Magnetic Potentials, Longitudinal Currents, and Magnetic Properties of Vacuum: All Implicit in Maxwell s Equations

Héctor A. Múnera
Centro Internacional de Física
A.A. 251955
Bogotá D.C., Colombia

Octavio Guzmán
Departamento de Física
Universidad Nacional
Bogotá D.C., Colombia

We have recently obtained new explicit nonperiodic solutions for the three-dimensional time-dependent wave equation in spherical coordinates. Since Maxwell field equation (MFE) is formed by four wave equations, our results also lead to nonperiodic solutions of the set of classical Maxwell s equations (ME). To understand the meaning of these new expressions, we revisited the standard derivation of MFE from ME. Firstly, we reviewed the standard representation of magnetic and electric fields in terms of potentials to conclude that the magnetic scalar potential is as fundamental as the conventional electric scalar term. Next we checked the conditions for the equivalence of the classical and the field representations of ME to conclude that the class of Lorentz invariant inductive phenomena may contain nonvanishing longitudinal currents. This result agrees with Evans recent discovery of a longitudinal photomagneton. Finally, invariance under Lorentz gauge transformations leads to identifying a new constraint for the magnetic properties of the vacuum.

I. Introduction

The conventional solutions for the classical Maxwell s equations (ME) are periodic electric and magnetic (EM) fields, transversal to the direction of propagation. However, in a recent series of papers,[(1)] Evans argued the existence of a longitudinal photomagnetic field. Conventionally, this would imply that the photon has a nonzero mass.[(2)] On the other hand, we have recently obtained generic nonconventional explicit solutions of the wave equation (WE), with a clear nonperiodic dependence.[(3)] It is well-known that ME are equivalent to WEs in a variety of problems;[(4,5)] for instance, to a relativistically invariant Maxwell field equation (MFE).[(5-7)] In this paper we want to establish the exact conditions for the classical ME to be represented by WEs in general, and by a MFE in particular. Insertion of our new solutions would then lead to nonperiodic solutions of ME.

In section II we revisit the well-known solution of ME in terms of potentials. It is concluded that the magnetic scalar potential (typically neglected) is, at least, as fundamental as the conventional electric scalar term. Next section III revisits the conventional gauges used to uncouple ϕ and **A**. It is found that some generalizations of the standard Coulomb gauge and Lorentz condition are possible. It is also noted that the introduction of different gauges amounts to defining subsets of solutions ME, that may be disjoint. In particular not all Lorentz invariant solutions are transversal, thus allowing the existence of longitudinal solutions of ME. This finding gives further support to Evans claims. Section IV revisits the invariance of the MFE under the Lorentz gauge transformation. It is found that full invariance implies constraints on the magnetic properties of the vacuum. Section V closes the paper.

II. Electromagnetic Fields in Terms of Potentials

In this section we review in detail the conventional method for obtaining solutions of the set of four Maxwell s equations (ME, eq. 1) in terms of *potentials*. In particular, we are interested in subsets of solutions that may be represented as nonhomogeneous wave equations (WE), a particular instance being Maxwell field equation (MFE, eq. 2). As a result, we pinpoint magnetic components in the solution of ME, and find that the magnetic scalar potential is as fundamental as the conventional electric potential.

In CGS units, ME are [(4)]

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial w} \tag{1a}$$

$$\nabla \times \mathbf{B} = +\frac{\partial \mathbf{E}}{\partial w} + \frac{4\pi J}{c} \tag{1b}$$

$$\nabla \bullet \mathbf{E} = 4\pi\rho \tag{1c}$$

$$\nabla \bullet \mathbf{B} = 0 \tag{1d}$$

where time is the geometric variable $w \equiv ct$. Other symbols and dimensions are: electric and magnetic field, **E**, **B** [=(dyne esu ^{1}[=(esu cm ^{2}, current density J [=(esu sec ^{1}cm ^{2}, charge density ρ [=(esu cm ^{3}.

A general solution of ME is a pair of three-dimensional time-dependent vectors ($\mathbf{B}(\mathbf{r},t)$, $\mathbf{E}(\mathbf{r},t)$). The set of all such pairs is $\mathcal{M}$ (the set of all solutions of ME):

$$\mathcal{M} = \left\{ \left[\mathbf{B}(\mathbf{r},t), \mathbf{E}(\mathbf{r},t) \right] \middle| \forall \text{ solutions eqs. (1)} \right\} \quad \text{(S1)}$$

At this level of generality we are not concerned with the physical meaning of the solutions, so that the individual components of $\mathbf{B}(\mathbf{r},t)$ and $\mathbf{E}(\mathbf{r},t)$ may be complex.

Let ϕ and $\mathbf{A}$ be the standard scalar and vector potentials respectively. In next section III we will identify the conditions under which the classical ME may be written as the MFE (eq. 2). In 4-(row) vector notation, let $x^\mu \equiv (w,x,y,z) = (w,\mathbf{r})$, $A^\mu \equiv (\phi, \mathbf{A})$, and $\mathbf{j}^\mu \equiv (\rho, \mathbf{J}/c)$; the MFE is [5-7]

$$\Box A^\mu = -4\pi j^\mu \quad (2)$$

where the d Alembertian operator $\Box \equiv \nabla^2 - \partial^2/\partial w^2$ [=(cm ^{2}, and j^μ is a 4-vector charge density [=(esu cm^{-3}. Therefore, the dimensions of $(\phi, \mathbf{A})$ *must be* charge per unit length [=(esu cm ^{1}.

The standard solution of ME in terms of potentials (*e.g.* Jackson[5], chap. 6, referred to henceforth as J followed by page number) *assumes* that the magnetic field is

$$\mathbf{B} = \nabla \times \mathbf{A} \quad \text{(3a)},$$

where $\mathbf{A}$ has dimensions of energy per unit charge [=(gauss cm = erg esu ^{1}. Via Coulomb s force law, these dimensions are consistent with the units in eq. (2).

Clearly, there is no *a priori* assurance that *all* vectors $\mathbf{B}(\mathbf{r},t) \in \mathcal{M}$ may be represented by eq. (3a). The set $\mathcal{P}$ of all solutions of ME in terms of *potentials* based in eq. (3a) is

$$\mathcal{P} = \left\{ \begin{matrix} \left[\mathbf{B}(\mathbf{r},t), \mathbf{E}(\mathbf{r},t) \right] \middle| \forall \text{ solutions eq. (1)} \\ \ni \mathbf{B} \text{ is consistent with eq. (3a)} \end{matrix} \right\} \quad \text{(S2)},$$

where $\mathcal{P} \subseteq \mathcal{M}$ (the equality holds if, and only if, eq. (3a) is a complete representation of all $\mathbf{B}(\mathbf{r},t) \in \mathcal{M}$).

Since the divergence of the curl is zero, eq. (3a) satisfies eq. (1d) without further ado. However, from the mathematical properties of the curl ($\nabla \times (\nabla f) = 0$), it is clear that vector $\mathbf{A}$ above is not unique. In fact, any gradient $\nabla \Phi$ may be added to $\mathbf{A}$, to get

$$\mathbf{B}' = \nabla \times (\mathbf{A} + \nabla \Phi) = \mathbf{B} \quad \text{(3b)}.$$

In field theory,[5,7] eqs. (3a) and (3b) correspond to the invariant transformation $\mathbf{B} \to \mathbf{B}' = \mathbf{B}$, implying

$$\mathbf{A} \to \mathbf{A}' = \mathbf{A} + \nabla \Phi = \mathbf{A} + \gamma \quad \text{(4a)},$$

which is the first component of the standard *Lorentz gauge transformation* (eq. 6.34, J176). The other member of the transformation being (eq. 6.35, J176)

$$\phi \to \phi' = \phi - \frac{\partial \Phi}{\partial w} = \phi + \mathcal{E} \quad \text{(4b)}.$$

All potentials appearing in eqs. (4a) and (4b) have dimensions of energy per unit charge. We *interpret* $\Phi(\mathbf{r},t)$ *as magnetic flux of the vacuum* [=(erg cm esu ^{1} = gauss cm^2 = esu (the latter for dimensional consistency with eq. 2), while

$$\gamma \equiv \nabla \Phi \quad \text{(5a)}$$

is a magnetic vector potential representing the normal to the surfaces of constant magnetic flux of the vacuum $\Phi = C$. The electromotive force (ref. 4, page 265) associated with temporal variations of magnetic flux of vacuum being

$$\mathcal{E} \equiv -\frac{\partial \Phi}{\partial w} \quad \text{(5b)}.$$

Clearly, $\mathcal{E}$ is a magnetic scalar potential. Eqs. (3a) and (3b) lead, upon substitution in Faraday s law (eq. 1a) to

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial w}(\nabla \times \mathbf{A}) = -\nabla \times \frac{\partial \mathbf{A}}{\partial w} \quad \text{(6a)}$$

$$\nabla \times \mathbf{E}_m = -\frac{\partial}{\partial w}\left[\nabla \times (\mathbf{A} + \gamma) \right] = -\nabla \times \left(\frac{\partial \mathbf{A}}{\partial w} + \frac{\partial \gamma}{\partial w} \right) \quad \text{(6b)}$$

Hence,

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial w} \quad \text{(7a)}$$

$$\mathbf{E}_m = -\frac{\partial \mathbf{A}}{\partial w} - \frac{\partial \gamma}{\partial w} = -\frac{\partial \mathbf{A}}{\partial w} + \nabla \mathcal{E}. \quad \text{(7b)}$$

Eqs. (7a) and (7b) describe the transformation $\mathbf{A} \to \mathbf{A}' = \mathbf{A} + \gamma \Rightarrow \mathbf{E} \to \mathbf{E}_m = \mathbf{E} + \nabla \mathcal{E}$, where the transformation $\mathbf{E} \to \mathbf{E}_m$ is clearly noninvariant, except for the trivial case $\nabla \mathcal{E} = 0$. However, both $\mathbf{E}$ and $\mathbf{E}_m$ satisfy Faraday s law.

Invoking the properties of the curl again, arbitrary electric scalar field gradients $\nabla \phi_E$ and $\nabla \phi_B$ (associated with the space variation of $\mathbf{E}$ and the time variation of $\mathbf{B}$, respectively) may be added to both sides of eqs. (6a) and (6b) to get

$$\mathbf{E}_e = -\nabla \phi_E - \frac{\partial \mathbf{A}}{\partial w} - \nabla \phi_B = -\frac{\partial \mathbf{A}}{\partial w} - \nabla \phi \quad \text{(8a)}$$

$$\begin{aligned} \mathbf{E}_{me} &= -\nabla (\phi_E)_m - \frac{\partial \mathbf{A}}{\partial w} + \nabla \mathcal{E} - \nabla (\phi_B)_m \\ &= -\frac{\partial \mathbf{A}}{\partial w} + \nabla \mathcal{E} - \nabla \phi_m \end{aligned} \quad \text{(8b)}.$$

where the effective electric scalar potential $\phi \equiv \phi_E + \phi_B$ [=(erg esu ^{1} originates in both spatial and temporal variations. Subscript m in ϕ_m (eq. 8b) reflects the fact that there is no *a priori* reason for the arbitrary gradients to be identical in both eqs. (8a) and (8b). Subscripts m and e in $\mathbf{E}_m$, $\mathbf{E}_e$, $\mathbf{E}_{me}$ refer to the origin of the scalar potential contained in the electric field $\mathbf{E}$. Notice that $\mathbf{E}_e$ (eq. 8a) is the conventional expression for electric field $\mathbf{E}$.

Eqs. (7a) and (8a) describe the transformation

$$-\frac{\partial \mathbf{A}}{\partial w} \to -\frac{\partial \mathbf{A}}{\partial w} - \nabla \phi \Rightarrow \mathbf{E} \to \mathbf{E}_e = \mathbf{E} + \nabla \phi \quad \text{(9a)},$$

where $\mathbf{E} \to \mathbf{E}_e$ is noninvariant, except for the trivial case $\nabla \phi = 0$. Of course, $\mathbf{E}_e$ also satisfies Faraday s law. Likewise, Eqs. (7a), (7b) and (8b) contain the novel sequence of transformations

$$-\frac{\partial \mathbf{A}}{\partial w} \rightarrow -\frac{\partial \mathbf{A}}{\partial w} + \nabla \mathcal{E} \rightarrow -\frac{\partial \mathbf{A}}{\partial w} + \nabla \mathcal{E} - \nabla \phi_m \quad (9b).$$
$$\Rightarrow \mathbf{E} \rightarrow \mathbf{E}_m \rightarrow \mathbf{E}_{me}$$

Although the first step $\mathbf{E} \rightarrow \mathbf{E}_m$ is noninvariant, the second step $\mathbf{E}_m \rightarrow \mathbf{E}_{me}$ may be made invariant. This transformation may be written in terms of scalar potentials only as

$$\mathcal{E} \rightarrow \mathcal{E}' = \mathcal{E} - \phi_m \quad (10),$$

where eq. (10) is the *dual* of the conventional second member of the Lorentz gauge transformation (recall eq. 4b). The ensuing dual-magnetic solution of ME is currently under consideration.[8] Previous discussion may be summarized in the following table of transformations.

making $\mathbf{E}_m$ invariant under transformations of the 4-vector $(\mathcal{E}, \mathbf{A})$.[8]

Current practice is to drop Φ right at the beginning (*i.e.*, just after eq. 3, above) thus missing eq. (7b). The standard reason for ignoring the magnetic flux being: (1) Φ is arbitrary; and, (2) Φ does not contribute to the fields $\mathbf{E}_e$ and $\mathbf{E}'_e$ associated with the complete Lorentz transformation (eqs. 4a and b): $\mathbf{A} \rightarrow \mathbf{A}'$, $\phi \rightarrow \phi' \Rightarrow \mathbf{E}_e \rightarrow \mathbf{E}'_e$. However, as demonstrated here, Φ and $\mathcal{E}$ appear in the expressions for **B** and **E** earlier than ϕ. Therefore, the second component of the Lorentz gauge transformation $(\phi \rightarrow \phi')$ that makes Φ apparently redundant is not even meaningful at the early stage that Φ is dropped.

ELECTRIC FIELDS CONSISTENT WITH FARADAY S LAW

starting potential	arbitrary gradient	intermediate potential	arbitrary gradient	Eq.	Final electric field
A	0	**A**	0	7a	$-\partial \mathbf{A}/\partial w$
A	$\nabla\mu$	$\mathbf{A}+\gamma$	0	7b	$-\partial \mathbf{A}/\partial w + \nabla\mathcal{E}$
A	0	**A**	$\nabla\phi$	8a	$-\partial \mathbf{A}/\partial w - \nabla\phi$
A	$\nabla\mu$	$\mathbf{A}+\gamma$	$\nabla\phi_m$	8b	$-\partial \mathbf{A}/\partial w + \nabla\mathcal{E} - \nabla\phi_m$

Comments and discussion. It is noted that both $\mathbf{E}_m$ (eq. 7b) and $\mathbf{E}_e$ (eq. 8a) result from the same mathematical artifact, *i.e.* arbitrary functions added to a rotational operator (associated with the magnetic field in eq. 1d and with the electric field in eq. 1a). Hence, $\nabla\mathcal{E}$ and $\nabla\phi$ have the same mathematical standing. There are no *a priori* compelling reasons to drop one form or another from further consideration.

Also, it may be seen that the most general expression $\mathbf{E}_{me}$ (eq. 8b) may be reached either from $\mathbf{E}_m$ (eq. 7b) *via* a transformation on $\mathcal{E}$ (as in eq. 10) or from $\mathbf{E}_e$ (eq. 8a) *via* the conventional transformation on ϕ (as in eq. 4b). However, it is stressed that contrary to what one could expect from conventional wisdom the natural result of introducing the vector potential **A** (eqs.3a and 3b) into Faraday s law leads to eq. (8b), *via* eq. (7b) (not *via* the standard eq. 8a).

As noted before, both transformations $\mathbf{E} \rightarrow \mathbf{E}_m$ and $\mathbf{E} \rightarrow \mathbf{E}_e$ are noninvariant. To make electric field invariant to the Lorentz gauge transformation it is necessary to *assign independent existence* to the arbitrarily introduced scalar potential ϕ, thus making it possible to define the second component to the Lorentz gauge transformation (eq. 4b, above).

Also, as seen above, $\mathbf{E}_m$ (eq. 7b) is similar to $\mathbf{E}_e$ (eq. 8a), with the new magnetic potential $\mathcal{E}$ playing the role of the conventional electric potential ϕ. Hence, by analogy, one can define two magnetic gauge transformations

III. Coupling Conditions in Solving Maxwell s Equations

In this section, we review the assumptions made to uncouple the standard potentials (ϕ and **A**) in Ampere s and Coulomb s laws (eq. 1b and 1c respectively). Obviously, different gauges lead to different subsets of solutions. In particular, the subset of solutions associated with the Coulomb gauge (transversal solutions) is disjoint with the subset of solutions associated with the Lorentz condition (except for electric static scalar phenomena), thus opening a door to longitudinal solutions. Since the subset of Coulomb solutions is not the same as the subset of Lorentz solutions, it is clear that they may represent different phenomena.

The nonhomogeneous components of ME are Ampere s law (eqs. 1b), and Coulomb s law (eq. 1c). Upon substitution of the conventional B (eq. 3) and E (eq. 8a) we get a pair of simultaneous second order equations in **A** and ϕ

$$\nabla^2\phi + 4\pi\rho = -\frac{\partial(\nabla \bullet \mathbf{A})}{\partial w} \quad (11)$$

$$\Box\mathbf{A} + \frac{4\pi\mathbf{J}}{c} = \nabla\left(\nabla \bullet \mathbf{A} + \frac{\partial\phi}{\partial w}\right) \quad (12)$$

Eqs. (11) and (12) may be uncoupled by introducing *additional constraints*. Some obvious possibilities are:

(1) **Extended Coulomb condition**. The LHS of eq. (11) may be made zero by constraining the RHS *via* the *extended Coulomb condition* (ECC):

$$\frac{\partial(\nabla \bullet \mathbf{A})}{\partial w} = 0 \qquad (13).$$

Substitution of eq. (13) into eq. (11) leads to a restricted solution of eq. (11):

$$\nabla^2 \phi = -4\pi\rho \qquad (11a),$$

Eq. (11a) is Poisson s equation, whose solution for $\phi(\mathbf{r},w)$ is well-known (eq. 6.44, J177). It is noted in passing that eq. (11a) also is a special case of a scalar WE $\Box\phi = -4\pi\rho$ when $\partial\phi/\partial w = 0$.

The general solution of eq. (13) is the *extended Coulomb gauge* (ECG) defined by

$$\nabla \bullet \mathbf{A} = f(\mathbf{r}) \qquad (14a).$$

Substituting eq. (14a) into eq. (12) one gets a nonhomogeneous WE

$$\Box\mathbf{A} = \nabla\left[f(\mathbf{r}) + \frac{\partial\phi}{\partial w}\right] - \frac{4\pi\mathbf{J}}{c} \qquad (12a),$$

which may be solved to get pairs (ϕ , **A**). The subset of solutions of eq. (12a) constrained by eq. (14a) is the set $\mathcal{C}$ of solutions consistent with ME under the extended Coulomb gauge:

$$\mathcal{C} = \left\{\left[\mathbf{B}(\mathbf{r},t),\mathbf{E}(\mathbf{r},t)\right] \middle| \begin{matrix}\text{eq.}(1) \wedge \text{eq.}(3) \\ \wedge\, \text{eq.}(8\text{a}) \wedge \text{eq.}(14\text{a})\end{matrix}\right\} \qquad (S3)$$

A particular solution of eq. (12a) is achieved by further specializing eq. (14a) to the *restricted Coulomb gauge* (RCG) defined by $f(\mathbf{r}) = K$ (*K* an arbitrary constant):

$$\nabla \bullet \mathbf{A} = K \qquad (14b),$$

This immediately leads to the conventional expression (eq. 6.46, J177)

$$\mathbf{A} = \nabla\left(\frac{\partial\phi}{\partial w}\right) - \frac{4\pi\mathbf{J}}{c} \qquad (12b),$$

which is a particular case of eq. (12a). It is widely-known that the solution of eq. (12b) may be cast in the form of transversal and longitudinal currents, the latter cancelling out the current associated with the solution of the scalar component (eq. 11a), thus leaving transversal currents only (eq. 6.52, J178). For any value of *K* there is a set $\mathcal{C}_r$ containing the restricted solutions associated with eq. (14b):

$$\mathcal{C}_r = \left\{\left[\mathbf{B}(\mathbf{r},t),\mathbf{E}(\mathbf{r},t)\right] \middle| \begin{matrix}\text{eq.}(1) \wedge \text{eq.}(3) \\ \wedge\, \text{eq.}(8\text{a}) \wedge \text{eq.}(14\text{b})\end{matrix}\right\} \qquad (S4).$$

Evidently, $\mathcal{C}_r \subset \mathcal{C} \subset \mathcal{P} \subseteq \mathcal{M}$. Therefore, it is crystal clear that the well-known transversal solutions associated with $\mathcal{C}_r$ simply are a subset of all possible solutions of ME (eq. 1). Also, since, Lorentz condition has not even been defined as yet, there is no guarantee that all the transversal solutions of $\mathcal{C}_r$ are Lorentz invariant. Indeed, as shown below, there exists a proper subset $\mathcal{C}_{rs} \subset \mathcal{C}_r$ of invariant solutions. Hence, the complement ($\mathcal{C}_r - \mathcal{C}_{rs}$) is not Lorentz invariant. The *standard Coulomb gauge* attains for $K = 0$, and introduces no further restrictions on the transversal set $\mathcal{C}_r$.

(2) **Extended invariant coupling condition**. Alternatively, let us us constrain the RHS of eq. (12) by introducing the *extended invariant coupling condition* (EICC):

$$\nabla\left(\nabla \bullet \mathbf{A} + \frac{\partial\phi}{\partial w}\right) = 0 \qquad (15),$$

Substituting eq. (15) into eq. (12), one immediately obtains another particular instance of eq. (12):

$$\Box\mathbf{A} + \frac{4\pi\mathbf{J}}{c} = 0 \qquad (12c).$$

Eq. (12c) is the vector potential component of eq. (2) that may be solved for **A** by standard methods. After obtaining the three components of vector $\mathbf{A}(\mathbf{r},w)$, we can explicitly calculate $F(\mathbf{r},w) = \partial(\nabla \bullet \mathbf{A})/\partial w$. Substituting in eq. (11) one gets

$$\nabla^2\phi = -\left[F(\mathbf{r},w) + 4\pi\rho\right] \qquad (11b),$$

which is a particular case of eq. (11), but a generalization of Poisson s eq. (11a) . In principle, eq. (11b) may be solved for ϕ , thus completing the set of pairs of potentials (ϕ , **A**). However, not all solutions of eqs. (12c) and (11b) are automatically solutions of ME. It is necessary to further constraint solutions (ϕ , **A**) to explicitly satisfy the EICC (eq. 15). The last subset defines the class $\mathcal{X}$ of solutions consistent with ME under the EICC :

$$\mathcal{X} = \left\{\left[\mathbf{B}(\mathbf{r},t),\mathbf{E}(\mathbf{r},t)\right] \middle| \begin{matrix}\text{eq.}(1) \wedge \text{eq.}(3) \\ \wedge\, \text{eq.}(8\text{a}) \wedge \text{eq.}(15)\end{matrix}\right\} \qquad (S5).$$

Clearly, there is no reason to entertain the idea that the set $\mathcal{C}$ (arising from the ECC) should be the same as the set χ resulting from the EICC. Also, $\mathcal{X} \subset \mathcal{P} \subseteq \mathcal{M}$.

(3) **Extended Lorentz condition**. Consider the *extendend Lorentz condition* (ELC) which is a special case of eq. (15) defined by

$$\nabla \bullet \mathbf{A} + \frac{\partial\phi}{\partial w} = f(w) \qquad (16a).$$

By substituting eq. (16a) into eq. (12), one gets eq. (12c), which was also obtained with the more general EICC. Substituting eq. (16a) into eq. (11):

$$\Box\phi = -\left[\frac{\partial f(w)}{\partial w} + 4\pi\rho\right] \qquad (11c)$$

which is a subcase of eq. (11b) with $F(\mathbf{r},w) = \partial f(w)/\partial w - \partial^2\phi/\partial w^2$. Eq. (11c) is another nonhomogeneous WE that may also be solved by standard methods for ϕ . As before, eqs. (12c), (11c) and (16a) lead to the set $\mathcal{L}$ of pairs (ϕ , **A**) consistent with ME under the ELC:

$$\mathcal{L} = \left\{\left[\mathbf{B}(\mathbf{r},t),\mathbf{E}(\mathbf{r},t)\right] \middle| \begin{matrix}\text{eq.}(1) \wedge \text{eq.}(3) \\ \wedge\, \text{eq.}(8\text{a}) \wedge \text{eq.}(16\text{a})\end{matrix}\right\} \qquad (S6).$$

From eq. (11c) it immediately follows that for the special case $f(w) = K$ (a constant) the scalar component of eq. (2) yields:

$$\Box\phi = -4\pi\rho \qquad (11d)$$

This means that MFE (eq. 2) is a particular solution of ME under the *restricted Lorentz condition* (RLC)

$$\nabla \bullet \mathbf{A} + \frac{\partial\phi}{\partial w} = K \qquad (16b).$$

Obviously, as demonstrated with the previous discussion, solutions of ME under conditions more general than the RLC are possible, but they are not represented by the MFE. It is also evident that not all solutions produced by the four individual WEs forming the MFE (eq. 2) are automatically solutions of ME. The individual solutions are interrelated and constrained by eq. (16b). They define the set $\mathcal{L}_r$ (solutions of ME, consistent with the RLC and represented by the MFE, eq. 2):

$$\mathcal{L}_r = \left\{ \left[\mathbf{B}(\mathbf{r},t),\mathbf{E}(\mathbf{r},t)\right] \middle| \begin{array}{l} \text{eq.}(1) \wedge \text{eq.}(3) \\ \wedge\, \text{eq.}(8\text{a}) \wedge \text{eq.}(16\text{b}) \end{array} \right\} \quad (\text{S7}),$$

The restricted Lorentz set above fulfills: $\mathcal{L}_r \subset \mathcal{L} \subset \mathcal{X} \subset \mathcal{P} \subseteq \mathcal{M}$. For $K = 0$, eq. (16b) leads to the *standard Lorentz condition* (SLC); it implies no further constraints for $\mathcal{L}_r$.

Comments and discussion. Firstly, it goes without saying that all our analysis has been classical, without the slightest mention of quantum mechanics. Secondly, from the point of view of ME, the role of the RLC (eq. 16b) is to select a subset of $\mathcal{P}$, representable by the 4 individual WEs conforming the MFE (eq. 2). This fact immediately implies that there are many solutions of ME that are not solutions of MFE, for instance, the complementary set under the EICC ($\mathcal{X} - \mathcal{L}_r$). See Fig. 1.

Conversely, from the viewpoint of the MFE, the role of the RLC is to select a subset out of the set of all solutions to the MFE. Hence, there are many solutions of the MFE that are not solutions of ME, *i.e.* the set of pairs (ϕ ,**A**) that do not fulfill eq. (16b). This finding precisely is the conclusion that Dirac, Fock and Podolsky reached more than sixty years ago: this condition [Lorentz(cannot be regarded as a quantum mechanical equation, but rather as a condition on permissible Ψ functions. [9] The invariance implicit in the individual WEs of the MFE (eq. 2) is a separate matter, addressed in section IV below.

From the view point of the electric scalar potential ϕ, let us partition $\mathcal{L}_r$ into two classes: static ($\mathcal{S}_\phi$) and induction phenomena ($\mathcal{I}_\phi$) defined by

$$\mathcal{S}_\phi = \left\{ \left[\mathbf{B}(\mathbf{r},t),\mathbf{E}(\mathbf{r},t)\right] \middle| \begin{array}{l} \text{eq.}(1) \wedge \text{eq.}(3) \wedge \text{eq.}(8\text{a}) \\ \wedge\, \text{eq.}(16\text{b}) \wedge \left(\partial\phi/\partial w = 0\right) \end{array} \right\} \quad (\text{S8}).$$

$$\mathcal{I}_\phi = \left\{ \left[\mathbf{B}(\mathbf{r},t),\mathbf{E}(\mathbf{r},t)\right] \middle| \begin{array}{l} \text{eq.}(1) \wedge \text{eq.}(3) \wedge \text{eq.}(8\text{a}) \\ \wedge\, \text{eq.}(16\text{b}) \wedge \left(\partial\phi/\partial w \neq 0\right) \end{array} \right\} \quad (\text{S9}).$$

Hence, by definition $\mathcal{S}_\phi \subset \mathcal{L}_r$, $\mathcal{I}_\phi \subset \mathcal{L}_r$, and $\mathcal{S}_\phi \cap \mathcal{I}_\phi = \varnothing$, *i.e.* the two classes are disjoint. Likewise, let $\mathcal{C}_{rs} \subset \mathcal{C}_r$ be the static subset of the restricted Coulomb set $\mathcal{C}_r$:

$$\mathcal{C}_{rs} = \left\{ \left[\mathbf{B}(\mathbf{r},t),\mathbf{E}(\mathbf{r},t)\right] \middle| \begin{array}{l} \text{eq.}(1) \wedge \text{eq.}(3) \wedge \text{eq.}(8\text{a}) \\ \wedge\, \text{eq.}(14\text{b}) \wedge \left(\partial\phi/\partial w = 0\right) \end{array} \right\} \quad (\text{S10}).$$

It is easy to see that solutions contained in $\mathcal{S}_\phi$ are consistent with both the Lorentz condition and the Coulomb gauge; hence, $\mathcal{S}_\phi$ is formed by transversal solutions only.

However, solutions contained in $\mathcal{I}_\phi$ need not be transversal too (see Fig. 1). Hence, the latter may contain longitudinal solutions. Previous assertions follow from the following two propositions:

Proposition 1. The set $\mathcal{S}_\phi$ is identical to $\mathcal{C}_{rs}$; hence, $\mathcal{S}_\phi \subset \mathcal{C}_r$.

Proof. Let us substitute the static condition $\partial\phi/\partial w = 0$ from (S8) into the equation for the scalar potential defining $\mathcal{S}_\phi$ (eq. 11d); one gets Poisson s eq. (11a), which defines the scalar potential in the extended set $\mathcal{C}$. Hence, the definition of the scalar potential for the sets $\mathcal{S}_\phi$ and $\mathcal{C}_{rs} \subset \mathcal{C}_r \subset \mathcal{C}$ is the same. Likewise, the vector potential defining the restricted Coulomb set $\mathcal{C}_r$ is given by eq. (12b). Since $\mathcal{C}_{rs}$ is defined by $\partial\phi/\partial w = 0$, then eq. (12b) reduces to eq. (12c). The latter defines the vector potential for the set $\mathcal{L}_r \supset \mathcal{S}_\phi$. Hence, the definition of the vector potential for $\mathcal{S}_\phi \subset \mathcal{L}_r$ and for $\mathcal{C}_{rs}$ is the same. Since the potentials defining the two sets $\mathcal{S}_\phi$ and $\mathcal{C}_{rs}$ fulfill exactly the same equations, then the two sets are identical $\mathcal{S}_\phi = \mathcal{C}_{rs}$. But $\mathcal{C}_{rs} \subset \mathcal{C}_r$; then $\mathcal{S}_\phi \subset \mathcal{C}_r$. Q.E.D.

Corollary 1. $\mathcal{C}_{rs}$ is Lorentz invariant.

Proof. It immediately follows from $\mathcal{S}_\phi = \mathcal{C}_{rs}$, $\mathcal{C}_{rs} \subset \mathcal{C}_r$, and $\mathcal{S}_\phi \subset \mathcal{L}_r \Rightarrow \mathcal{C}_{rs} \subset (\mathcal{L}_r \cap \mathcal{C}_r)$.

Proposition 2. The induction class $\mathcal{I}_\phi$ associated with the SLC (eq. 16b, $K = 0$) is disjoint with the $\mathcal{C}_r$ associated with the conventional Coulomb gauge (eq. 14b, $K = 0$).

Proof: For this we need to show that no solution in $\mathcal{I}_\phi$ belongs in $\mathcal{C}_r$, and conversely, that no solution in $\mathcal{C}_r$ belongs in $\mathcal{I}_\phi$. The RLC (eq. 16b), leads to

$$\nabla \bullet \mathbf{A} = K - \frac{\partial\phi}{\partial w} = -\frac{\partial\phi}{\partial w} = \text{variable for } K = 0, \text{ and } \mathcal{I}_\phi$$

$$\left(\partial\phi/\partial w \neq 0\right) \qquad (17).$$

Since eq. (17) is not the same as the RCC (eq. 14b, for $K = 0$), no solution in $\mathcal{I}_\phi$ belongs in $\mathcal{C}_r$. Now for the converse part. Solutions in $\mathcal{C}_r$ are described by eq. (14b). The set $\mathcal{C}_r$ may be partitioned into a static subset $\mathcal{C}_{rs}$ and its complement $\mathcal{C}_r - \mathcal{C}_{rs}$ Subset $\mathcal{C}_{rs}$ is the same as $\mathcal{S}_\phi$, which is disjoint from $\mathcal{I}_\phi$ by definition. Subset

$\mathcal{C}_r - \mathcal{C}_{rs}$ is defined by $\nabla \bullet \mathbf{A} = 0$ and $\partial\phi/\partial w \neq 0$. Consequently, solutions in $\mathcal{C}_r - \mathcal{C}_{rs}$ fulfill $\nabla \bullet \mathbf{A} + \partial\phi/\partial w \neq 0$, which is the exact opposite of the SLC (eq. 16b with $K = 0$). Hence, no term in $\mathcal{C}_r - \mathcal{C}_{rs}$ may belong in $\mathcal{L}_r$, and even less in its proper subset $\mathcal{I}_\phi \subset \mathcal{L}_r$. Since $\mathcal{I}_\phi$ is disjoint both with $\mathcal{C}_r - \mathcal{C}_{rs}$ and its complement $\mathcal{C}_{rs}$, then $\mathcal{I}_\phi \cap \mathcal{C}_r = \varnothing$. Q.E.D.

Summarizing, the class $\mathcal{C}_r$ only contains solutions with transversal currents; hence, longitudinal fields are not allowed within $\mathcal{C}_r$. Classically, there are no explicit restrictions for the existence of longitudinal solutions within the complementary class ($\mathcal{M} - \mathcal{C}_r$). If one restricts consideration to phenomena consistent with Lorentz condition, there are no limitations to the type of solutions in the induction class $\mathcal{I}_\phi$.

IV. Invariance Under Lorentz Gauge Transformations

In this section we explicitly carry the neglected magnetic potentials through the *Lorentz gauge transformation* (LGT). It is found that both pairs (ϕ ,**A**) and (ϕ',$\mathbf{A}'$) of a LGT explicitly including magnetic potentials are invariant, provided that a new constraint be introduced on the vacuum s magnetic flux $\Phi(\mathbf{r},t)$.

Consider the conventional LGT defined by eqs. (4a) and (4b) above (eqs. 6.34 and 35, J176), repeated here for convenience. For arbitrary $\Phi(\mathbf{r},t)$,

$$\mathbf{A} \to \mathbf{A}' = \mathbf{A} + \nabla\Phi = \mathbf{A} + \gamma \qquad (4a),$$

$$\phi \to \phi' = \phi - \frac{\partial\Phi}{\partial w} = \phi + \mathcal{E} \qquad (4b).$$

The question is whether the transformed pair (ϕ',$\mathbf{A}'$) automatically fulfills invariance in three regards: (a) the fields B and E, (b) the standard Lorentz condition (SLC), and (c) the MFE (eq. 2). Jackson[5] only addresses the first two questions.

(a) Invariance of fields. Invariance $\mathbf{E} \to \mathbf{E}' = \mathbf{E}$ and $\mathbf{B} \to \mathbf{B}' = \mathbf{B}$ immediately follows by substituting eqs. (4a) and (4b) into eqs. (3) and (8a) initially written for the transformed $\mathbf{E}'$, $\mathbf{B}'$, $\mathbf{A}'$ and ϕ'.

(b) Invariance of Lorentz condition. Let us consider the more general EICC (eq. 15), written for the transformed potentials (ϕ',$\mathbf{A}'$):

$$\nabla\left(\nabla \bullet \mathbf{A}' + \frac{\partial\phi'}{\partial w}\right) = 0 \qquad (15).$$

Substituting eqs. (4a and b) back into eq. (15), we get the invariant transformation:

$$\left[\nabla\left(\nabla \bullet \mathbf{A}' + \frac{\partial\phi'}{\partial w}\right) = 0\right]$$
$$\to \left[\nabla\left(\nabla \bullet (\mathbf{A} + \gamma) + \frac{\partial(\phi - \partial\Phi/\partial w)}{\partial w}\right) = 0\right]$$
$$\to \left[\nabla\left(\nabla \bullet \mathbf{A} + \frac{\partial\phi}{\partial w}\right) = 0\right]$$

if, and only if,

$$\nabla(\Box\Phi) = 0 \qquad (18a),$$

A special solution assures invariance of the extended Lorentz condition (ELC),

$$\Box\Phi = f(w) \qquad (18b).$$

Invariance of the restricted Lorentz condition (RLC) obtains for $f(w) = K$ = constant:

$$\Box\Phi = K \qquad (18c).$$

The SLC attains for $K = 0$. The condition $\Box\Phi = 0$ appears in textbooks as a requirement of the SLC (eq. 6.42, J177), that restricts severely the gauge arbitrariness (ref. 7, p. 12). Hence, all coupling conditions related to the EICC are fully invariant under the LGT. The conventional analysis ends here.

(c) **Invariance of field equation**. Ampere s and Coulomb s laws (eqs. 11 and 12) in terms of transformed potentials are

$$\nabla^2\phi' + 4\pi\rho = -\frac{\partial(\nabla \bullet \mathbf{A}')}{\partial w} \qquad (11)$$

$$\Box\mathbf{A}' + \frac{4\pi\mathbf{J}}{c} = \nabla\left(\nabla \bullet \mathbf{A}' + \frac{\partial\phi'}{\partial w}\right) \qquad (12)$$

Consider the vector part first. Substituting eq. (15) into (12) we get

$$\Box\mathbf{A}' + \frac{4\pi\mathbf{J}}{c} = 0 \qquad (12c),$$

which is the vector part of the transformed eq. (2). Substituting eq. (4a) we get the invariant transformation $[\Box\mathbf{A}' + 4\pi\mathbf{J}/c = 0] \to [\Box(\mathbf{A} + \gamma) + 4\pi\mathbf{J}/c = 0]$
$\to [\Box\mathbf{A} + 4\pi\mathbf{J}/c = 0]$, if and only if

$$\Box\gamma = \Box(\nabla\Phi) = 0 \qquad (19).$$

Consider now the scalar component of eq. (2). Substitution of the RLC into eq. (11) leads to $\Box\phi' = -4\pi\rho$. Substituting eq. (4b) we obtain the invariant transformation

$[\Box\phi' = -4\pi\rho] \to [\Box(\phi - \partial\Phi/\partial w) = -4\pi\rho]$
$\to [\Box\phi = -4\pi\rho]$,

if, and only if,

$$\Box\frac{\partial\Phi}{\partial w} = \frac{\partial(\Box\Phi)}{\partial w} = 0 \qquad (20).$$

Eq. (20) immediately follows from the RLC eq. (18c), above. This completes the proof of the complete invariance of EM fields, the coupling conditions (EICC, ELC, and RLT) and the MFE to Lorentz gauge transformations in the presence of magnetic potentials.

Comments and discussion. In summary, the subset of solutions of ME formed by the solutions of the MFE constrained by the RLC ($\mathcal{L}_r \subset \mathcal{M}$) is completely invariant to the LGT (including magnetic potentials) provided that the magnetic flux of vacuum is constrained by the usual constraint eq. (18c) *plus* the new constraint, eq. (19). The weaker eqs. (18a) and (18b) define other subsets of solutions of ME (larger, and represented by WEs other than eq. 2), also completely invariant under LGT.

To assure invariance of MFE under LGT, the initially arbitrary vacuum must fulfill the two d Alembertian constraints eqs. (18c) and (19). If ME are constrained by the standard Lorentz condition ($K = 0$), we get the homogeneous conditions $\Box\Phi = 0$, and $\Box\gamma = 0$; that may be put together as a 4-vector equation $\Box V^{\mu} = 0$ with $V^{\mu} \equiv (H\Phi, \gamma)$, where H is a constant in inverse length units (with additional conditions defining the coupling of Φ and γ, $\Box V^{\mu} = 0$ may become the field equation for the vacuum). Since $\gamma = \nabla\Phi$ is the normal to the surfaces of isoflux in vacuum, eq. (19) represents a sort of continuity in the deformation of the surfaces of magnetic flux.

To our knowledge, eq. (19) is a new constraint. However, Itzykson and Zuber (ref. 7, page 12) hint at its existence when they incorporate the Lorentz condition in the formalism of MFE (*via* a Lagrange multiplier) to get $\Box\partial_{\mu}\mathbf{A}^{\mu} = 0$. When applied to the vacuum Φ this would imply $V^{\mu} \equiv (\mathcal{E}, \gamma)$, rather than $V^{\mu} \equiv (H\Phi, \gamma)$ as above.

Clearly, eqs. (18c) and (19) admit, in principle, solutions other than the trivial $\Phi = 0$. According to conventional wisdom, the noncovariant Coulomb gauge reduces eq. (18c) to $\nabla^2\Phi = 0$ everywhere, and hence $\Phi = 0$ (J138). However, as noted by Gribov,[10] this is true only in the Abelian case (cited in ref. 7, p. 576). On the contrary, in our Lorentz invariant case $\nabla^2\Phi = 0$ is a result of $\partial\mathcal{E}/\partial w = 0$, or, $\mathcal{E} = f(\mathbf{r})$ (which may hold under a variety of circumstances). Thus, $\Phi = f(\mathbf{r})w + C$. The initial condition $\Phi = 0$ at $w = 0$, leads to $\Phi = f(\mathbf{r})w$. Hence, in general, $\Phi \neq 0$ even for $\nabla^2\Phi = 0$.

Finally, it goes without saying that the completely Lorentz invariant MFE may be written in manifestly covariant form (see, for instance, J370 or ref. 7, page 7 ff.).

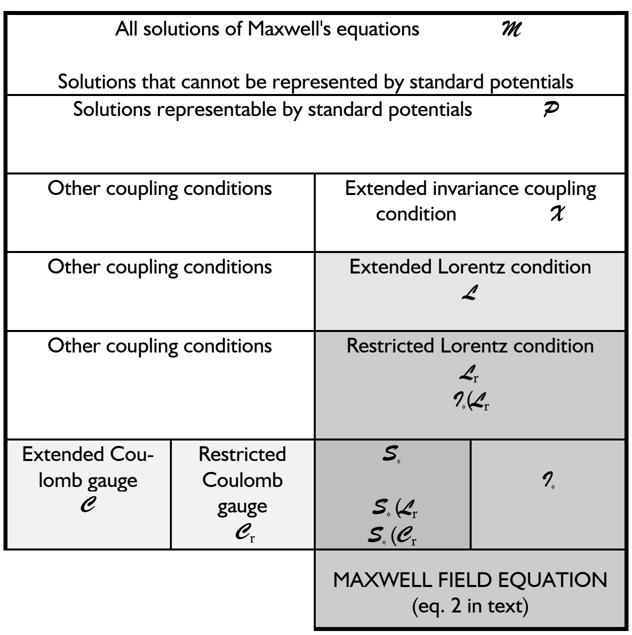

Figure 1. Venn diagram for the set of solutions of Maxwell s equations (1) according to different constraints. Connection to set of solutions of Maxwell field equation (2) under the restricted Lorentz condition.

V. Concluding Remarks

We revisited the well-known derivation of Maxwell field equation (MFE) from Maxwell equations (ME). From the viewpoint of classical ME, the set of solutions of MFE simply are a subset of all possible solutions of ME. The subset of transversal solutions (defined by the Coulomb gauge) is a different subset. In general, longitudinal solutions of ME are not prohibited (except for the Coulomb class).

From the point of view of MFE, not all solutions correspond to solutions of ME too (only those satisfying the RLC). This is the class of Lorentz invariant solutions that may be partitioned into two groups of phenomena, relative to the electric scalar potential: static, and inductive. Electromagnetic fields for the static class $\mathcal{S}_\phi$ must be transversal, but no condition arises for the inductive class $\mathcal{I}_\phi$, thus allowing longitudinal fields. This finding wholly supports Evans claims regarding the existence of a longitudinal photomagnetic field;[1] moreover, it suggests that the empirical evidence cited by Evans[11] belongs in $\mathcal{I}_\phi$.

Invariance of the MFE to the LGT (under the RLC) introduces severe limitations to the initially arbitrary magnetic flux of the vacuum. Solutions of ME obeying the standard Lorentz condition ($K = 0$) are represented by the fully invariant MFE provided that the vacuum s Φ fulfills two conditions: the standard $\Box\Phi = 0$ and the new $\Box\gamma = \Box(\nabla\Phi) = 0$. The magnetic vector potential γ may be interpreted as the normal to the surfaces of constant Φ. In general, our solutions correspond to $\Phi \neq 0$.

According to Gribov,[10] the fact $\Phi \neq 0$ implies that the solutions are non-Abelian. According to Ryder,[13] $\Phi \neq 0$ implies a non-simply connected topology, that leads to an explanation of the observable Aharonov-Bohm effect.

The existence of the magnetic flux Φ and the associated vector and scalar potentials (γ and $\mathcal{E}$) allows definition of a magnetic scalar field $\mathbf{B}_{gs}$ associated with spatial fluctuations of the vacuum as $\mathbf{B}_{gs} = \nabla \bullet \gamma = \nabla^2\Phi$. From eq. (18c) we get $\mathbf{B}_{gs} = K + \partial^2\Phi/\partial w^2 = K - \partial\mathcal{E}/\partial w$.

Excepting magnetostatic situations characterized by $\mathcal{E}=$ constant, $\mathbf{B}_{gs}$ is a *bona fide* function $\mathbf{B}_{gs}(\mathbf{r},w)$. This magnetic scalar field may be connected to Evans $\mathbf{B}^{(0)}$ by $B_{gs}(\mathbf{r},w) \propto B^{(0)} \exp[-i(\omega t - \kappa \bullet \mathbf{r})]$ (12).

Alternatively, one may define a magnetic scalar field associated with temporal oscillations of the vacuum $\mathbf{B}_{gt} = \partial\mathcal{E}/\partial w = -\partial^2\Phi/\partial w^2$. From eq. (18c) we get $\mathbf{B}_{gt} = K - \nabla^2\Phi$, which is also a *bona fide* function $\mathbf{B}_{gt}(\mathbf{r},w)$, provided that spatial oscillations of vacuum be permitted. Again, this may be connected to Evans $B^{(0)}$. Note the spatial-temporal duality exhibited by $\mathbf{B}_{gs}$ and $\mathbf{B}_{gt}$.

Last, but not least, it was found that magnetic scalar potentials are as fundamental as the electric scalar term for the representation of the electric field. Indeed, a magnetic transformation of the scalar potentials naturally arises. This leads to a magnetic-dual solution of ME in terms of potentials, to be addressed in a separate paper.(8) The connection of the scalar potentials uncovered here to the solutions of Maxwell s equations recently reported by Chubykalo and Smirnov-Rueda(14) is deferred to forthcoming papers. Also, the implications of the new nonperiodic solutions(3) regarding magnetic properties, mass, spin, and quantum properties of photon is left for further work.

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