the expression (32). We consider accelerated motion in a gravitational field, as discussed by Einstein (1907). Let an object fall in the gravitational field of another object. We then have $(v^2/c^2) = 2\Phi/c^2$, where Φ is a gravitational potential. The effect of a gravitational force on time is described, according to (31), by the relation

$$t = \tau\, e^{\frac{\Phi}{c^2}} \quad (36)$$

The formula (36) was first presented by Einstein (1907) and has not been used since. In the above paper only the approximation value was used

$$t = \tau\left(1 + \frac{\Phi}{c^2}\right), \quad (37)$$

which further led to the Riemann approach to the gravitation theory. The relation (36) follows from the Finsler approach to gravitation theory with the interval (32) at

$g_{00} = (1 - \Phi/c^2)$, $n_0 = 1$, $dx = dy = dz = n_1 = n_2 = n_3 = 0$. In the general case it is necessary to consider a curved anisotropic Finsler space with the fields $g_{ij}(x)$ and $n_i(x)$. This is an interesting and separate problem.

In conclusion we note that the experimental determination of the effect of a gravitational force on a clock moving according to the formulae (36) and (37) will enable us to determine which kind of geometry should be considered.

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