The Equivalence Principle as a Consequence of the Third Law

D. F. Roscoe
Department of Applied Mathematics
University of Sheffield
Sheffield S1O 2TN, UK
Fax: 742-739826, Email: D.Roscoe@uk.ac.shef.pa

Introduction

One of the enduring mysteries of classical and modern physics arises from the fact that the value obtained for the gravitational-mass ratio of two particles compared in a weighing experiment is identical to the value obtained for the inertial-mass ratio of the same two particles compared in a collision experiment (to within experimental error). For this reason we speak of the equivalence between inertial and gravitational mass, and tend to use the concepts interchangeably. However, whilst practitioners of science and engineering have been content to accept this equivalence as a matter of fact, there has, until recently, been no rational understanding of it. However, recently, Ghosh, Assis and I have each given the basis for such an understanding.

By treating the Universe as composed of an homogeneous part + a non-homogeneous part, Assis, and Ghosh by similar arguments, have argued that inertial effects in a body arise from the gravitational influence of the homogeneous part of the Universe acting on the body, whilst the manifest gravitational effects arise from the non-homogeneous part of the Universe acting on the body. Thus, by assuming a particular (Galilean invariant) theory of gravitation as given, Assis and Ghosh are able to arrive at the Equivalence Principle for Galilean invariant definitions of mass. One can imagine that the basic ideas used in these analyses could be extended, by choice of an appropriate theory of gravitation, to cover the Equivalence Principle for Lorentz invariant definitions of mass.

By contrast, I have effectively established a rational foundation for the Equivalence Principle in the reverse way by showing how, by treating inertia as given, it is possible to arrive at gravitation; specifically, by assuming the conservation of three-momentum (Galilean inertia), it is possible to arrive at a Galilean invariant theory of gravitation, whilst by assuming the conservation of four-momentum

(Lorentzian inertia), it is possible to arrive at a Lorentzian invariant theory of gravitation.

However, both of these approaches to establishing a rational foundation for the Equivalence Principle are difficult to absorb and so, in the following, we present a. simple Newtonian analysis which can equally well be said to provide such a foundation. Specifically, we show that an appropriate consideration of the general nature of Newtonian instantaneous action-at-a-distance shows us that the gravitational-mass ratio of two particles determined in a weighing experiment can be interpreted directly as the inertial-mass ratio of the same two particles determined in a particular kind of 'at-a-distance collision experiment'.

A Newtonian analysis

A common formulation of Newton's Third Law states 'for every action there is an equal and opposite reaction' and so, for two particles in collision (observed from an inertial frame) we necessarily have

$$m\Delta\mathbf{v} = -M\Delta\mathbf{V}$$

where m, M are the respective inertial masses and $\Delta\mathbf{v}, \Delta\mathbf{V}$ are the respective velocity changes through the collision. However, this formal rendering of the Third Law for a two-particle system is specific to the case of *collision*, in which interaction occurs through direct contact. Now, the various phenomena of gravitation and electromagnetism inform us we must admit the notion of a more general case in which the interaction between two particles can be characterized as being continual, and 'at-a-distance'. In this case, the foregoing formal statement which summarizes the case of 'collision' is inadequate, and must be generalized to become

$$m\ddot{\mathbf{x}} \equiv +\mathbf{G}(t); \quad M\ddot{\mathbf{X}} = -\mathbf{G}(t) \qquad (1)$$

where $\mathbf{G}(t)$ can be, interpreted as the programme of Newtonian forces exerted on each particle in the system. However, (1) by itself is not sufficient to express the full meaning of the Third Law for the at-a-distance case since it merely asserts the equality of the opposing forces, but neglects to say that these forces act in the same straight line—a condition which is generally taken to be implied in any statement of the Third Law. This condition of colinearity can be expressed as

$$\mathbf{x} - \mathbf{X} = \lambda\mathbf{G}(t) \qquad (2)$$

where λ is an unknown Galilean scalar.

If we now note from (1) that $\ddot{\mathbf{x}} = \mathbf{G}/m$ and $\ddot{\mathbf{X}} = -\mathbf{G}/M$ then we find

$$\ddot{\mathbf{x}} - \ddot{\mathbf{X}} = \left(\frac{m+M}{mM}\right)\mathbf{G}(t)$$

If we now write $\mathbf{Y} \equiv (\mathbf{x} - \mathbf{X})$ then $\mathbf{G}(t)$ can be eliminated between this latter statement and (2) to obtain

$$\mathbf{Y} = \lambda\left(\frac{mM}{m+M}\right)\ddot{\mathbf{Y}} \qquad (3)$$

Now consider, the general nature of a procedure which is required to compare two test masses, m_1 and m_2, for equality only, via measurements made on their respective interactions with the mass M, which occur according to (3). Suppose these latter measurements are made simultaneously and in the same location—implying $\mathbf{Y}$ and λ are identical for both cases—and suppose they simply consist of instantaneous measurements of the two relative acceleration vectors, denoted as $\ddot{\mathbf{Y}}_1$ and $\ddot{\mathbf{Y}}_2$ respectively. For each set of measurements we obtain, by taking the inner-product of (3) with itself, the scalar equations

$$\mathbf{Y}^T\mathbf{Y} = \lambda^2\left(\frac{m_1 M}{m_1 + M}\right)^2 \ddot{\mathbf{Y}}_1^T\ddot{\mathbf{Y}}_1$$

$$\mathbf{Y}^T\mathbf{Y} = \lambda^2\left(\frac{m_2 M}{m_2 + M}\right)^2 \ddot{\mathbf{Y}}_2^T\ddot{\mathbf{Y}}_2$$

where superscript 'T' denotes 'vector transpose', and it follows immediately that

$$\left(\frac{m_1}{m_2}\right)\left(\frac{m_2 + M}{m_1 + M}\right) = \sqrt{\frac{\ddot{\mathbf{Y}}_2^T\ddot{\mathbf{Y}}_2}{\ddot{\mathbf{Y}}_1^T\ddot{\mathbf{Y}}_1}}$$

It is clear from this latter expression that m_1, and m_2 can only be equal when the magnitudes of the two relative acceleration vectors are also equal. In this general way, it can be seen how Newtonian action-at-a-distance can be used to calibrate the relative inertial masses of particles, and how—significantly—the process is independent of the specific form of λ which defines the quantitative nature of the action; that is, the process could be electromagnetic, gravitational, or anything else one cares to imagine.

Newton's Law of Universal Gravitation can now be seen as simply a special case which starts with the assumption that M (say) is so large compared to m that, by (1), $\ddot{\mathbf{x}} >> \ddot{\mathbf{X}} \approx 0$ with the consequence $\mathbf{X}$ can be considered fixed in some subclass of 'inertial frames; consequently, relative to those particular frames in this subclass for which $\mathbf{X} \approx 0$, (3) becomes $x \approx \lambda m\ddot{\mathbf{x}}$. Consistency with Kepler's Laws is then obtained if $\lambda \equiv -\lambda_0|\mathbf{x}|^3$, where λ_0 is an appropriately determined positive constant. With this perspective of Newtonian gravitation, we can see how the process described above to test for equality between inertial masses in a Newtonian at-a-distance interaction defines the essential features of the classical pan-scale weighing experiment. In other words, relative masses determined in weighing experiments can be interpreted simply as measures of relative *inertial* masses determined in at-a-distance 'collision' experiments, and Newtonian gravitation can be interpreted as simply a special case of an 'at-a-distance inertial interaction'.

This qualitative approach to understanding the nature of Newtonian gravitation leads very quickly to a parallel understanding of the 'potential energy' concept which arises in the Newtonian prescription: Consider a binary inertial interaction, described from a frame which is at rest with respect to the system's centre of gravity, and suppose the particles concerned have masses and velocities (m,v), (M,V). Then we have:

$$mv + MV = 0$$

from which it follows

$$\tfrac{1}{2}mv \cdot v = \left(\frac{M}{m}\right)\tfrac{1}{2}MV \cdot V,$$

so that, using $(KE)_m$ and $(KE)_M$ to denote the respective kinetic energies, we finally obtain

$$(KE)_m - \left(\frac{M}{m}\right)(KE_M) = 0$$

The important thing to remember is that, since this relationship follows directly from the Third Law, it *must be* true for any Newtonian binary relationship, including, specifically, a classical Newtonian gravitating pair for which $M >> m$. In such a case, m is the mass of the gravitating particle, and M is the mass of the 'gravitational source'. The kinetic energy/potential energy relationship for the gravitating particle in such a system, given, of course, by $(KE)_m + (PE)_m = 0$, is, like (4), only true in that class of frames which is at rest with respect to the system's centre of gravity; consequently, it can be compared directly with (4), with the result

$$(PE)_m = -\left(\frac{M}{m}\right)(KE)_M.$$

That is, the potential energy of a gravitating mass is proportional to the negative kinetic energy of the source mass, and this conclusion is based purely on the physics of momentum conservation. In other words, the Newtonian concept of 'potential energy' is simply a modelling device which, in cases for which $M >> m$, allows the dynamical behaviour of the gravitating source to be ignored.

Apart from providing Newtonian gravitation with an inertial context, and thereby already eliminating the need for the equivalence principle, this simple analysis reminds us of the essentially approximate nature of the inverse square law (that is, the source is assumed to be in an inertial frame) as a description of a Galilean-invariant gravitation theory and, correspondingly, suggests the existence of a general Galilean-invariant gravitation theory which accounts also for the inertial motions of gravitational sources. The Assis and Ghosh theories, being based on Weber's Law, which is truly relativistic in the sense of being applicable in all frames, are of this type.

References

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