A Quantum of Light Shed on Classical Potentials and Fields*

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The quantum concept of light as photon, elaborated with possibly non-zero rest mass, is used to re-examine the classical problem of retarded electromagnetic potentials. A conflict with the classical Liénard-Wiechert formulation is revealed. An alternative formulation is recommended.

1. Introduction

The classical theory of retarded electromagnetic potentials is usually attributed to Liénard (1898) and Wiechert (1900), who developed a mathematical model based on Euclidean geometry and general signal retardation analogous to that occurring with light propagation. The Liénard-Wiechert model represents the amalgamation of dual concepts: the discrete particle (the source) and the continuous wave (the signal). The duality of the concepts was evident at the time, in competition between the Lorentz electron theory and the Maxwell field theory for electricity (see O'Rahilly 1938), and it remains evident today in occasional as-yet unresolved conflicts between relativity theory and quantum theory.

Was the wave-particle duality really resolved adequately within the Liénard-Wiechert model? It seemed so in terms of turn-of-the-century Lorentz electrons and Maxwell waves. But if still so today, the model would have to be consistent with both relativity theory and quantum theory.

The Liénard-Wiechert model predates Einstein's special relativity theory (SRT, 1905 and 1907), but has since been incorporated into that theory by means of various mathematical arguments which are not the main topic of the present paper.

The corresponding question of consistency between the Liénard-Wiechert model and quantum theory seems not to have been addressed. That is the topic of the present paper. If the Liénard-Wiechert model is indeed consistent with quantum mechanics, then a new argument based on modern light quanta ought to reproduce it.

The present paper argues that in fact such consistency is *not* achieved. We are therefore forced to reconsider, and possibly alter, both the classical arguments and the potentials that result from them. In addition, we need to review the currently believed consistency with SRT.

2. The Classical Formulation

The classical Liénard-Wiechert potentials are

$$\left(\Phi,\vec{A}\right)=Q\left[\frac{\left(1,\vec{b}\right)}{kR}\right]_{ret} \quad (1)$$

where Φ and $\vec{A}$ are, respectively, scalar and vector potentials, Q is source charge, $\vec{b}$ is $\vec{v}/c$ with $\vec{v}$ being source velocity, k is $1-\vec{n}\bullet\vec{b}$ with $\vec{n}$ being $\vec{R}/R$, $\vec{R}$ is source-to-observer spatial vector, and "*ret*" means source parameters are evaluated at retarded time, *i.e.* causally connected time, *i.e.* $t_{ret}=t-R_{ret}/c$.

Interesting features of the Liénard-Wiechert potentials include the following:

1. Source coordinates are defined implicitly. That is, t_{ret} is a function of R_{ret}, and R_{ret} is a function of t_{ret}. This feature may serve to warn the cautious user that constructing an analysis from the point of view of the observer may be tricky.
2. The scalar potential Φ is not a symmetric function of the source speed v, and the vector potential $\vec{A}$ is not an antisymmetric function of v. But they *would* be if the four-vector $\left(\Phi,\vec{A}\right)$ were obtained by Lorentz transformation of the four-vector $\left(\Phi_0,\vec{0}\right)$ from the coordinate frame of the source. This conflict should warn the user to take care.

The classical Liénard-Wiechert potentials are nevertheless incorporated into SRT. The first step is rewriting them as

$$\left(\Phi,\vec{A}\right)=Q\left[\frac{g\left(1,\vec{b}\right)}{gkR}\right]_{ret} \quad (2)$$

Then the numerator $\left[g\left(1,\vec{b}\right)\right]_{ret}$ is recognized as the unit-normalized source-velocity four-vector V^m_{ret}. As-

* Dedicated to Ruth E. Kolb, 1909-1996, for her encouragement of independent thinking.

suming uniform motion, the subscript "*ret*" has no effect on V^m. The denominator $[gkR]_{ret}$ is recognized as the frame-invariant scalar inner product $\left[V_m R^m\right]_{ret}$, where the four-vector $R^m_{ret} = \left(R, \vec{R}\right)_{ret}$ is the zero-length source-to-observer coordinate difference. Thus

$$\left(\Phi, \vec{A}\right) = Q\left[\frac{V^m}{V_m R^m}\right]_{ret} \qquad (3)$$

A noteworthy feature of this formulation is that the inner product $\left[V_m R^m\right]_{ret}$ in the denominator is not a frame-invariant scalar *number*, but rather a frame-invariant scalar *function* that varies with some appropriate time argument. We shall return to this point later.

3. The Quantum Approach

If we were to address the problem of retarded potentials within the context of quantum photons instead of classical waves, we would change many things. For example, instead of infinite wave extent, we would have a localized blip, and instead of phase velocity, we would have centroid velocity. In place of an energy density $\left(E^2 + B^2\right)/2$ defined by electric field $\vec{E}$ and magnetic field $\vec{B}$, we would have a total energy hn defined by Planck's constant h and frequency n, and in place of momentum density $\vec{E} \times \vec{B}/c$ defined by fields, we would have total momentum hn/c defined by frequency. Indeed, we would not have fields at all; we would have only photons, real and virtual, to mediate forces.

It is clear that the quantum photons are very different from classical electromagnetic fields. Polarization is a good example: standard quantum electrodynamics admits not two but four polarization states (see Feynman, 1988). It is also clear that quantum photons are very similar to quantum particles. Interference is a good example: both kinds of entities experience interference.

Photons would become even more particle-like if we were to consider possibly non-zero photon rest mass. This concept traces back to deBroglie and extends forward to cutting-edge experimental physics (see Pecker 1991, Evans and Vigier 1994 and 1995). It involves a two-part system for light propagation, with a propagating pilot wave and a wave-guided photon particle. The wave can propagate at c, but the particle could carry non-zero mass and, if so, could not travel at c.

This concept would add something really decisive to direct the analysis. Note that if photons have mass, then the population of photons shed from a source has a *center* of mass. Center of mass was always a useful concept in classical mechanics, and it could well be an equally useful concept in a quantum-based review of retarded electromagnetic potentials.

In the observer's coordinate frame, this center of mass moves as time elapses. But in the source's coordinate frame, it is permanently positioned right at the source. This suggests that the analysis should definitely begin in the source coordinate frame.

4. In the Source Coordinate Frame

In the source coordinate frame, let time and space coordinates be t_0 and x_0, y_0, z_0. Let the observer move at some uniform velocity $\vec{v}$ past the source. Let $t_0 = 0$ be the time of closest approach between source and observer. Let a photon leave the source at $t_0 = t_{emit0}$ and arrive at the observer at $t_0 = t_{abs0}$. The distance the photon travels is

$$R_0 = c(t_{abs0} - t_{emit0}) \qquad (4a)$$

In this same interval, the observer travels distance

$$v(t_{abs0} - t_{emit0}) = bR_0 \qquad (4b)$$

Without meaningful loss of generality, we can make the direction of $\vec{v}$ be along the x_0 axis, and we can make the observer path lie in the x_0, y_0 plane with y_0 non-negative. The situation in the source coordinate frame is then essentially like that illustrated by Fig. 1a.

In the case illustrated, b is 1/2. The observer is located at some positive x_0 when the photon is emitted, and progresses to a larger x_0 by the time the photon is absorbed. Negative initial x_0 is also possible, and so is negative final x_0. The latter condition would correspond to negative t_{abs0}.

As t_{abs0} evolves, R_0 evolves with it, first decreasing as t_{abs0} passes trough negative values to zero, and then increasing as t_{abs0} progresses through positive values. It is possible to track the entire history of R_0 as a function of t_{abs0} with v as a parameter. The R_0 is a symmetric function of both t_{abs0} and v, and it

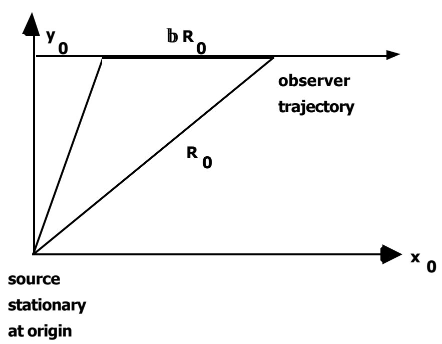

Figure 1a. Situation in source coordinate frame

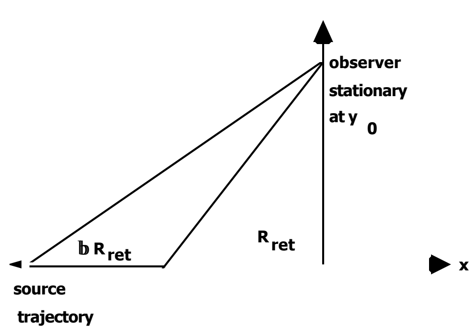

Figure 1b. Situation in observer coordinate frame.

has asymptotes with slopes $\pm v$. A plot of $R_0(t_{abs0})$ and its asymptotes looks like Fig. 2a.

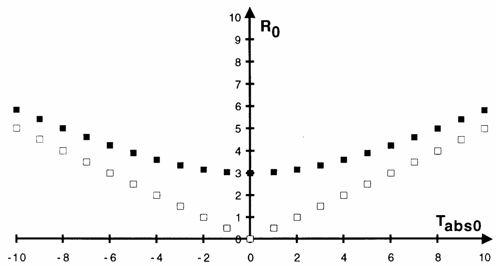

Figure 2a. Evolution of R_0 with t_{abs0} in the source coordinate frame.

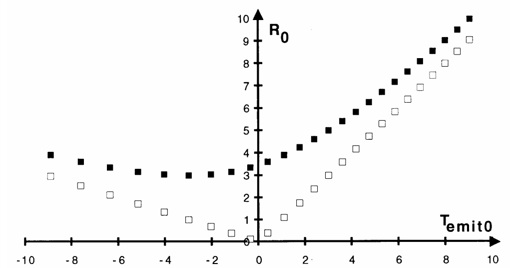

Figure 2b. Evolution of R_0 with t_{emit0} in the source coordinate frame.

In this illustration, and the several similar ones to follow, the units of time are seconds, and the units of distance are light seconds. The separation at closest approach is three such distance units. Smaller separation at closest approach would produce a curve closer to its asymptotes, and zero separation would produce a "curve" coincident with the asymptotes. Higher relative velocity would produce steeper asymptotes.

It is also possible to track R_0 as a function of t_{emit0}, again with v as a parameter. The R_0 is then not a symmetric function of either t_{emit0} or v. Its asymptotes have slopes $-v(1+b)$ and $+v(1-b)$. Figure 2b illustrates $R_0(t_{emit0})$ and its asymptotes.

Observe that every numerical value that occurs in $R_0(t_{abs0})$ (Fig. 2a) also occurs in $R_0(t_{emit0})$ (Fig. 2b), only for a different value of time argument. If we were to suppress the distinguishing subscripts "*abs0*" and "*emit0*" on the time arguments, then the function depicted on Fig. 2b would appear to anticipate the function depicted on Fig. 2a. The magnitude of the time shift would be $t_{abs0} - t_{emit0}$, the time required for light propagation over the distance R_0. This observation foreshadows a similar but more troubling one later on.

5. In the Observer Coordinate Frame

Clearly, R_0 provides the denominator for the scalar potential Φ_0 in the source coordinate frame, and it is R_0 that the denominator $[gkR]_{ret}$ in (2) or $[V_m R^m]_{ret}$ in (3) promises to recover while using only variables evaluated in the observer coordinate frame.

Let us consider plots of R_0 analogous to Figs. 2a and 2b, but constructed in the observer coordinate frame. For this exercise, we require new time variables t_{abs} and t_{emit} defined in the observer coordinate frame. These have to be determined by Lorentz transformations from the source coordinate frame.

First we have

$$t_{abs} = g\left(t_{abs0} - v x_{abs0} / c^2\right) \tag{5a}$$

with $x_{abs0} = v t_{abs0}$, so that

$$\begin{aligned} t_{abs} &= g\left(t_{abs0} - v^2 t_{abs0} / c^2\right) \\ &\equiv t_{abs0} / g \end{aligned} \tag{5b}$$

So when transformed to the observer coordinate frame, Figure 2a just shrinks in width by a factor of g. The essential symmetry of the plots does not change. The only change is that the slopes of the asymptotes change from $\pm v$ to $\pm gv$. Figure 3a shows the new plots.

Similarly we have

$$t_{emit} = g\left(t_{emit0} - v x_{emit0} / c^2\right) \tag{6a}$$

with $x_{emit0} = 0$, so that

$$t_{emit} = g t_{emit0} \tag{6b}$$

So when transformed to the observer coordinate frame, Figure 2b just stretches in width by a factor of g. It remains asymmetrical. The slopes of its asymptotes change from $-v/(1+b)$ and $+v/(1-b)$ to

$$\frac{-v}{g(1+b)} = -v\sqrt{\frac{1-b}{1+b}} \tag{7a}$$

and

$$\frac{+v}{g(1-b)} = +v\sqrt{\frac{1+b}{1-b}} \tag{7b}$$

Figure 3b shows the new plots.

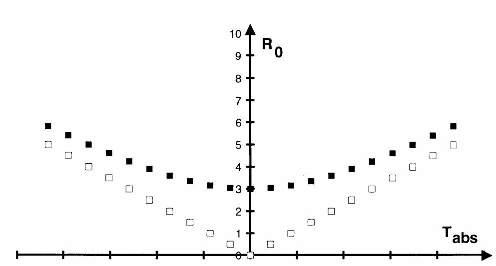

Figure 3a. Evolution of R_0 with t_{abs} in the observer coordinate frame.

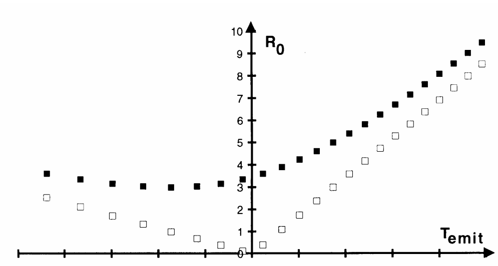

Figure 3b. Evolution of R_0 with t_{emit} in the observer coordinate frame.

6. The Classical Expression

Now consider the classical $[gkR]_{ret}$ denominator for the potentials in (2). To construct this, we must begin with a situation display similar to

Fig. 1a, but in the observer coordinate frame. Figure 1b provides this. The source is located at some negative x when the photon is emitted, and progresses to a more negative x by the time the photon is absorbed.

As t_{emit} evolves, R_{ret} evolves with it, first decreasing as t_{emit} passes through negative values to zero, and then increasing as t_{emit} progresses through positive values. Except for the names of the variables, a plot of $R_{ret}(t_{emit})$ is identical to $R_0(t_{abs})$, Fig. 2a. (Just to complete this description, note that the corresponding plot of $R_{ret}(t_{abs})$ is a mirror image of $R_0(t_{emit})$, Fig. 2b.)

Since $R_{ret}(t_{emit})$ is symmetric in t_{emit} and v, it follows that $[gkR]_{ret}$ is *not* symmetric in t_{emit} *or* v. The asymmetry of $[gkR]_{ret}(t_{emit})$ is similar to that of $R_0(t_{emit})$. Furthermore, $[gkR]_{ret}(t_{emit})$ has asymptotes

$$-vg\,(1-b) = -v\sqrt{\frac{1-b}{1+b}} \tag{8a}$$

and

$$+vg\,(1+b) = +v\sqrt{\frac{1+b}{1-b}} \tag{8b}$$

which are identical to those of $R_0(t_{emit})$, given by (7a) and (7b). Indeed, upon careful examination there is a perfect match between the entire function $[gkR]_{ret}(t_{emit})$ and the function $R_0(t_{emit})$, Fig. 3b.

In practice $[gkR]_{ret}$ is evaluated by an observer whose clock reads only t_{abs}, *not* t_{emit}. The corresponding plot of $[gkR]_{ret}(t_{abs})$ matches $R_0(t_{abs})$, Fig. 3a. Viewed in this way, $[gkR]_{ret}$ has symmetry in v and t_{abs}.

7. The Required Symmetry

Suppose that Coulomb-Ampere forces are *really* mediated by virtual photons, and that the potentials are but a computation device to predict the effects of the photons. That is, suppose the physical reality consists of radial photon paths rather than spherical wavefronts of potential. Then there then has to be some one-to-one relationship between the lateral dispersion of photons and the denominator of potentials.

Suppose further that the photons could have non-zero rest mass. Clearly the center of mass of all the photons shed at t_{emit} has to lie at the retarded source position. So every photon emitted toward the observer has to be matched by another photon emitted in the opposite direction, toward an imaginary observer whose situation is opposite to that of the real observer; *i.e.* whose t_{emit} has the opposite sign. This means that for given R_{ret}, the lateral dispersion of photons has to be symmetric in t_{emit}.

So the denominator for potentials also needs symmetry in t_{emit}. So the symmetry requirement for the denominator of potentials reduces to symmetry in v and t_{emit}.

Unfortunately, symmetry in v and t_{emit} is a combination simply not provided by the classical $[gkR]_{ret}$, no matter how we look at it. So in short, $[gkR]_{ret}$ does *not* properly recover the needed R_0 as promised.

In order to produce the R_0 with the needed symmetry in v and t_{emit}, rather than symmetry in v and t_{abs}, we have to evaluate $[gkR]_{ret}$ with t_{emit} replacing t_{abs}. This means that the classical function $[gkR]_{ret}$ *anticipates* the sought R_0 function, and the time shift is just $t_{abs} - t_{emit}$, the time required to propagate light over the distance R_{ret}.

Thus the potentials seem to require denominator $[gkR]_{retret}$. Except for the names of the variables, a plot of $[gkR]_{retret}$ versus t_{emit} is identical to $R_0(t_{abs})$, Fig. 3a, and so certainly has the required symmetry in v and t_{emit}.

But $[gkR]_{retret}$ is certainly a *very* peculiar looking expression. Fortunately, it can be replaced with something much more acceptable-looking by applying simple trigonometry. Figure 4 shows R_0 and $[gkR]_{ret}$ as hypotenuses of right triangles. The perpendicular sides are as needed to satisfy the definition of R_0 as $[gkR]_{retret}$.

Because coordinates perpendicular to motion are unaffected by coordinate-frame change, we have

$$R_{0\perp} = R_{ret\perp} \tag{9a}$$

Because $t_{emit} = gt_{emit0}$, we have the second retardation, bR_{ret} in the observer frame, given in the source frame as

$$(bR_{ret})_0 = gbR_{ret} \tag{9b}$$

It then follows that

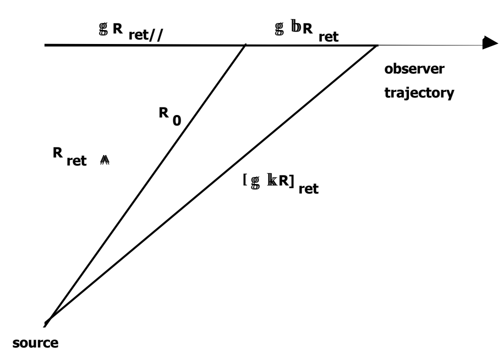

Figure 4. Determination of R_0 from R_{ret} and v.

$$R_{0//} = gR_{ret//} \qquad (9c)$$

and

$$R_0 = \sqrt{R_{ret\perp}^2 + g^2 R_{ret//}^2} \qquad (9d)$$

The validity of this solution can be confirmed by applying the Pythagorean theorem to the larger right triangle that has $[gkR]_{ret}$ as its hypotenuse.

It is *this* R_0 that belongs in the potentials. In place of (2) or (3), we need

$$\begin{aligned}(\Phi, \vec{A}) &= Q\left[g\left(1, \vec{b}\right)\right]_{ret} / R_0 \\ &= QV_{ret}^m / R_0\end{aligned} \qquad (10)$$

with R_0 evaluated by (9d). We must *abandon* the usual assumption that the needed R_0 is the classical $[gkR]_{ret}$ as in (2), or its relativistic equivalent $\left[V_m R^m\right]_{ret}$ as in (3).

8. Discussion

How could all this be so? How *could* there be an error embedded in the accepted body of knowledge that constitutes SRT? There appear to be several contributing factors making this possible, and they are discussed below.

First of all, the standard Liénard-Wiechert results are *very* nearly correct. For uniform source motion, the error is only a time shift, and in many practical scenarios would be undetectable.

Secondly, the Liénard-Wiechert results are in fact entirely pre-relativistic. Einstein *used* them (1922), but he did *not* in fact re-derive them within the context of his own theory. We cannot now guess if he would ever have done that. Indeed, the anonymous referee for this paper remarked that Einstein's thinking was more along the lines of Section 7.

For the benefit of the present author and the audience that first saw some of the plots presented here, J.P. Vigier recounted his own knowledge of the history of privately expressed doubts about the Liénard-Wiechert results. The history traces through Louis deBroglie and indeed back to Einstein. But nobody articulated the doubts in print.

The fact that Einstein used the Liénard-Wiechert results conferred on them unwarranted authority. With the end results assumed not subject to question, modern authors have generally just retro-fitted modern mathematical methods onto them, without seizing the opportunity to delve into questions that the modern methods might have exposed.

For example, the modern concept of invariant scalar inner product underlies the formulation (3). But the fact that $[gkR]_{ret}$ is equivalent to the inner product $\left[V_m R^m\right]_{ret}$, means only that it is *an* invariant; it does *not* mean that it is *the* invariant that corresponds to the correct time argument; *i.e.*, the correct proper time of the correct entity in the problem. The slipperiness of the construct $\left[V_m R^m\right]_{ret}$ has been demonstrated. For example, Whitney (1989) shows that the operations of retardation and Lorentz transformation can lead to ambiguity by failing to commute.

Another of the modern approaches uses generalized functions: the Dirac delta and the Heaviside step. [See, for example, Jackson (1975), Sections 12.11 and 14.1.] The problem with the generalized functions is that they lack the mathematical property of uniform convergence, and as a result they can produce apparently pathological behaviors. Worst among these is failure in operator commutation: as the generalized functions are used in field theory, the operations of differentiation, integration, and generalized-function limit do not commute (Whitney 1988). This means the evaluation of fields in terms of generalized functions is not unique.

The Liénard-Wiechert formulae for fields would have revealed a problem had they been exhaustively tested. Whitney (1991) offers a scenario specially contrived to reveal problems. A charge traverses a circular orbit centered in the *x*-*y* plane, and an observer sits somewhere on the *z* axis. The field formulae are easy to evaluate because $k_{ret} = 1$ and $R_{ret} = R$, a constant. In this scenario, the velocity-determined part of the E field has a g^2 in the denominator, and so tends to zero as b tends to unity. The acceleration-dependent part of the *E* field contains both radiative and non-radiative parts. Only the radiative part has any component along the *z* axis. The only way to produce *any E* field that is Coulomb-like in its direction is to conscript this radiative *E*, pretend that the $\vec{E} \times \vec{B}$ radiation does not have a prior claim on it, and time-average it. Such conscription is, of course, *not* legitimate. But without it, the total Coulomb field comes out $-Q / R^2$ times $\vec{b}_{ret}$; *i.e.* a vector lying entirely in the *x*-*y* plane and time-averaging to zero. This means the rapidly circulating charge looses its Coulomb attraction for this observer. Such a prediction is not believable.

The standard Liénard-Wiechert potentials have stood unchallenged for a very long time. But they do lead to unacceptable problems, and a new formulation definitely seems needed. The present paper offers a new formulation that is consistent with the concept of photons that are similar to quantum particles with non-zero rest mass. It should be noted, however, that photon rest mass is not specified here, and is not even actually necessary; the new theory holds in the limit as photon mass approaches zero.

The proposed new formulation does have at least one feature somewhat reminiscent of the classical formulation; namely, a characteristic "shrinkage" phenomenon. Standard theory exhibits such a phenomenon in regard to the pattern of E field lines. In the source frame, the E field lines are straight and uniformly spaced in angle. Standard theory says that in the observer frame, the whole straight-line pattern just shrinks by γ in the direction of source motion. [See Jackson (1975), Fig. 11.9] Somewhat similarly, the present theory says that, in the observer coordinate frame, the photon propagation path just shrinks by g in the direction of source motion.

Acknowledgment

The author is grateful to Prof. J. P. Vigier for having suggested that the concept of non-zero photon rest mass might shed light on the classical problem of retarded potentials. A brief and preliminary version of the present paper was presented under the title "A Quantum of Light Shed on a Classical Problem" at the Symposium entitled "The Present Status of the Quantum Theory of Light", which was given at York University in Toronto, Canada, in August of 1995 to honor Prof. Vigier's career. This version of the paper will appear in its proper context with the other Conference papers in the Proceedings published by Kluwer Academic Publishers.

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General Information

The meeting will take place at the York University campus, which is located in the north-west of metropolitan Toronto.

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