General Mechanics of a Photon in the Gravitational Field of a Stationary Homogeneous Spherical Body

S.X.K. Howusu
Department of Physics
University of Jos
PMB 2084 Jos, Plateau State
Nigeria

In an earlier paper (Howusu 1991) the new Theory of General Mechanics was constructed. In this paper, it is shown how a photon is accelerated according to the Theory of General Mechanics.

1. Introduction

In an earlier paper (Howusu 1991), I presented a new Theory of General Mechanics. The Law of General Mechanics states that:

> *In all inertial reference frames, the instantaneous general time rate of change of the instantaneous general linear momentum $\vec{P}_g$ of a particle is equal to the instantaneous total general external force $\vec{F}_g^e$ acting on it.*

$$\frac{\mathrm{d}}{\mathrm{d}t_g}\vec{P}_g = \vec{F}_g^e$$

In the Law of General Mechanics, the general time of a clock is its time after it has been affected by its state of motion and surrounding force fields, and for a particle of non-zero rest mass m_o moving with instantaneous linear velocity $\vec{u}$

$$\vec{P}_g = m_o\left(1-\frac{u^2}{c^2}\right)^{-1/2}\vec{u} \tag{1}$$

while for a photon of instantaneous frequency ν and linear velocity $\vec{u}$

$$\vec{P}_g = \frac{\hbar\nu}{c^2}\vec{u} \tag{2}$$

where $\hbar$ is Dirac's constant. And it may be noted that in the Law of General Mechanics the instantaneous total energy external force may include any combination of force fields in nature simultaneously.

It was also shown (Howusu 1991) that the Law of General Mechanics is in perfect agreement with experiment in the null and electromagnetic fields. Furthermore, the Law of General Mechanics was shown to resolve the problem of the anomalous orbital precession of the planets in the solar system with excellent agreement with experiment (and the Theory of General Relativity), as well as the problem of the gravitational frequency shift of the photon, in complete agreement with experiment (and General Relativity).

Finally, it was shown that for a photon moving in a gravitational field having a scalar potential Φ_g, the Law of General Mechanics becomes (interchanging sides for convenience)

$$-\nabla\Phi_g(\vec{r},t) = \frac{\mathrm{d}}{\mathrm{d}t}\vec{u}(\vec{r},t) - \frac{1}{c^2}\left\{\left[1+\frac{1}{c^2}\Phi_g(\vec{r},t)\right]^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\Phi_g(\vec{r},t)\right\}\vec{u}(\vec{r},t) \tag{3}$$

This is the general mechanical equation of motion of the photon in a gravitational field.

It may be noted that, to the order of c^o, the general mechanical equation for the motion of a photon in a gravitational field (3) reduces to

$$-\nabla\Phi_g(\vec{r},t) = \frac{\mathrm{d}}{\mathrm{d}t}\vec{u}(\vec{r},t)$$

which is the same as the classical Newtonian equation for the motion of a particle of non-zero rest mass in a gravitational field. This equation will describe the motion of the photon correctly to the order of c^o, *i.e.* as a classical particle. It is well known that this description of the motion of a photon in a gravitational field is not sufficiently accurate. Therefore, in this paper we shall investigate the full general mechanical equation (3) for the motion of the photon in the gravitational field exterior to a stationary homogeneous body.

2. Full General Mechanical Equations

Consider a photon moving in the gravitational field of a stationary homogeneous sphere of radius R and rest mass M^o. Let S be the reference frame with origin O at the centre of the sphere. Thus, in spherical coordinates of S, the gravitational potential Φ_g^+ in the region exterior to the sphere is given by

$$\Phi_g^+(\bar{r}, t) = -\frac{a}{r} \tag{4}$$

where

$$a = GM_o \tag{5}$$

and G is the universal gravitational constant. Hence, the components of the full general mechanical equation (3) for the motion of the photon are given by

$$-\frac{a}{r^2} = \ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2 \sin\theta - \frac{a}{c^2r^2}\left(1 - \frac{a}{c^2r}\right)^{-1}\dot{r}^2 \tag{6}$$

and

$$0 = r\ddot{\theta} + 2\dot{r} - r\dot{\phi}^2 \sin\phi\,\cos\theta - \frac{a}{c^2r}\left(1 - \frac{a}{c^2r}\right)^{-1}\dot{r}\dot{\theta} \tag{7}$$

and

$$0 = r\ddot{\phi} + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta - \frac{a\sin\theta}{c^2r}\left(1 - \frac{a}{c^2r}\right)^{-1}\dot{r}\dot{\phi} \tag{8}$$

To solve these equations, let us consider, without loss of generality, a photon moving in the plane $\theta = \pi/2$ for the sake of mathematical convenience. Then the general mechanical equations (6), (7) and (8) reduce to

$$-\frac{a}{r^2} = \ddot{r}\dot{\phi}^2 - \frac{a}{c^2r^2}\left(1 - \frac{a}{c^2r}\right)^{-1}\dot{r}^2 \tag{9}$$

and

$$0 = r\dot{\phi} + 2\dot{r}\dot{\phi} - \frac{a}{c^2r}\left(1 - \frac{a}{c^2r}\right)^{-1}\dot{r}\dot{\phi} \tag{10}$$

If the photon moves purely radially in the plane $\theta = \pi/2$, then

$$\dot{\phi} \equiv 0$$

and, therefore, the equations (9) and (10) reduce to

$$-\frac{a}{r^2} = \ddot{r} - \frac{a}{c^2r^2}\left(1 - \frac{a}{c^2r}\right)^{-1}\dot{r}^2 \tag{11}$$

Let w be the new variable defined by

$$w(r) = \dot{r}(r) \tag{12}$$

Then equation (11) transforms as

$$-\frac{a}{r^2}w^{-1} = \frac{\mathrm{d}}{\mathrm{d}r}w - \frac{a}{c^2r}\left(1 - \frac{a}{c^2r}\right)^{-1}w$$

which is Bernoulli's differential equation. Therefore, by the transformation

$$z(r) = w^2(r) \tag{13}$$

it reduces to the homogeneous first-order differential equation

$$Q(r) = \frac{\mathrm{d}}{\mathrm{d}r}z + P(r)z \tag{14}$$

where

$$P(r) = -\frac{2a}{c^2r^2}\left(1 - \frac{a}{c^2r}\right)^{-1} \tag{15}$$

and

$$Q(r) = -\frac{2a}{r^2} \tag{16}$$

The equation 14 is a homogeneous first-order differential equation whose general solution is given by

$$z(r) = 2c^2\left(1 - \frac{a}{c^2r}\right) + A\left(1 - \frac{a}{c^2r}\right)^2 \tag{17}$$

where A is an arbitrary constant. Hence, from (12) and (13) we obtain

$$\dot{r}^2 = 2c^2\left(1 - \frac{a}{c^2r}\right) + A\left(1 - \frac{a}{c^2r}\right)^2 \tag{18}$$

Thus, for a photon emitted at an infinite distance away from the body, we have the condition

$$\dot{r} = c \quad \text{at} \quad r = \infty \tag{19}$$

from which it follows that

$$A = -c^2$$

and hence (18) becomes

$$\dot{r}^2 = \left(1 - \frac{a}{c^2r}\right)\left(1 + \frac{a}{c^2r}\right)c^2 \tag{20}$$

This is the general mechanical expression for the instantaneous radial speed of a photon which is emitted at infinity and which moves purely radially in the gravitational field of the body. This expression may also be written equivalently as

$$\dot{r}^2 = \left(1 - \frac{a^2}{c^4r^2}\right)c^2 \tag{21}$$

Consequently, according to the Theory of General Mechanics, the linear speed of a photon is reduced below the null gravitational field value c as it moves radially from infinity towards the body. In particular, at the surface of

Table 1 — The speed of a photon on the surfaces of some astronomical gravitating bodies, taking $c = 2.9979 \times 10^8$ $\mathrm{ms^{-1}}$ and $G = 6.672 \times 10^{-11}$ $\mathrm{Nkg^{-2}m^2}$. Astronomical data from Moore (1980).

Body	R (m)	M_o (kg)	$\frac{a^2}{c^4R^2}$	$\frac{\dot r^2(R)}{c^2}$
Earth	6.378×10^6	5.976×10^{24}	4.83838×10^{-19}	0.999 999 999 999 999 999 516 162
Sun	6.960×10^8	1989×10^{30}	4.50084×10^{-12}	0.999 999 999 954 992

the body, $r = R$, the linear speed, $\dot r(R)$, of the photon is given by

$$\dot r^2(R) = \left(1 - \frac{a^2}{c^4R^2}\right)c^2 \tag{22}$$

Conversely, from the general solution (18) and the fact that its speed at infinity (in a null gravitational field) should be c, it follows that a photon is emitted in a gravitational field with a linear speed given by (21), which is below c. In particular, at the surface of the body, a photon is emitted with a linear speed given by (22). The linear speeds of a photon at the surfaces of some astronomical bodies (assumed to be stationary and homogeneous) are shown in Table 1.

It may be noted from (21) that the variation of the linear speed of a photon in a gravitational field is a general mechanical effect of order at most c^{-4}. Moreover, from equation (22), the linear speed with which a photon is emitted at the surface of the body vanishes if

$$\frac{a}{c^2r} = 1 \tag{23}$$

Physically, this implies that, under the condition (23) of a homogeneous spherical mass distribution, photons cannot be emitted from the surface of the body. Consequently, the body constitutes an astronomical object that may be called a “black electromagnetic hole”.

Furthermore, from (22) it can be seen that the linear speed with which a photon is emitted at the surface of the body becomes purely imaginary if

$$\frac{a}{c^2R} > 1 \tag{24}$$

We note from the astronomical data given in Table 1 that the sun is very far from satisfying the general mechanical condition (23) for a black electromagnetic hole. In fact, according to the condition, a black electromagnetic hole having the radius of the sun will have a rest mass greater than 10^5 solar masses! Conversely, according to the condition, a black electromagnetic hole having the rest mass of the sun will have a radius of order 10^5 m.

Now, in the second place, consider the general motion of the photon in the plane. In this case, its angular equation (10) integrates exactly as

$$\dot\varphi(r) = \frac{l}{r^2}\left(1 - \frac{a}{c^2r}\right) \tag{25}$$

where l is its general angular momentum per unit general mass. This is the general mechanical expression for the instantaneous angular momentum of the photon (in terms of the radial coordinate). And, substituting (25) into the radial equation (19), we obtain (after rearranging terms)

$$\ddot r - \frac{a}{c^2r^2}\left(1 - \frac{a}{c^2r}\right)^{-1}\dot r^2 = -\frac{a}{r^2} + \frac{l^2}{r^3}\left(1 - \frac{a}{c^2r}\right)^2 \tag{26}$$

By the method used for solving (11) above, we obtain the general solution of (26) as

$$\dot r^2(r) = \left(1 - \frac{a}{c^2r}\right)^2\left\{B - \frac{l^2}{r^2} + 2c^2\left(1 - \frac{a}{c^2r}\right)^{-1}\right\} \tag{27}$$

where B is an arbitrary constant. Hence, for a photon emitted tangentially to the surface of the body, we have the condition

$$\dot r = 0 \quad \text{at} \quad r = R \tag{28}$$

Thus, (27) becomes

$$\dot r^2(r) = \left(1 - \frac{a}{c^{2r}}\right)^2\left\{\frac{l^2}{R^2} - 2c^2\left(1 - \frac{a}{c^2R}\right)^{-1} - \frac{l^2}{r^2} + 2c^2\left(1 - \frac{a}{c^2r}\right)^{-1}\right\} \tag{30}$$

This is the full general mechanical expression for the radial speed of a photon emitted tangentially from the surface of the body (in terms of the radial coordinate). And substituting into (26), we obtain, after some rearrangement

$$\begin{aligned}\ddot r(r) = {} & \frac{a}{r^2}\left\{1 + \frac{l^2}{c^2R^2} - 2\left(1 - \frac{a}{c^2R}\right)^{-1}\right\} \\ & + \frac{l^2}{r^3}\left\{1 - \frac{a^2}{c^4R^2} + \frac{2a}{l^2c^2}\left(1 - \frac{a}{c^2R}\right)^{-1}\right\} \\ & - \frac{3al^2}{c^2r^4} + \frac{3a^2l^2}{c^4r^5}\end{aligned} \tag{31}$$

This is the full general mechanical expression for the radial acceleration of the photon emitted tangentially at the surface of the body (in terms of the radial coordinate).

In principle, the results (25), (30) and (31) constitute a complete solution of the problem of the photon emitted tangentially at the surface of the body. But it is instructive to express the motion in terms of the angular coordinate. Hence, let v be the new variable defined by

$$v(\phi) = \frac{q}{r(\phi)} \tag{32}$$

Then it follows, using the expression (25) for the angular speed of the photon, that the radial equation (31) transforms as

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}v(\phi) = &-\frac{a}{l^2}\left\{1+\frac{l^2}{c^2R^2}-2\left(1-\frac{a}{c^2R}\right)^{-1}\right\}\\ &-\left\{1-\frac{a^2}{c^4R^2}+\frac{2a}{l^2c^2}\left(1-\frac{a}{c^2R}\right)^{-1}\right\}v(\phi)\\ &+\frac{3a}{c^2}v^2(\phi)-\frac{3a^2}{c^4}v^3(\phi)\end{aligned} \tag{33}$$

This is the general mechanical equation for the motion of the photon emitted tangentially at the surface of the body (in terms of the angular coordinate). But since the linear speed of emission of the photon at the surface is given by (22), it follows that

$$l^2 = c^2R^2\left(1-\frac{a^2}{c^4R^2}\right) \tag{34}$$

Hence, substituting into (33) and assuming that

$$\frac{a}{c^2R} < 1 \tag{35}$$

as is the case for the sun and all bodies in the solar system, it reduces to

$$\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}v(\phi)+v(\phi) = \frac{3a}{c^2}v^2(\phi)-\frac{3a^2}{c^4}v^3(\phi) \tag{36}$$

This is the general mechanical equation for the motion of a photon emitted tangentially from the surface of the body in terms of the angular coordinate and correct to the order of c^{-4} under the assumption (35).

It may be noted that, to the order of c^{-2}, the general mechanical equation for the motion of a photon emitted tangentially from the surface of the body in terms of angular coordinate under the assumption (35), which is valid for the sun, reduces to the full general relativistic equation for the photon (Rindler 1972) in the solar system.

$$\frac{\mathrm{d}^2}{\mathrm{d}\phi^2}v(\phi)+v(\phi) = \frac{3a}{c^2}v^2(\phi) \tag{37}$$

Thus, the general mechanical equation for the motion of a photon resolves the problem of the gravitational deflection of a photon in the solar system just as well as the Theory of General Relativity.

3. Concluding Remarks

In this paper, we have shown that according to the Theory of General Mechanics, a photon is not only accelerated angularly by a gravitational field—as is well known experimentally (and in agreement with the Theory of General Relativity)—but also, to the order of c^{-2}, linearly. We have shown that a photon emitted with a linear speed c at a point without a gravitational field is decelerated as it moves in the direction of a decreasing gravitational field. And finally, we have demonstrated that a photon is emitted within a gravitational field with such a speed that it will accelerate in the direction of an increasing gravitational field towards a linear speed c at points without a gravitational field.

References

Howusu, S.X.K., 1991. On the gravitation of moving bodies, *Phys. Essays* 4(1), 81.

Moore, P.N., 1980. *The Guinness Book of Astronomy—Facts and Feats*, London, Guinness.

Rindler, W., 1972. *Essential Relativity*, New York, Springer Verlag.