Is Thomas Rotation a Paradox?

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In this paper the author proves that the identification of the Thomas rotation angle of the Cartesian axis with the angle between the relativistic composite non-collinear velocities leads to a conflict with the concept of inertial motion. As a consequence, the Thomas rotation is unable to extend the 1+1 Lorentz transformation to 1+3 dimensions and does not confirm the Thomas half.
Key words: Special relativity, Thomas rotation.

Introduction

In a previous paper (Mocanu 1991) it was shown that the attempt to extend the 1+1 Lorentz transformation to 1+3 dimensions, (inevitably implying the Thomas rotation (TR)) , leads to contradictions within the framework of relativistic electrodynamics; in Beckmann's words: "Thomas may save Einstein's kinematics. but then his electrodynamics go to pot" (Beckmann 1991). It is the aim of this paper to prove that, even within the frame of relativistic kinematics, TR comes into conflict with the property of inertial motion, according to which the velocity of a moving inertial reference frame is radially oriented with respect to the origin of the inertial reference frame at relative rest; in Phipps's words: "Thomas rotation contaminates the inertial motion" (Phipps 1990). Nevertheless, TR may be reformulated in such a way that it complies with the concept of inertial motion, but in turn two contradictions are involved. The first is related to the ambiguity regarding the rotation of the Cartesian axes as they are viewed from the reference frame at rest. It is well known that, when applied to the gyration of the rotation axis of a spinning mass, TRs give rise to the so-called Thomas precession (also known as Thomas half) (Thomas 1926, 1982). Thomas precession was proposed by Thomas as a means of reconciling a conflict in the spinning electron of the Goudsmit-Uhlenbeck model that gave twice the observed precession effect. Concerning this proposal Uhlenbeck writes (Uhlenbeck 1976):

> *I remember that when I first heard about it, it seemed unbelievable that a relativistic effect would give a factor of 2 instead of something of the order v/c ... Even the cognoscenti of the relativity theory (Einstein included!) were quite surprised.*

However, it is shown in this paper that the second consequence of the above-mentioned reformulation of the Thomas rotation problem disproves the Thomas precession; Uhlenbeck's suspicion was fully justified. The property of two successive inertial translations is discussed in the first section and a brief review of the present resolution of Thomas rotation is presented in the second section. The third section is devoted to identifying the angle between the relativistic composite velocities $\mathbf{v}_1 \oplus \mathbf{v}_2$ and $\mathbf{v}_2 \oplus \mathbf{v}_1$ of the non-collinear velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ with the Thomas rotation angle of the Cartesian axis. To the best of the author's knowledge, this problem has not drawn the attention of previous investigators. A modified formulation of the Thomas rotation problem from the viewpoint of the property of the two successive inertial translations is discussed in the comments.

1. The property of two successive inertial translations

If an event located at a point P is referred to a reference frame Σ_o, with respect to which P is at rest at this time, Σ_o is a proper reference frame. Therefore, to refer an event located at P which moves with respect to a reference frame Σ, is equivalent to assuming the existence, at least in the close neighborhood of P, of a proper reference frame with respect to which P is at rest. As a consequence, it is advantageous to organize the set of reference frames $\Sigma_1, \Sigma_2, \ldots \Sigma_j, \ldots \Sigma_k, \ldots$ into pairs, one of them being the proper reference frame. By convention, two inertial reference frames Σ_j and Σ_k, with parallel homologous axes, form a directly ordered pair of reference frames (Σ_j, Σ_k) if the first frame of the pair Σ_j moves with the velocity $\mathbf{v}_{kj} = \mathbf{v}_k$ with respect to the second frame of the pair Σ_k. If $t_j = t_k = 0$ corresponds to the event when the origins O_j of Σ_j and O_k of Σ_k coincide, the inertial motion of Σ_j assumes a radial orientation of the velocity $\mathbf{v}_k$ with respect to O_k, that is, $\mathbf{v}_k$ is collinear with the position vector $\mathbf{r}_k$ (Figure 1). This is the standard configuration of a pair of inertial reference frames. Consequently, the fundamental property of a pair of inertial reference frames in standard configuration is the radial orientation of the velocity (ROV) of the moving inertial frame with respect to the origin of the inertial motion at relative rest.

Let Σ_o, Σ_1 and Σ_2 be three inertial frames. The velocity of $\Sigma_o(\Sigma_1)$ relative to $\Sigma_1(\Sigma_2)$ is $\mathbf{v}_1(\mathbf{v}_2)$, collinear with the position vector radius $\mathbf{r}_1(\mathbf{r}_2)$ of the origin $O_o(O_1)$ of $\Sigma_o(\Sigma_1)$ with respect to $\Sigma_1(\Sigma_2)$. Now, with Σ_o, Σ_1 and Σ_2 let us form the set $S_d^{(3)}$ of the three directly ordered pairs of reference frames,

$$S_d^{(3)} = \{(\Sigma_o, \Sigma_1), (\Sigma_1, \Sigma_2), (\Sigma_o, \Sigma_2)\} \tag{1}$$

Since the pairs (Σ_o, Σ_1) and (Σ_1, Σ_2) satisfy the ROV condition (following the collinearity of $\mathbf{v}_1$ with $\mathbf{r}_1$ and $\mathbf{v}_2$ with $\mathbf{r}_2$), then the pair (Σ_o, Σ_2) fulfills the ROV condition, and as a consequence, it is a pair of inertial frames in standard configuration, if and only if the composite velocity $\mathbf{u}_{2*1} = \mathbf{v}_2 * \mathbf{v}_1$ of Σ_o with respect to Σ_2 has a radial orientation with respect to O_2, *i.e.* it is collinear with the position vector radius $\mathbf{r}$ (Figure 2a). The velocity $\mathbf{u}_{2*1}$ is called the direct composite velocity of $\mathbf{v}_2$ and $\mathbf{v}_1$. Let us next consider the set $S_i^{(3)}$ of three inversely ordered pairs of inertial frames,

$$S_i^{(3)} = \{(\Sigma_2, \Sigma_1), (\Sigma_1, \Sigma_o), (\Sigma_2, \Sigma_o)\} \tag{2}$$

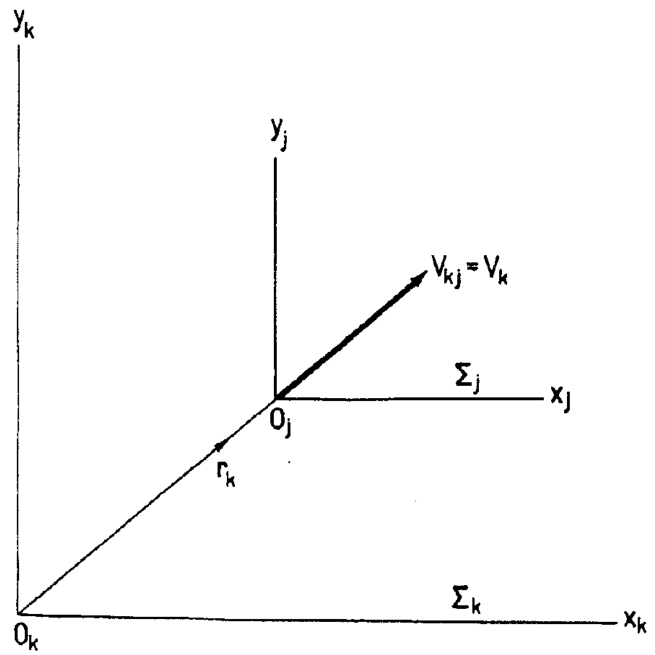

Figure 1

Similarly, the pairs of frames (Σ_2, Σ_1) and (Σ_1, Σ_o) satisfy the ROV condition and (Σ_2, Σ_o) is a pair of inertial frames in standard configuration if and only if the composite velocity $(-\mathbf{v}_1)*(-\mathbf{v}_2) = -(\mathbf{v}_1 * \mathbf{v}_2) = \mathbf{u}_{1*2}$ has a radial orientation with respect to O_o and, in this way, is collinear with $-\mathbf{r}$, (Figure 2b). The velocity $\mathbf{u}_{1*2}$ is called the inverse composite velocity of $-\mathbf{v}_1$ and $-\mathbf{v}_2$. Since $(\mathbf{v}_2 * \mathbf{v}_1)$ is collinear with $\mathbf{r}$ and $-(\mathbf{v}_1 * \mathbf{v}_2)$ is collinear with $-\mathbf{r}$, then $(\mathbf{v}_2 * \mathbf{v}_1)$ is collinear with $(\mathbf{v}_1 * \mathbf{v}_2)$,

$$\mathbf{v}_1 * \mathbf{v}_2 \uparrow \mathbf{v}_2 * \mathbf{v}_1 \uparrow \mathbf{r} \tag{3}$$

In the derivation of the collinearity of $(\mathbf{v}_1 * \mathbf{v}_2)$ and $(\mathbf{v}_2 * \mathbf{v}_1)$, no reference has been made to a specific composition law of velocities (*) which may be Galilean (+) or Lorentzian $\oplus$ (it may also be something else: *cf.* Mocanu 1986, 1987).

In the case of the Galilean velocity composition law, equation (3) becomes (Figure 3a),

$$\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}_2 + \mathbf{v}_1 \uparrow \mathbf{r}, \ \mathbf{v}_1, \mathbf{v}_2 \in R^3, \tag{4}$$

respectively,

$$\mathbf{e}_r = \mathbf{e}_{\mathbf{v}_1+\mathbf{v}_2} = \mathbf{e}_{\mathbf{v}_2+\mathbf{v}_1}, \tag{5}$$

where R^3 is the set of 3-vectors in the Euclidean 3-space and $\mathbf{e}_r$ denotes the unit vector of $\mathbf{r}$, identical with the unit vectors $\mathbf{e}_{\mathbf{v}_1+\mathbf{v}_2}$ and $\mathbf{e}_{\mathbf{v}_2+\mathbf{v}_1}$ of the Galilean composite velocities $\mathbf{v}_1 + \mathbf{v}_2$ and $\mathbf{v}_2 + \mathbf{v}_1$,

$$\mathbf{e}_{\mathbf{v}_1+\mathbf{v}_2} = \frac{\mathbf{v}_1 + \mathbf{v}_2}{|\mathbf{v}_1 + \mathbf{v}_2|}, \ \mathbf{e}_{\mathbf{v}_2+\mathbf{v}_1} = \frac{\mathbf{v}_2 + \mathbf{v}_1}{|\mathbf{v}_2 + \mathbf{v}_1|}. \tag{6}$$

Equation (5) confirms the adequacy of the Galilean composition law of velocities with the inertial motion.

According to the relativistic composition law $\oplus$, the composite velocities $\mathbf{u}_{2\oplus1} = \mathbf{v}_2 \oplus \mathbf{v}_1$ and $-\mathbf{u}_{1\oplus2} = \mathbf{v}_1 \oplus \mathbf{v}_2$ are given by (Mocanu 1986, 1991a; Ungar 1988a)

$$\mathbf{u}_{2\oplus1} = \mathbf{v}_2 \oplus \mathbf{v}_1 = \frac{\gamma_2 \mathbf{v}_2 + \kappa_2 \mathbf{v}_2 (\mathbf{v}_1 \cdot \mathbf{v}_2) v_2^{-2} + \mathbf{v}_1}{\gamma_2 (1 + \mathbf{v}_1 \cdot \mathbf{v}_2 / c^2)}, \tag{7a}$$

$$\mathbf{v}_1, \mathbf{v}_2 \in R_c^3,$$

$$-\mathbf{u}_{1\oplus2} = \mathbf{v}_1 \oplus \mathbf{v}_2 = \frac{\gamma_1 \mathbf{v}_1 + \kappa_1 \mathbf{v}_1 (\mathbf{v}_1 \cdot \mathbf{v}_2) v_1^{-2} + \mathbf{v}_2}{\gamma_1 (1 + \mathbf{v}_1 \cdot \mathbf{v}_2 / c^2)}, \tag{7b}$$

where $R_c^3 = \{\mathbf{v} \in R^3; |\mathbf{v}| < c\}$ is the set of 3-vectors in the Euclidean 3-space R^3 with magnitude smaller than the speed of light c in free space, called the relativistically admissible velocity space, and γ_1, γ_2 are the Lorentz factors,

$$\gamma_1 = \left(1 + \frac{v_1^2}{c^2}\right)^{-1/2}, \ \gamma_2 = \left(1 + \frac{v_2^2}{c^2}\right)^{-1/2}, \ \kappa_1 = \gamma_1 - 1, \ \kappa_2 = \gamma_2 - 1. \tag{8}$$

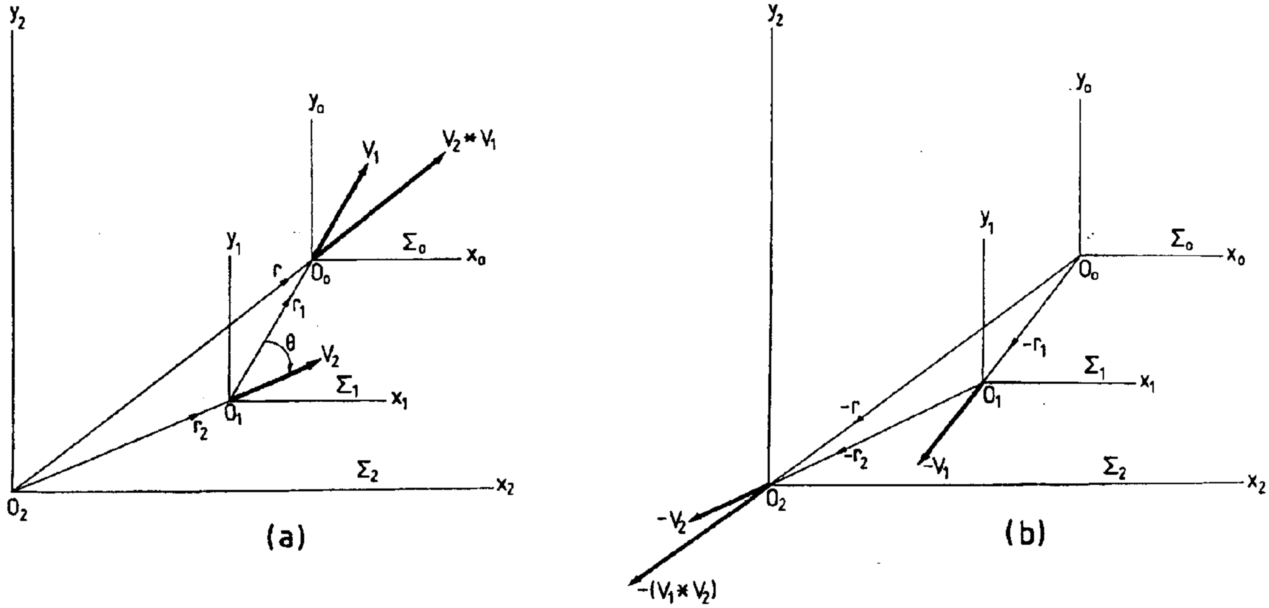

Figure 2

Although the resulting velocities $\mathbf{u}_{2\oplus 1}$ and have $\mathbf{u}_{1\oplus 2}$ equal magnitudes (Mocanu 1986, 1991)

$$(\mathbf{v}_2 \oplus \mathbf{v}_1)^2 = (\mathbf{v}_1 \oplus \mathbf{v}_2)^2 = \left[\frac{\mathbf{v}_1 + \mathbf{v}_2}{1 + \mathbf{v}_1 \cdot \mathbf{v}_2 / c^2}\right]^2 - \frac{1}{c^2}\left[\frac{\mathbf{v}_1 \times \mathbf{v}_2}{1 + \mathbf{v}_1 \cdot \mathbf{v}_2 / c^2}\right]^2 \quad (9)$$

they have different orientations,

$$\mathbf{v}_1 \oplus \mathbf{v}_2 \neq \mathbf{v}_2 \oplus \mathbf{v}_1. \quad (10)$$

From the relations (5) and (10), where $*$ is replaced by $\oplus$, the following inequalities are derived,

$$\mathbf{e}_\mathbf{r} \neq \mathbf{e}_{\mathbf{v}_1 \oplus \mathbf{v}_2} \neq \mathbf{e}_{\mathbf{v}_2 \oplus \mathbf{v}_1}, \quad (11)$$

where $\mathbf{e}_{\mathbf{v}_1 \oplus \mathbf{v}_2}$ and $\mathbf{e}_{\mathbf{v}_2 \oplus \mathbf{v}_1}$ denote the unit vectors of $\mathbf{v}_1 \oplus \mathbf{v}_2$ and $\mathbf{v}_2 \oplus \mathbf{v}_1$,

$$\mathbf{e}_{\mathbf{v}_1 \oplus \mathbf{v}_2} = \frac{\mathbf{v}_1 \oplus \mathbf{v}_2}{|\mathbf{v}_1 \oplus \mathbf{v}_2|}, \ \mathbf{e}_{\mathbf{v}_2 \oplus \mathbf{v}_1} = \frac{\mathbf{v}_2 \oplus \mathbf{v}_1}{|\mathbf{v}_2 \oplus \mathbf{v}_1|}. \quad (12)$$

Comparison of equations (5) and (11) shows that the relativistic composition law of velocities leads to a conflict with the concept of inertial motion. To illustrate this point beyond any doubt it is sufficient to represent graphically the velocity vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_1 + \mathbf{v}_2, \mathbf{v}_2 + \mathbf{v}_1, \mathbf{v}_1 \oplus \mathbf{v}_2, \mathbf{v}_2 \oplus \mathbf{v}_1$. Since the equations

$$\mathbf{e} = \frac{\mathbf{v}_1 \times \mathbf{v}_2}{|\mathbf{v}_1 \times \mathbf{v}_2|} = \frac{(\mathbf{v}_1 \oplus \mathbf{v}_2) \times (\mathbf{v}_2 \oplus \mathbf{v}_1)}{|(\mathbf{v}_1 \oplus \mathbf{v}_2) \times (\mathbf{v}_2 \oplus \mathbf{v}_1)|}, \quad (13)$$

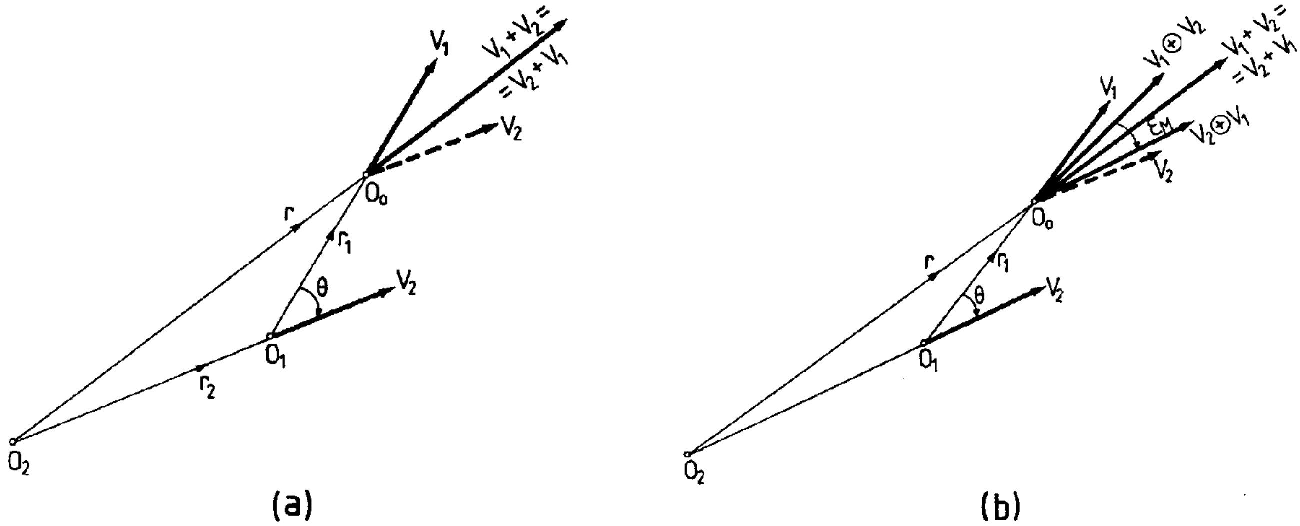

Figure 3

hold, the above-mentioned velocities are coplanar, as depicted in Figure 3b.

Since $\mathbf{v}_1 \oplus \mathbf{v}_2$ and $\mathbf{v}_2 \oplus \mathbf{v}_1$ are not radially oriented with respect to Σ_2, then neither the direct pair of inertial frames (Σ_o, Σ_2) nor the inverse pair (Σ_2, Σ_o) fulfill the condition of the standard configuration of an inertial reference frame (Figure 1). Consequently, the inequality (10) prevents us from extending the 1+1 Lorentz transformation to one with 1+3 dimensions. The incompatibility of the two successive Lorentz transformations implying non-collinear velocities may be reformulated as follows: two successive relativistic inertial translations in two non-collinear directions are not equivalent to a single inertial translation. The paradox is evident; it relies upon the contradictory relations (5) and (11). The resolution of the paradox within the Special Theory of Relativity implies the Thomas rotation, as will be shown in the next section.

2. Brief review of the Thomas rotation problem

According to Ungar (1988a), in Figure 2a, the axes of both frames Σ_o and Σ_2 have been constructed parallel to those of Σ_1 as seen by observers accompanying the moving system Σ_1. Nevertheless, an observer in Σ_2 sees the axis of Σ_o rotated relative to his own axis by a Thomas rotation angle ϵ, as shown in Figure 4.

There have been various attempts in the literature to express the TR in terms of its generating velocity parameters, resulting in expressions having different forms. Since the Lorentz transformation in one time dimension and one space dimension can be represented by complex numbers, there are expressions of TR in terms of hypercomplex numbers (Ben-Menahem 1985). Expressions of TR were presented by Ben-Menahem (1985, 1986) and others (Belloni and Reina 1986, Dancoff and Inglis 1936, Fischer 1972, Furry 1955, Hirschfeld and Metzger 1986, MacKenzie 1972, Rowe 1984, Schelupski 1957). Various approximations were obtained by Salingaros in a series of papers in which he used the Baker-Campbel-Hausdorff formula, which corrects the product of non-commuting exponentials (Salingros 1986, 1988). Salingaros's results are discussed by Baylis and Jones using the Pauli algebra (Baylis and Jones 1988). The determination of the TR by matrix algebra is well described by Goldstein: "In general, the algebra involved is quite forbidding, more than enough, usually, to discourage any actual demonstration of the rotation matrix" (Goldstein 1980).

Thomas rotation is referred to as a Wigner rotation by several authors (*e.g.* Ben-Menahem 1985). Noz and Kim (1988) reserve the term Wigner rotation for the space rotation defined in the Lorentz frame in which the boosted particle under consideration is at rest. About Wigner rotation Jones writes: "It is well to realize, however, that the Wigner rotation is a quite complicated object. It depends ... on three Lorentz transformations rather in an intricate way" (Baylis and Jones 1988). Recently, Ungar, in a series of papers (1988a,b, 1989a,b, 1990, 1991) not only resolved the TR by means of matrix algebra, but elaborated an interesting formalism underlying a non-associative group structure for relativistically admissible velocities. The superiority of Ungar's resolution over other existing ones rests on the fact that it appears in a rotation matrix form to which standard results may be applied. In general, two successive Lorentz transformations (also called boosts) are not equivalent to a boost; the combination of two successive boosts $B(\mathbf{v}_1)B(\mathbf{v}_2)$ is equivalent to a net boost preceded or followed by a Thomas rotation,

$$B(\mathbf{v}_1)B(\mathbf{v}_2)=\begin{cases}B(\mathbf{v}_1\oplus\mathbf{v}_2)\mathrm{Tom}[\mathbf{v}_1;\mathbf{v}_2]\\ \mathrm{Tom}[\mathbf{v}_1;\mathbf{v}_2]B(\mathbf{v}_2\oplus\mathbf{v}_1)\end{cases} \tag{14}$$

where $\mathrm{Tom}[\mathbf{v}_1;\mathbf{v}_2]$ is a 4×4 matrix representing space rotation of time space coordinates. If the rotation angle from $\mathbf{v}_1$ to $\mathbf{v}_2$ is denoted by θ (Figure 2a), the TR angle ϵ is given by (Ungar 1988a, Ben-Menahem 1985)

$$\cos\epsilon=\frac{(k+\cos\theta)^2-\sin^2\theta}{(k+\cos\theta)^2+\sin^2\theta}, \qquad \sin\epsilon=\frac{2(k+\cos\theta)\sin\theta}{(k+\cos\theta)^2+\sin^2\theta}, \tag{15}$$

where k denotes,

$$k^2=\frac{(\gamma_1+1)(\gamma_2+1)}{(\gamma_1-1)(\gamma_2-1)} \tag{16}$$

Replacing k from equation (16) into (15) leads to the expressions of $\cos\epsilon$ and $\sin\epsilon$ in terms of γ_1, γ_2 and θ,

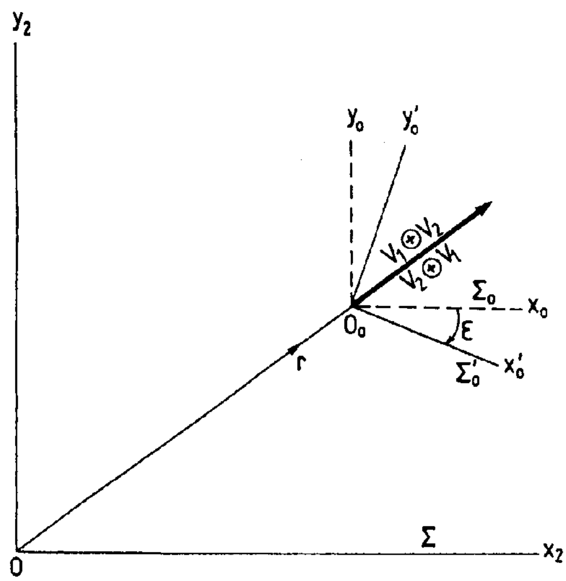

Figure 4

$$\cos\epsilon = \frac{c_o + c_1\cos\theta + c_2\cos^2\theta}{n_o + n_1\cos\theta},$$
$$\sin\epsilon = \frac{(s_o + s_1\cos\theta)\sin\theta}{n_o + n_1\cos\theta} \qquad (17)$$

where $c_i(i = 0,1,2)$, n_j and $s_j(j = 0,1)$ are given by (A.7).

Thus, in the literature, the study of the TR by means of vector and matrix algebra (several authors circumvent the difficulties by implying the Clifford formalism) is restricted by a severe complexity. In the next section of this paper the author copes with the complexity of the TR by abstraction, thus obtaining an astonishing simplicity with which TR can be derived. The reward is that we discover that the TR angle ϵ of the Cartesian space coordinate (Figure 1a) is identical with the angle ϵ_M between the composite velocities $\mathbf{v}_1 \oplus \mathbf{v}_2$ and $\mathbf{v}_2 \oplus \mathbf{v}_1$ (Figure 3b). This result allows us to render evident the ambiguities related to the TR angle, and disproves the Thomas precession.

3. Determining the angle between the composite velocities

According to the actual interpretation of TR given by Ungar ((1988a), Ben-Menahem (1985) and Salingaros (1986, 1988), the rotation of the Cartesian axis of Σ_o with respect to Σ_2 implies the coincidence of $\mathbf{v}_1 \oplus \mathbf{v}_2$ and $\mathbf{v}_2 \oplus \mathbf{v}_1$ (Figure 4). This suggests an examination of the dependence of the TR angle ϵ upon the angle ϵ_M between the composite velocities $\mathbf{v}_1 \oplus \mathbf{v}_2$ and $\mathbf{v}_2 \oplus \mathbf{v}_1$

$$\epsilon_M = \angle(\mathbf{v}_1 \oplus \mathbf{v}_2, \mathbf{v}_2 \oplus \mathbf{v}_1). \qquad (18)$$

In order to express the angle ϵ_M as simply as possible, the scalar and vector products between $\mathbf{v}_1 \oplus \mathbf{v}_2$ and $\mathbf{v}_2 \oplus \mathbf{v}_1$ may be used

$$\cos\epsilon_M = \frac{(\mathbf{v}_1 \oplus \mathbf{v}_2)\cdot(\mathbf{v}_2 \oplus \mathbf{v}_1)}{(\mathbf{v}_1 \oplus \mathbf{v}_2)^2}$$
$$\sin\epsilon_M = \frac{|(\mathbf{v}_1 \oplus \mathbf{v}_2)\times(\mathbf{v}_2 \oplus \mathbf{v}_1)|}{(\mathbf{v}_1 \oplus \mathbf{v}_2)^2} \qquad (19)$$

After simple operations, equations (19) expressed in terms of γ_1, γ_2 and θ are identical with (17), as it is derived in (A.9), *i.e.* the Thomas rotation angle ϵ of the Cartesian axis (as it is viewed from Σ_2) is identical with the angle ϵ_M between the composite velocities $\mathbf{v}_1 \oplus \mathbf{v}_2$ and $\mathbf{v}_2 \oplus \mathbf{v}_1$

$$\epsilon = \epsilon_M \qquad (20)$$

According to the identity (20), the statement resulting from equations (14), formulated as follows: two successive Lorentz transformations are not equivalent to a pure Lorentz transformation, but to a pure Lorentz transformation preceded (followed) by a rotation angle $\epsilon(-\epsilon)$ of the Cartesian coordinate axis Ox_o (Figure 4)—may be replaced by the statement: two successive Lorentz transformations are not equivalent to a pure Lorentz transformation but to a pure Lorentz transformation followed by a rotation angle $\epsilon_M(-\epsilon_M)$ of $\mathbf{v}_1 \oplus \mathbf{v}_2(\mathbf{v}_2 \oplus \mathbf{v}_1)$ to coincide with $\mathbf{v}_2 \oplus \mathbf{v}_1(\mathbf{v}_1 \oplus \mathbf{v}_2)$, (Figure 5) such that equations (14) become the following equations:

$$\begin{aligned} B(\mathbf{v}_1)B(\mathbf{v}_2) &= R(\epsilon_M)B(\mathbf{v}_1 \oplus \mathbf{v}_2) \\ B(\mathbf{v}_2)B(\mathbf{v}_1) &= R(-\epsilon_M)B(\mathbf{v}_2 \oplus \mathbf{v}_1) \end{aligned} . \qquad (21)$$

where $R(\alpha)$ denotes the rotation operator of angle α.

In this last formulation, the TR quite clearly indicates the conflict with the fundamental property of two succes-

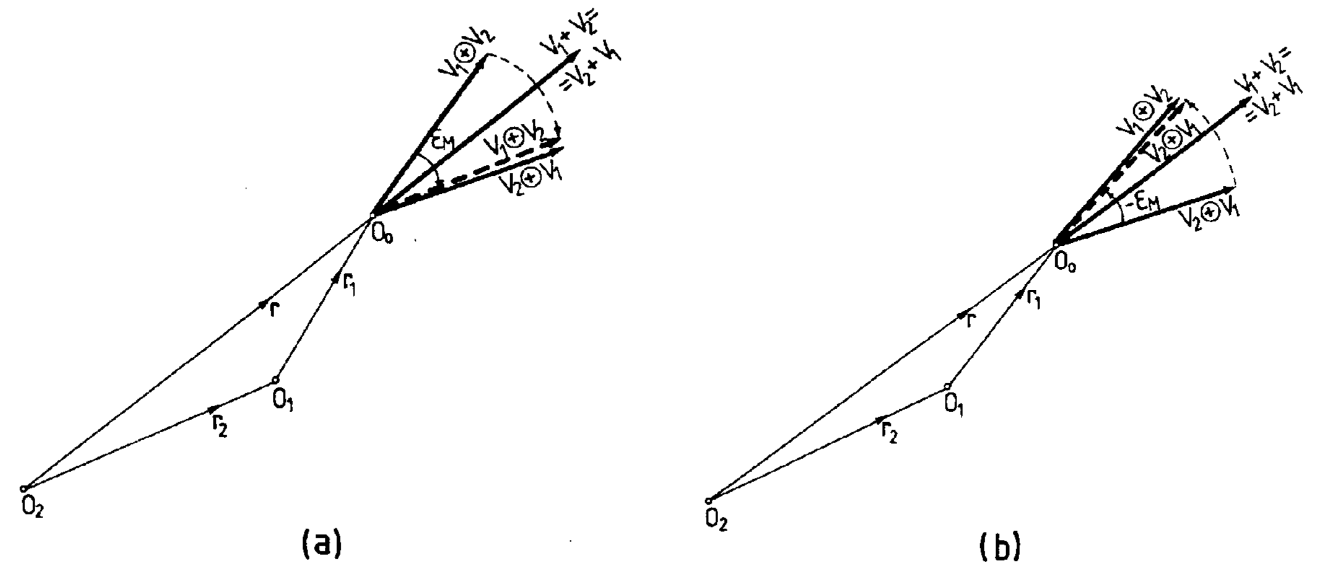

Figure 5

sive inertial translations. Indeed, since neither $\mathbf{v}_1 \oplus \mathbf{v}_2$ nor $\mathbf{v}_2 \oplus \mathbf{v}_1$ satisfy the ROV condition, *i.e.* they are not radially oriented with respect to the origin O_2 of Σ_2 (Figure 5), then the Thomas rotation angle $\epsilon_M(-\epsilon_M)$ of $\mathbf{v}_1 \oplus \mathbf{v}_2(\mathbf{v}_2 \oplus \mathbf{v}_1)$ coinciding with $\mathbf{v}_2 \oplus \mathbf{v}_1(\mathbf{v}_1 \oplus \mathbf{v}_2)$ does not ensure the extension of 1+1 Lorentz transformation to 1+3 dimensions. In fact, the matrix $\mathrm{Tom}[\mathbf{v}_1;\mathbf{v}_2]$ involved in equations (14) rotates the vector $\mathbf{v}_1 \oplus \mathbf{v}_2(\mathbf{v}_2 \oplus \mathbf{v}_1)$ from a position where it contradicts the ROV condition into the position of the vector $\mathbf{v}_2 \oplus \mathbf{v}_1(\mathbf{v}_1 \oplus \mathbf{v}_2)$, where it also contradicts the ROV condition.

Thomas employed his rotation angle ϵ to calculate the so-called Thomas precession which give rise to the Thomas half, that is, a factor ½ that was necessary to resolve a conflict between theory and observation (Thomas 1926, 1982). As Møller has pointed out, if $\mathbf{v}_1 = \mathbf{v}$ and $\mathbf{v}_2 = \mathbf{v} + \mathrm{d}\mathbf{v}$ the rotation vector $\mathrm{d}\bar{\Omega}$ given by (Møller 1987)

$$\mathrm{d}\bar{\Omega} = \frac{1}{v^2}\left\{\frac{1}{\sqrt{1-v^2/c^2}} - 1\right\}(\mathbf{v}\times\mathrm{d}\mathbf{v}), \tag{22}$$

represents the infinitesimal rotation which has to be applied to the axis of Σ_2, at the time $t + \mathrm{d}t$, in order to give the same orientation as has the axis in Σ_o. If we put $\mathrm{d}\mathbf{v} = \mathbf{v}\mathrm{d}t$ and when $v \ll c$, equation (22) becomes,

$$\bar{\omega} = \frac{\mathbf{v}\times\dot{\mathbf{v}}}{2c^2},\ \dot{\mathbf{v}} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t}, \tag{23}$$

where $\bar{\omega}$ denotes the Thomas precession. Since $\bar{\Omega}$ assumes inertial motion which in turn is contradicted by the identity (20), then a disproof of the Thomas rotation implies a disproof of the Thomas precession.

Comments

Summarizing, we can state that, identifying the Thomas rotation angle ϵ of the Cartesian axis of the moving reference frame Σ_o, as it is viewed from the origin of the reference frame Σ_2, at relative rest, with the angle ϵ_M between the composite velocities $\mathbf{v}_1 \oplus \mathbf{v}_2$ and $\mathbf{v}_2 \oplus \mathbf{v}_1$ (Figure 4) leads to a conflict between the Thomas rotation and the fundamental property of inertial motion. As a consequence TR is unable to extend the 1+1 Lorentz transformation to 1+3 dimensions.

However, from the fact that the rotation, either of the Cartesian axis or of the composite velocities, does not change the position vector radius, $\mathbf{r}_1, \mathbf{r}_2$ and $\mathbf{r}$, it suggests a modified formulation of Thomas rotation problem. Instead of the rotation of $\mathbf{v}_1 \oplus \mathbf{v}_2$ coinciding with $\mathbf{v}_2 \oplus \mathbf{v}_1$, (Figure 5) we shall consider the rotation of $\mathbf{v}_1 \oplus \mathbf{v}_2$ to coincide with the position vector radius $\mathbf{r}$, that is, with the unit vector $\mathbf{e}_{\mathbf{v}_1+\mathbf{v}_2}$ of the Galilean composite velocity $\mathbf{v}_1 + \mathbf{v}_2$, as it is shown in Figure 6a, where the rotation angle $\epsilon_{\mathbf{v}_1\oplus\mathbf{v}_2} = \angle\left(\mathbf{v}_1 \oplus \mathbf{v}_2, \mathbf{e}_{\mathbf{v}_1+\mathbf{v}_2}\right)$ is given by

$$\cos\epsilon_{\mathbf{v}_1\oplus\mathbf{v}_2} = \frac{(\mathbf{v}_1 \oplus \mathbf{v}_2)\cdot\mathbf{e}_{\mathbf{v}_1+\mathbf{v}_2}}{|\mathbf{v}_1 \oplus \mathbf{v}_2|}, \tag{24}$$

$$\sin\epsilon_{\mathbf{v}_1\oplus\mathbf{v}_2} = \frac{|(\mathbf{v}_1 \oplus \mathbf{v}_2)\times\mathbf{e}_{\mathbf{v}_1+\mathbf{v}_2}|}{|\mathbf{v}_1 \oplus \mathbf{v}_2|}. \tag{25}$$

Similarly, instead of the rotation of $\mathbf{v}_2 \oplus \mathbf{v}_1$ coinciding with $\mathbf{v}_1 \oplus \mathbf{v}_2$ we limit the rotation of $\mathbf{v}_2 \oplus \mathbf{v}_1$ to coincidence with $\mathbf{r}$, that is, with unit vector $\mathbf{e}_{\mathbf{v}_2+\mathbf{v}_1}$ of the Galilean composite velocity $\mathbf{v}_2 + \mathbf{v}_1$, as it is depicted in Figure 6b, where the rotation angle $\epsilon_{\mathbf{v}_2\oplus\mathbf{v}_1} = \angle\left(\mathbf{e}_{\mathbf{v}_2+\mathbf{v}_1}, \mathbf{v}_2 \oplus \mathbf{v}_1\right)$ is given by

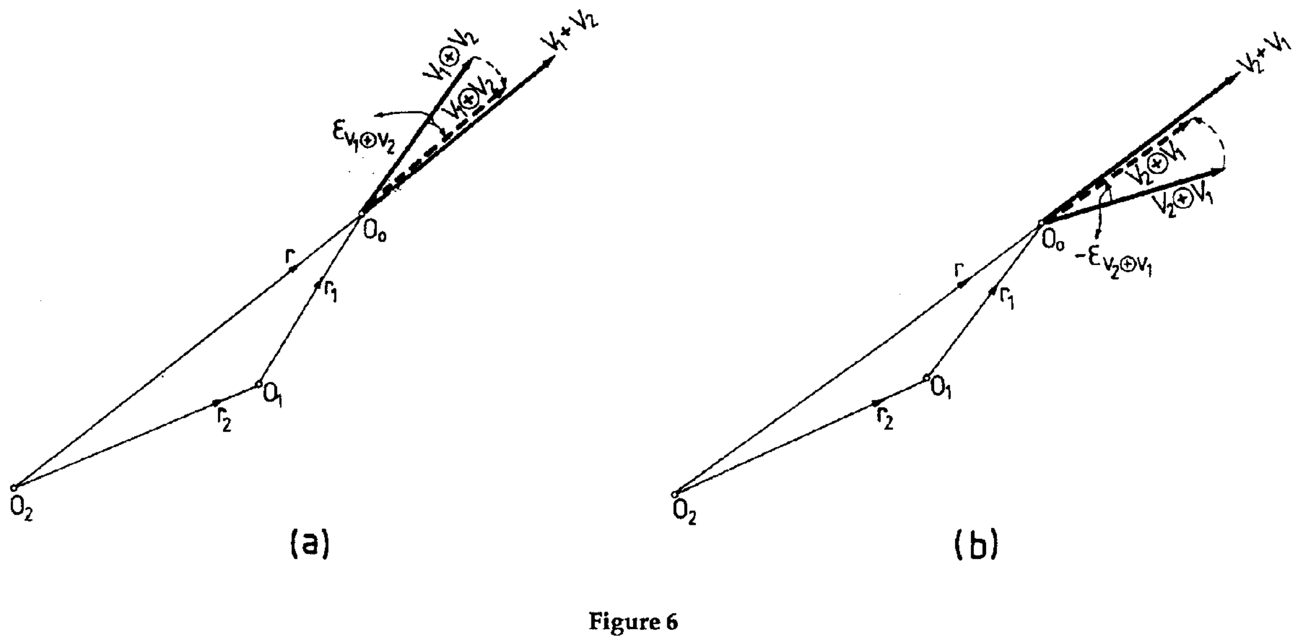

Figure 6

$$\cos\epsilon_{\mathbf{v}_2\oplus\mathbf{v}_1} = \frac{\mathbf{e}_{\mathbf{v}_2+\mathbf{v}_1}\cdot(\mathbf{v}_2\oplus\mathbf{v}_1)}{|\mathbf{v}_2\oplus\mathbf{v}_1|}, \tag{26}$$

$$\sin\epsilon_{\mathbf{v}_2\oplus\mathbf{v}_1} = \frac{|\mathbf{e}_{\mathbf{v}_2+\mathbf{v}_1}\times(\mathbf{v}_2\oplus\mathbf{v}_1)|}{|\mathbf{v}_2\oplus\mathbf{v}_1|}. \tag{27}$$

Since the rotation angles $\epsilon_{\mathbf{v}_1\oplus\mathbf{v}_2}$ and $\epsilon_{\mathbf{v}_2\oplus\mathbf{v}_1}$ imply the ROV condition, at first sight, this new formulation of the Thomas rotation problem appears correctly solved. Nevertheless, corresponding to the different rotations $\epsilon_{\mathbf{v}_1\oplus\mathbf{v}_2}$ and $\epsilon_{\mathbf{v}_2\oplus\mathbf{v}_1}$, there are two different rotations of the Cartesian axis and, consequently we are in the situation of a paradox: which one is the "correct" Thomas rotation angle? is it $\epsilon_{\mathbf{v}_1\oplus\mathbf{v}_2}$ or $\epsilon_{\mathbf{v}_2\oplus\mathbf{v}_1}$? The answer is that both of them satisfy the ROV condition, *i.e.* they comply with the fundamental property of inertial motion and, as a consequence, even in this new formulation the Thomas rotation is not able to derive "a unique and coherent rotation angle" in order to ensure the extension of the Lorentz transformation from 1+1 to 1+3 dimensions. It is also evident that the non-associativity and non-commutativity of the composition law of non-collinear velocities is an internal contradiction of the Special Theory of Relativity.

Conclusions

The Thomas rotation paradox revealed earlier by Mocanu (1991) following two successive Lorentz transformations of the electromagnetic field is confirmed within the framework of relativistic kinematics following two successive boosts.

Appendix

Denoting by N_c the numerator of $\cos\epsilon_M$ (19), and taking into account equations (7) we have

$$\begin{aligned} N_c &= \left[\gamma_1\mathbf{v}_1+\mathbf{v}_2+\kappa_1\mathbf{v}_1(\mathbf{v}_1\cdot\mathbf{v}_2)v_1^{-2}\right]\left[\gamma_2\mathbf{v}_2+\mathbf{v}_1+\kappa_2\mathbf{v}_2(\mathbf{v}_1\cdot\mathbf{v}_2)v_2^{-2}\right] \\ &= \gamma_1 v_1^2+\gamma_2 v_2^2+(\gamma_1+\gamma_2+\gamma_1\gamma_2)v_1 v_2\cos\theta+ \\ &+(\gamma_1\kappa_2 v_1^2+\gamma_2\kappa_1 v_2^2)\cos^2\theta+\kappa_1\kappa_2 v_1 v_2\cos^3\theta \end{aligned} \tag{A.1}$$

Similarly, the numerator N_s of $\sin\epsilon_M$ (19) becomes

$$\begin{aligned} N_s &= \left|\begin{matrix}(\gamma_1\mathbf{v}_1+\mathbf{v}_2+\kappa_1\mathbf{v}_1(\mathbf{v}_1\cdot\mathbf{v}_2)v_1^{-2}) \\ \times(\gamma_2\mathbf{v}_2+\mathbf{v}_1+\kappa_2\mathbf{v}_2(\mathbf{v}_1\cdot\mathbf{v}_2)v_2^{-2})\end{matrix}\right| \\ &= \left\{\begin{matrix}(\gamma_1\gamma_2-1)v_1v_2+\left[\gamma_1(\gamma_2-1)v_1^2+\gamma_2(\gamma_1-1)v_2^2\right] \\ \times\cos\theta+(\gamma_1-1)(\gamma_2-1)v_1v_2\cos^2\theta\end{matrix}\right\}\sin\theta \end{aligned} \tag{A.2}$$

Denoting by n the denominator of $\cos\epsilon_M$ and $\sin\epsilon_M$ and taking into account equation (9) we get

$$\begin{aligned} n &= \gamma_1\gamma_2\left[(v_1+v_2)^2-c^{-2}(v_1+v_2)^2\right]= \\ &\gamma_1\gamma_2\left[v_1^2+v_2^2-c^{-2}v_1^2v_2^2+2v_1v_2\cos\theta+c^{-2}v_1^2v_2^2\cos^2\theta\right] \end{aligned} \tag{A.3}$$

From equations (8), the magnitudes of v_1 and v_2 as functions of γ_1 and γ_2 are given by,

$$v_1 = c\gamma_1^{-1}\sqrt{\gamma_1^2-1},\; v_2 = c\gamma_2^{-1}\sqrt{\gamma_2^2-1}$$

such that N_c (A.1), N_s (A.2) and n (A.3) expressed in terms of γ_1 and γ_2 take the forms

$$N_c = c^2\gamma_1^{-1}\gamma_2^{-1}(c_o+c_1\cos\theta+c^2\cos^2\theta)\psi(\gamma_1,\gamma_2,\theta), \tag{A.4}$$

$$N_s = c^2\gamma_1^{-1}\gamma_2^{-1}[(s_o+s_1\cos\theta)\sin\theta]\psi(\gamma_1,\gamma_2,\theta), \tag{A.5}$$

$$n = c^2\gamma_1^{-1}\gamma_2^{-1}(n_o+n_1\cos\theta)\psi(\gamma_1,\gamma_2,\theta), \tag{A.6}$$

where $c_i(i=0,1,2)$, $n_j, s_j(j=0,1)$ and $(\gamma_1,\gamma_2,\theta)$ are given by

$$\begin{aligned} & c_o = \gamma_1+\gamma_2;\quad c_1^2 = (\gamma_1^2-1)(\gamma_2^2-1) \\ & c_2 = (\gamma_1-1)(\gamma_2-1) = s_1 \\ & n_0 = \gamma_1\gamma_2+1,\; n_1 = c_1 = s_0 \\ & \psi(\gamma_1,\gamma_2,\theta) = \gamma_1\gamma_2-1+c_1\cos\theta \end{aligned} \tag{A.7}$$

Inserting the expressions for N_c, N_s and n (A.4–A.6) into equations (19) yields

$$\begin{aligned} \cos\epsilon_M &= \frac{c_o+c_1\cos\theta+c_2\cos^2\theta}{n_o+n_1\cos\theta} \\ \sin\epsilon_M &= \frac{(s_o+s_1\cos\theta)\sin\theta}{n_o+n_1\cos\theta} \end{aligned} \tag{A.8}$$

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