A Formulation of the Gravitational Equation of Motion

T. Chang
Department of Physics
University of Alabama in Huntsville
Huntsville, AL 35899, USA

According to Einstein's principle of equivalence, inertial forces in an accelerated reference system are equivalent to the existence of a gravitational field. In order to formulate the gravitational force as well as inertial forces in explicit form, we introduce two conditions into the 4-D line element and transformations. As a consequence, the equation of motion for gravitational force or inertial force has a form similar to the equation of Lorentz force on a charge in electrodynamics. The inertial forces in non-inertial systems are calculated for two special cases: a uniformly accelerated system, and a uniformly rotating system.

PACS: 04.20., 04.50

1. Introduction

In general relativity, space-time coordinates can be chosen arbitrarily (Einstein 1916, 1955). However, in order to get unique solutions of the Einstein field equation, certain coordinate conditions must be imposed. Two recent examples (Duan 1992, Torre 1992) have discussed the choice of coordinate conditions. In this paper, we emphasize that constraints on coordinate conditions are imposed by the principle of equivalence.

Einstein (1916) illustrated the principle of equivalence using the following simple example. Let Σ be an inertial reference system; let another system K be uniformly accelerated with respect to Σ. Then relative to K, all free bodies have equal and parallel accelerations. They behave just as if a gravitational field were present and K were unaccelerated. In order to formulate the above simple equivalent gravitational field suggested by Einstein, let us start with an inertial system Σ. Let $X^m = (cT, X, Y, Z) = (cT, \mathbf{R})$ be the pseudo-Cartesian coordinate in the system Σ. Then the 4-D line element takes a form

$$ds^2 = (c\,dT)^2 - dX^2 - dY^2 - dZ^2 \qquad (1)$$

We now consider a system K with coordinates $x^m = (ct, x, y, z) = (x^\circ, \mathbf{r})$, which has a constant acceleration, $\mathbf{a}$, with respect to Σ. Since the Lorentz transformations are not valid between Σ and K, the following quasi-Galilean transformations are usually introduced (Møller, 1972)

$$\mathbf{R} = \mathbf{r} + \frac{1}{2}\mathbf{a}t^2; \qquad T = t \qquad (2)$$

A simple calculation for Eqs.(1) and (2) gives

$$ds^2 = \left(1 - \frac{[at]^2}{c^2}\right)(dx^\circ)^2 - 2\frac{\mathbf{a}t}{c}\cdot d\mathbf{r}\,dx^\circ - d\mathbf{r}^2 \qquad (3)$$

We substitute these components of g_{mn} into the geodesic equation

$$\frac{d^2x^m}{dt^2} + \Gamma^m_{rl}\frac{dx^r}{dt}\frac{dx^l}{dt} = 0 \qquad (4)$$

In the case of a low velocity approximation, Eq.(4) reduces to $d^2\mathbf{r}/dt^2 = -\mathbf{a}$. This indicates that a free particle with a rest mass m_o in a uniformly accelerated system should experience a uniformly inertial force $(-m_o\mathbf{a})$. Although this result is well known, the above calculation shows that certain constraints on space-time coordinates and transformations should be imposed by the principle of equivalence.

2. Two Proposed Conditions Based on the Principle of Equivalence

We propose the following two conditions in our formulation:

(i) In any reference system, the 4-D line element takes a standard form

$$ds^2 = \Psi_o(dx^\circ)^2 - 2(\overline{\Psi}\cdot d\mathbf{r})dx^\circ - d\mathbf{r}^2 \qquad (5)$$

Comparing Eq.(5) with $ds^2 = g_{mn}dx^m dx^n$, we have $g_{oo} = \Psi_o$, $g_{oi} = -\Psi_i$, $g_{ij} = -\delta_{ij}$.

(ii) Between two reference systems, K and K', the value of ds^2 is an invariant, but space-time coordinates take infinitesimal Galilean transformations

$$d\mathbf{r}' = d\mathbf{r} - \mathbf{v}dt; \qquad t' = t \tag{6}$$

where $\mathbf{v} = \mathbf{v}(\mathbf{r},t)$ is the 3-D relative velocity between two systems K and K'.

The physical implication of the above proposed conditions is as follows: Weinberg (1972) has emphasized that a local gravitational field has two kinds of sources. One is nearby mass; another is all the mass in the universe. We can anticipate that the inertial systems are determined by the mean background gravitational field produced by all the matter of the universe. This background gravitational field provides a base for us to propose the above two specific conditions.

In Eq.(5), Ψ_o and $\overline{\Psi}$ are components of metric tensor g_{mn}. They have properties of gravitational potentials, but they do not form a 4-D vector. From Eq.(6), it is easy to derive the transformations of these potentials

$$\Psi_o' = \Psi_o - \frac{v^2}{c^2} - 2\overline{\Psi}\cdot\frac{\mathbf{v}}{c}; \quad \overline{\Psi}' = \overline{\Psi} + \frac{\mathbf{v}}{c} \tag{7}$$

These transformations form a group with parameter $\mathbf{v}$.

3. Gravitational Equation of Motion

In this section, we will rewrite the general equation of motion, Eq.(4), into an explicit form in terms of the proposed two conditions. We consider the variational principle (Landau 1975)

$$d\int(-m_o c)ds = d\int L(x^i,u^i,t)dt \tag{8}$$

where $L(x^i,u^i,t)$ is the Lagrangian. Using Eq. (5), we obtain

$$L(x^i,u^i,t) = (-m_o c^2)\sqrt{\Psi_o - 2\overline{\Psi}\cdot\frac{\mathbf{u}}{c} - \frac{u^2}{c^2}} \tag{9}$$

where $\mathbf{u} = d\mathbf{r}/dt$ is the velocity of a particle in system K.

Let $\Psi_o = (1+2f)$. Substituting the Lagrangian (9) into the Lagrange equation (Landau 1975), $d/dt(\partial L/\partial \mathbf{u}) = \partial L/\partial \mathbf{r}$, we obtain the equation of motion:

$$\frac{d(m\mathbf{u})}{dt} + c\overline{\Psi}\frac{dm}{dt} = m\left(\mathbf{g} + \frac{\mathbf{u}}{c}\times\mathbf{h}\right) \tag{10}$$

where $\mathbf{g}$ and $\mathbf{h}$ are defined as

$$\mathbf{g} = c^2\left(-\nabla f - \frac{1}{c}\frac{\partial\overline{\Psi}}{\partial t}\right); \qquad \mathbf{h} = c^2(\nabla\times\overline{\Psi}) \tag{11}$$

Here, $\mathbf{g}$ is gravitational intensity and $\mathbf{h}$ is a magnetic-type gravitational intensity. Both have units of $\mathrm{cm\,sec^{-2}}$. Therefore, the gravitational force is analogous to the Lorentz force on a charged particle in an electromagnetic field except for two different terms: (i) a moving mass $m = m_o\Gamma$, where $\Gamma = \left[\Psi_o - 2\overline{\Psi}\cdot(\mathbf{u}/c) - u^2/c^2\right]^{-1/2}$; (ii) a small additional term on the left side of Equation (10), which is related to the changing rate of a moving mass. Eq.(10) is our formulation of the gravitational equation of motion, which is also valid to describe a magnetic-type gravitational force as well as the inertial force in non-inertial systems.

4. Inertial Force in Accelerated Systems

(a) A Uniformly Linear Accelerated System

In the introduction of this paper, we discussed the inertial force in a uniformly linear accelerated system. From Eqs.(3) and (5), we have $f = -\frac{1}{2}(a^2t^2/c^2)$, $\overline{\Psi} = \mathbf{a}t/c$. By using Eq.(11), we obtain

$$\mathbf{g} = -c\frac{\partial\overline{\Psi}}{\partial t} = -\mathbf{a}; \qquad \mathbf{h} = 0 \tag{12}$$

Notice that this equivalent gravitational intensity $\mathbf{g}$ in Eq.(12) is produced by the 3-D gravitational vector potential.

(b) A Uniformly Rotating System

Suppose a uniformly rotating system has a constant angular velocity $\overline{\Omega} = \Omega\mathbf{k}$ with respect to an inertial system Σ. Let the infinitesimal Galilean transformation be

$$d\mathbf{r} = d\mathbf{R} - (\overline{\Omega}\times\mathbf{R})dT; \qquad t = T \tag{13}$$

Substituting Eq.(13) into (1), we obtain the 4-D line element in this rotating system K as follows:

$$ds^2 = \left(1 - \frac{[\Omega\times\mathbf{r}]^2}{c^2}\right)(dx^\circ)^2 - \frac{2}{c}(\Omega\times\mathbf{r})\cdot d\mathbf{r}dx^\circ - d\mathbf{r}^2 \tag{14}$$

From Eq.(14), we have $f = -1/2c^2(\Omega\times\mathbf{r})^2$, $\overline{\Psi} = (\Omega\times r)/c$. Using Eq.(11), we obtain

$$\mathbf{g} = \frac{1}{2}\nabla(\Omega\times\mathbf{r})^2 = \Omega^2\mathbf{r}_\parallel; \quad \mathbf{h} = \nabla\times(\Omega\times\mathbf{r}) = 2c\Omega \tag{15a,b}$$

where $\mathbf{r}_\parallel = x\mathbf{i} + y\mathbf{j}$. Eq.(15a) gives the centrifugal force, $\mathbf{F}_a = m\mathbf{g} = m\Omega^2\mathbf{r}_\parallel$, in the rotating system, while Eq.(15b) also gives the Coriolis force, $\mathbf{F}_b = m\mathbf{u}\times\mathbf{h}/c = 2m\mathbf{u}\times\overline{\Omega}$. Therefore, the Coriolis force is produced by a magnetic-type gravitational intensity.

5. Summary

Generally speaking, the gravitational equation of motion Eq.(10) has a form similar to the equation of the Lorentz force on a charge in electrodynamics. On the other hand, the gravitational potentials Ψ_o and $\overline{\Psi}$ do not

form a 4-D vector. This property is different from the electromagnetic potentials in electromagnetic fields. In our formulation, the gravitational equation of motion has a 3-D vector form. Therefore, in a given reference system, one still has freedom to choose 3-D curvilinear spatial coordinates. For further investigation, the gravitational field equations and other related problems in terms of the two proposed conditions will be studied in another paper.

Acknowledgments

The author thanks Drs. D.G. Torr and J.P. Hsu for their valuable discussions.

References

Duan, Y., *et al.*, 1992. *Gen. Rel. and Gravi*. 24:1033.

Einstein, A., 1916. *Annal. Phys*. 49:769.

Einstein, A., 1955. *The Meaning of Relativity*, Princeton Univ. Press, New Jersey, pp. 55-64.

Landau, L. D. and Lifshitz, E. M., 1975. *The Classical Theory of Fields*, Fourth English Edition, Pergamon Press, Chap. 3-4, 10-11.

Møller, C., 1972. *The Theory of Relativity*, Second edition, Clarendon Press, Oxford, Chap. 8-10.

Torre, C.G., 1992. *Phys. Rev. D* 46:R3231.

Weinberg, S., 1972. *Gravitation and Cosmology*, John Wiley and Sons, Inc., New York, Chap. 3.