Remarks on the Transformations of Space and Time

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Space and time transformations from a "stationary" isotropic inertial system S_0 to any other inertial system S are shown to imply complete physical equivalence between the three possible pairs of assumptions chosen among the following: A1. Lorentz contraction of bodies moving with respect to S_0; A2. Larmor retardation of clocks moving with respect to S_0; A3. Two-way velocity of light equal to c in all inertial systems and in all directions. The empirical evidence supporting A2 and A3 is therefore in favour of A1 as well.

1. Introduction

We suppose that a ("stationary") inertial reference frame S_0 exists in which Maxwell's equations hold. A well known consequence of these equations is that the velocity of light in the vacuum is c in all directions. Therefore in S_0 clocks must be synchronised by using Einstein's procedure [1]. Space and time variables x_0, y_0, z_0, t_0 are thus available in all points of S_0 and the velocities of moving bodies (and of other inertial systems) can be measured. The most general form of space-time transformation from S_0 to a different inertial system $S(x, y, z, t)$ is:

$$\begin{cases} x &= f(x_0, y_0, z_0, t_0) \\ y &= g(x_0, y_0, z_0, t_0) \\ z &= h(x_0, y_0, z_0, t_0) \\ t &= e(x_0, y_0, z_0, t_0) \end{cases} \tag{1}$$

where f, g, h, e are four functions of the S_0 space co-ordinates x_0, y_0, z_0, and time t_0. In reducing the generality of (1) one can follow two guiding principles:

A. *Empty space is homogeneous, that is all points have the same physical properties, and is isotropic, that is all directions are physically equivalent.*
B. *Time is homogeneous, that is all properties of space remain the same with passing time.*

In Ref. [2] it was shown that, given A and B, the transformations (1) must be linear in all variables:

$$\begin{cases} x &= f_1 x_0 + f_2 y_0 + f_3 z_0 + f_4 t_0 + f_5 \\ y &= g_1 x_0 + g_2 y_0 + g_3 z_0 + g_4 t_0 + g_5 \\ z &= h_1 x_0 + h_2 y_0 + h_3 z_0 + h_4 t_0 + h_5 \\ t &= e_1 x_0 + e_2 y_0 + e_3 z_0 + e_4 t_0 + e_5 \end{cases} \tag{2}$$

The twenty coefficients appearing in (2) are constant with respect to x_0, y_0, z_0, t_0 but can naturally depend on the velocity v of S relative to S_0. One can choose the co-ordinate systems in S and in S_0 in such a way that the observer in S sees his origin ($x = y = z = 0$) coincident with that of S_0 at $t = 0$, and *vice versa* that the observer in S_0 sees his origin $(x_0 = y_0 = z_0 = 0)$ coincident with that of S at time $t_0 = 0$. This is equivalent to a suitable choice of the origin of the space co-ordinates and of time in S. Symbolically one can write:

$$[x_0 = y_0 = z_0 = t_0 = 0] \quad \Rightarrow \quad [x = y = z = t = 0]$$

Inserted in (2) this gives:

$$f_5 = g_5 = h_5 = e_5 = 0 \tag{3}$$

Assume next that plane (x_0, y_0) coincides with plane (x, y) at all times t_0:

$$[z_0 = 0] \quad \Rightarrow \quad [z = 0]$$

The third of (2) gives:

$$h_1 = h_2 = h_4 = 0 \tag{4}$$

Assume also that plane (x_0, z_0)coincides with plane $(x, z)$for all times t_0:

$$[y_0 = 0] \quad \Rightarrow \quad [y = 0]$$

The second of (2) gives:

$$g_1 = g_3 = g_4 = 0 \tag{5}$$

Finally assume that *at time* $t_0 = 0$ plane (y_0, z_0) coincides with plane (y, z). This is like saying that the relative velocity is parallel to the x_0 axis [and then perpendicular to plane (y_0, z_0), if the Cartesian co-ordinates are orthogonal]. The condition is then:

$$[t_0 = x_0 = 0] \quad \Rightarrow \quad [x = 0]$$

Given (3), from the first of (2) it follows:

$$f_2 = f_3 = 0 \tag{6}$$

Consider next to relative velocity condition by assuming that the origin of S ($x = 0$) seen from S_0 satisfies the equation $x_0 = vt_0$. By substituting $x = 0$ and $x_0 = vt_0$ in the first Eq. (2) and taking into account (3) and (6) it follows

$$f_4 = -f_1 v \tag{7}$$

One can now rewrite transformations (2) using (3)-(7):

$$\begin{cases} x &= f_1(x_0 - vt_0) \\ y &= g_2\, y_0 \\ z &= h_3\, z_0 \\ t &= e_1\, x_0 + e_2 y_0 + e_3 z_0 + e_4 t_0 \end{cases} \tag{8}$$

There is a complete equivalence of the axes y_0 and z_0, since the relative velocity is parallel to the x_0 axis. These axes are chosen arbitrarily, and nothing physical can distinguish them if space is isotropic. Therefore:

$$g_2 = h_3 \quad ; \quad e_2 = e_3 \tag{9}$$

It must furthermore be considered that all points in a plane perpendicular to the x-axis in S are physically equivalent because the whole co-ordinate system translates rigidly with a local velocity $\vec{v}$ which is everywhere the same. It follows that one is free to assume

$$e_2 = 0 \tag{10}$$

Thus one can replace (8) by

$$\begin{cases} x &= f_1(x_0 - vt_0) \\ y &= g_2\, y_0 \\ z &= g_2\, z_0 \\ t &= e_1\, x_0 + e_4 t_0 \end{cases} \tag{11}$$

The transformation (11) allows one to calculate the velocity of light relative to S: One must: (i) write it down in terms of space and time intervals Δx, Δy, ... Δt_0; (ii) invert it, expressing as functions of Δx, ... Δt; (iii) substitute the new result in the equation

$$c\Delta t_0 = \left[\Delta x_0^2 + \Delta y_0^2 + \Delta z_0^2\right]^{1/2}$$

(iv) introduce polar co-ordinates. As shown in [2], the one-way velocity of light $\tilde{c}_1(q)$ and the two-way velocity $\tilde{c}_2(q)$ turn out to be given by:

$$\frac{1}{\tilde{c}_1(q)} = \frac{e_1 c + e_4 b}{c f_1(1-b^2)} \cos q + \frac{e_4 + e_1 b c}{c}\left[\frac{\cos^2 q}{f_1^2(1-b^2)^2} + \frac{\sin^2 q}{g_2^2(1-b^2)}\right]^{1/2} \tag{12}$$

and

$$\frac{1}{\tilde{c}_2(q)} = \frac{e_4 + e_1 b c}{c}\left[\frac{\cos^2 q}{f_1^2(1-b^2)^2} + \frac{\sin^2 q}{g_2^2(1-b^2)}\right]^{1/2} \tag{13}$$

if q is the angle between the propagation direction of light and the "absolute" velocity of S (parallel to the x axis).

2. Three basic assumptions

The separate consequences of the following three assumptions will now be deduced:

A1. Lorentz-Fitzgerald contraction. A body at rest in S between the points of co-ordinates x_1 and x_2 when seen from S_0 appears contracted along the x direction according to the equation

$$x_{02} - x_{01} = \sqrt{1-b^2}\,(x_2 - x_1) \tag{14}$$

A body at rest in S between the points of co-ordinates y_1 and y_2 does not appear contracted along the y direction, that is it satisfies

$$y_{02} - y_{01} = y_2 - y_1 \tag{15}$$

A2. Larmor retardation. A clock at rest in any point of S, when seen from S_0 appears retarded according to the equation

$$t_{02} - t_{01} = \frac{1}{\sqrt{1-b^2}}(t_2 - t_1) \tag{16}$$

A3. Invariance of two-way velocity of light. A flash of light propagating forth and back on any segment AB at rest in S does so with a two-way velocity

$$\tilde{c}_2(q) = c \tag{17}$$

independent of S and of the angle q formed by the light propagation direction and the velocity of S relative to S_0.

Consequences of A1. Consider the extreme points of the body in the directions x and y seen from S_0 at the same time t_0. From the first of (11) one gets:

$$x_1 = f_1(x_{01} - vt_0);\; x_2 = f_1(x_{02} - vt_0) \tag{18}$$

and from the second

$$y_1 = g_2\, y_{01}\,; \qquad y_2 = g_2\, y_{02} \tag{19}$$

By subtracting from one another the two equations (18), and doing the same with (19), and comparing the so obtained results with (14) and (15) one obviously gets

$$f_1 = \frac{1}{\sqrt{1-b^2}} \quad ; \quad g_2 = 1 \tag{20}$$

Consequences of A2. Consider a clock at rest in some given point of S. Seen from S_0 the clock will obey the equation:

$$x_0(t_0) = b c\, t_0 + x_0(0)$$

so that the fourth of (11) becomes

$$t = (e_4 + e_1 b c)\, t_0 + e_1 x_0(0) \tag{21}$$

Considering any two times t_1 and t_2 marked by the moving clock and the corresponding S_0 times t_{01} and t_{02}, one easily gets from (21):

$$t_2 - t_1 = (e_4 + e_1 b c)\,(t_{02} - t_{01}) \tag{22}$$

Comparison with (16) gives

$$e_4 + e_1 b c = \sqrt{1-b^2} \tag{23}$$

Consequences of A3. Obviously the angular dependence in (13) disappears only if

$$g_2 = f_1\sqrt{1-b^2} \tag{24}$$

after which (13) becomes

$$\frac{1}{\tilde{c}_2(q)} = \frac{1}{c}\,\frac{e_4 + e_1 b c}{f_1\,(1-b^2)}$$

Clearly $\tilde{c}_2(q) = c$ implies:

$$f_1(1-b^2) = e_4 + e_1 bc \qquad (25)$$

3. Equivalence of three pairs of assumptions

One can examine the consequences of A1, A2 and A3, by taking them two at a time in all possible ways.

Consequences of A1 + A2. Considering together (20) and (23) one has:

$$f_1 = \frac{1}{\sqrt{1-b^2}}; \quad g_2 = 1; \quad e_4 = \sqrt{1-b^2} - e_1 b\, c \qquad (26)$$

Consequences of A1 + A3. Considering together (20), (24) and (25) one has:

$$f_1 = \frac{1}{\sqrt{1-b^2}}; \quad g_2 = 1; \quad e_4 = \sqrt{1-b^2} - e_1 b\, c \qquad (27)$$

Consequences of A2 + A3. By inserting (23) in (25) one gets

$$f_1 = \frac{1}{\sqrt{1-b^2}}$$

so that (24) gives

$$g_2 = 1$$

The last two conditions together with (23) are then

$$f_1 = \frac{1}{\sqrt{1-b^2}}; \quad g_2 = 1; \quad e_4 = \sqrt{1-b^2} - e_1 b\, c \qquad (28)$$

The identity of (26), (27) and (28) constitutes the main point of the present paper: The three possible pairs of assumptions (A1, A2), (A1, A3) and (A2, A3) lead exactly to the same consequences. This is relevant to relativistic physics because (A1, A2) were the basic assumptions of Lorentz's reformulation of relativity [3]. Objections have been raised [4] against the validity of A1, for which there is indeed no *direct* experimental basis. There are, however, rather good experimental indications that A2 and A3 are true properties of nature [5-6]. Given the theorem just proved, the same indications can be taken as a rather convincing basis for the validity of A1 as well.

To this one can add that the Ehrenfest paradox [7], invoked as an argument against length contraction, is not a very serious problem in the real physical world. It is enough that the rotating disk becomes dome-shaped, in order to have a contracted circumference with a constant radius. Only in the abstract world of ideas, the symmetry of the problem between the two faces of the disk can be perfect. Their equivalence breaks down for a real disk, which is bound to have small irregularities. Furthermore, the reality of length contraction has been convincingly argued for with Bell's example of the thread connecting two equally accelerating spaceships [8].

4. The inertial transformations

The meaning of (26), (27) and (28) is that the transformations of space and time relevant to the physical world are necessarily of the form:

$$\begin{cases} x = \dfrac{x_0 - b c t_0}{\sqrt{1-b^2}} \\ y = y_0 \\ z = z_0 \\ t = \sqrt{1-b^2}\, t_0 + e_1(x_0 - b c t_0) \end{cases} \qquad (29)$$

where only one undetermined coefficient is left, e_1. As a consequence of (12) the inverse (one-way) velocity of light obtained for the values of the coefficients f_1, g_2, and e_4 adopted in (29) is:

$$\frac{1}{\tilde{c}_1(q)} = \frac{1}{c} + \left[\frac{b}{c} + e_1\sqrt{1-b^2}\right]\cos q \qquad (30)$$

The Lorentz transformations are recovered if one assumes $\tilde{c}_1(q) = c$. From (30) it follows:

$$e_1 = -\frac{1}{c}\frac{b}{\sqrt{1-b^2}}$$

Different values of e_1 are obtained by using different clock synchronisation procedures. The so-called absolute synchronisation [9] is based on the idea that all clocks of S are set to time $t = 0$ when the passing clock at rest in the absolute system S_0 shows the time $t_0 = 0$. This means that from all positions in S_0 the time in S will be seen to be the same, and therefore that no position dependent time-lag factor will be present in the transformation of time. Therefore $e_1 = 0$, condition which gives rise to a particularly simple transformation, different from the Lorentz one:

$$\begin{cases} x = \dfrac{x_0 - b c t_0}{\sqrt{1-b^2}} \\ y = y_0 \\ z = z_0 \\ t = \sqrt{1-b^2}\, t_0 \end{cases} \qquad (31)$$

The velocity of light relevant to a theory based on (31) is found by taking $e_1 = 0$ in (29):

$$\frac{1}{\tilde{c}(q)} = \frac{1 + b\cos q}{c} \qquad (32)$$

If (31) is inverted, it gives:

$$\begin{cases} x_0 = \sqrt{1-b^2}\left[x + \dfrac{b c}{1-b^2} t\right] \\ y_0 = y \\ z_0 = z \\ t_0 = \dfrac{1}{\sqrt{1-b^2}} t \end{cases} \qquad (33)$$

There is a formal difference between (31) and (33). The latter implies, for example, that the origin of S_0 (satisfying $x_0 = y_0 = z_0 = 0$) is described in S by $y = z = 0$ and by

$$x = -\frac{bc}{1-b^2}t$$

This origin is thus seen to move with speed $bc/(1-b^2)$ which can exceed c, but cannot be superluminal. In fact a light pulse seen from S to propagate in the same direction as S_0 has $q = p$, and thus [using (32)] has velocity $\tilde{c}(p) = c/(1-b)$, which satisfies

$$\frac{c}{1-b} \geq \frac{cb}{1-b^2}$$

One of the typical features of these transformations is the presence of velocities which can grow without limit *when they are relative to moving systems* having absolute velocities bc near to c. Absolute velocities can instead never exceed c [10]. In STR one is used to relative velocities that are always equal and opposite, but this symmetry is a consequence of the particular synchronisation used and cannot be expected to hold more generally [10].

Consider now a third inertial system S' moving with velocity $b'c$ and its transformation from S_0, which of course is

$$\begin{cases} x' = \dfrac{x_0 - b'ct_0}{\sqrt{1-b'^2}} \\ y' = y_0 \\ z' = z_0 \\ t' = \sqrt{1-b'^2}\,t_0 \end{cases} \qquad (34)$$

By eliminating the S_0 variables from (34) and (33) one obtains the transformation between the two moving systems S and S':

$$\begin{cases} x' = \dfrac{\sqrt{1-b^2}}{\sqrt{1-b'^2}}\left[x - \dfrac{(b'-b)c}{1-b^2}t\right] \\ y' = y \\ z' = z \\ t' = \dfrac{\sqrt{1-b'^2}}{\sqrt{1-b^2}}t \end{cases} \qquad (35)$$

The transformation (31) was first written down by Tangherlini [11]. For (33) and (35) see Ref. [10]. A possible name for (31)-(33)-(35) is "inertial transformations". In its most general form (35) an inertial transformation depends on two velocities (v and v'). When one of them is zero, either S or S' coincide with the privileged system S_0 and the transformation (35) becomes either (31) or (33). The inertial transformations have been shown not to form a group [10].

5. Some consequences of the inertial transformations

A feature characterising the transformations (31)-(33)-(35) is the existence of *absolute simultaneity*: two events taking place in different points of S but at the same t are judged to be simultaneous also in S' (and vice versa), this property being consequence of the absence of space variables in the transformation of time. Of course the existence of absolute simultaneity does not imply that time is absolute: in fact, the b-dependent factor in the transformation of time gives rise to time-dilation phenomena similar to those of STR. *Time dilation* in another sense is however also absolute: a clock at rest in S is seen from S_0 to run slower, but a clock at rest in S_0 is seen from S to run *faster*. Both observers agree that motion relative to S_0 slows down the pace of clocks, and the phenomenon loses the relativistic flavour it has in STR, becoming so to say absolute. Quantitatively one has for both situations:

$$\Delta t = \sqrt{1-b^2}\,\Delta t_0 \qquad (36)$$

where Δt and Δt_0 are the time intervals between any two given events as measured with clocks at rest in S and in S_0, respectively. The difference with STR is however more apparent than real: a meaningful comparison of rates implies that a clock T_0 at rest in S_0 must be confronted with clocks at rest in different points of S. The result is thus dependent on the adopted convention for synchronising the latter clocks.

Absolute length contraction can also be deduced from (31)-(33). A rod at rest on the x axis of S between the points with co-ordinates x_2 and x_1 is seen in S_0 to have end points x_{02} and x_{01} at a common time t_0, where from (31):

$$x_2 = \frac{x_{02} - vt_0}{\sqrt{1-b^2}};\; x_1 = \frac{x_{01} - vt_0}{\sqrt{1-b^2}} \qquad (37)$$

From this one obtains

$$x_2 - x_1 = \frac{1}{\sqrt{1-b^2}}\left(x_{02} - x_{01}\right) \qquad (38)$$

The reasoning can be inverted by considering the rod at rest in S and observed from S_0, and using (33). One gets, after a few simple steps:

$$x_{02} - x_{01} = \sqrt{1-b^2}\left(x_2 - x_1\right) \qquad (39)$$

which could be obtained by inverting (38). The two results are thus mathematically equivalent and lead to the conclusion (with which both observers agree) that motion relative to S_0 leads to contraction. This is obviously an absolute effect, but again the discrepancy with the STR is due to the different conventions concerning clock synchronisation: the length of a moving rod can only be obtained by marking the *simultaneous* positions of its end points, and is therefore dependent on the very definition of simultaneity of distant events.

The assumed indifference of physical reality concerning synchronisation of clocks exists only insofar as one neglects accelerations: when these come into play every inertial system exists, so to say, only for a vanishingly small time interval and it is physically impossible to adopt any time-consuming procedure for the synchronisation of distant clocks in the accelerated frame (such as Einstein's procedure). Yet physical events take place and synchronisation is fixed by nature itself: the choice is $e_1 = 0$. How this happens was shown in Ref. [10].

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